From a 2D Shape to a String Structure Using the Symmetry Set*

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Abstract. Many attempts have been made to represent families of 2D shapes in a simpler way. These approaches lead to so-called structures as the Symmetry Set ($\mathcal{SS}$) and a subset of it, the Medial Axis ($\mathcal{MA}$). In this paper a novel method to represent the $\mathcal{SS}$ as a string is presented. This structure is related to so-called arc-annotated sequences, and allows faster and simpler query algorithms for comparison and database applications than graph structures, used to represent the $\mathcal{MA}$. Example shapes are shown and their data structures derived. They show the stability and robustness of the $\mathcal{SS}$ and its string representation.

1 Introduction

In 2D shape analysis the simplification of shapes into a skeleton-like structure is widely investigated. The Medial Axis ($\mathcal{MA}$) skeleton presented by Blum [1] is commonly used, since it is an intuitive representation that nowadays can be calculated in a fast and robust way. In so-called Shock Graphs, an $\mathcal{MA}$ skeleton is augmented with information of the distance from the boundary at which special skeleton points occurs, as suggested by Blum. Many impressive results on simplification, reconstruction and database search are reported, see e.g. [2,3, 4,5,6,7,8].

The $\mathcal{MA}$ is a member of a larger family, the Symmetry Set ($\mathcal{SS}$) [9], exhibiting nice mathematical properties, but more difficult to compute than the $\mathcal{MA}$. It also yields distinct branches, i.e. unconnected "skeleton" parts, which makes it hard to fit into a graph structure (like the $\mathcal{MA}$) for representation. In section 2 the definitions of these sets and related properties are given.

To overcome the complexity of the $\mathcal{SS}$ with respect to the $\mathcal{MA}$, we introduce in section 3 a sequentional data structure containing both the symmetry set and

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the evolute of the shape, resulting in a representational structure that is less complex and more robust than the $\mathcal{MA}$-based structure, a graph. It is related to the so-called *arc-annotated sequence* [10,11], and allows faster and simpler query algorithms for comparison of objects and all kinds of object database applications.

Examples are given on a convex shape, showing the stability and robustness of the new data structure, in section 4, followed by the conclusions in section 5.

2 Background on Shapes

In this section we give the necessary background regarding properties of shapes, the Medial Axis, the Symmetry Set, and the labeling of points on these sets. For more details, see e.g. [9,6].

Let $\mathcal{S}(x(t), y(t))$ denote a closed 2D shape and $(.)_t = \frac{\partial(.)}{\partial t}$, then $\mathcal{N}(t) = (-y_t, x_t)/\sqrt{x_t^2 + y_t^2}$ denotes its unit normal vector, and $\kappa(t) = (x_t y_{tt} - y_t x_{tt})/\sqrt{x_t^2 + y_t^2}^3$ is its curvature. The evolute $\mathcal{E}(t)$ is given by the set $\mathcal{S} + \mathcal{N}/\kappa$. Note that as κ can traverse through zero, the evolute moves "through" (minus) infinity. This occurs by definition only for concave shapes. An alternative representation can be given implicitly: $\mathcal{S}(x, y) = \{(x, y)|L(x, y) = 0\}$ for some function $L(x, y)$. Then the following formulae can be derived for $\mathcal{N}(x, y)$ and $\kappa(x, y)$: $\mathcal{N}(x, y) = (L_x, L_y)/\sqrt{L_x^2 + L_y^2}$ and $\kappa(x, y) = -(L_x^2 L_{yy} - 2L_x L_y L_{xy} + L_y^2 L_{xx})/\sqrt{L_x^2 + L_y^2}^3$ Although the curve is smooth and differentiable, the evolute contains non-smooth and non-differentiable *cusp* points, viz. those where the curvature is zero or takes a local extremum, respectively.

2.1 Medial Axis and Symmetry Set

The Medial Axis ($\mathcal{MA}$) is defined as the closure of the set of centers of circles that are tangent to the shape at at least two points and that contain no other tangent circles: they are so-called maximal circles. The Symmetry Set $\mathcal{SS}$ is defined as the closure of the set of centers of circles that are tangent to the shape at at least two points [9,12,13,14]. Obviously, the $\mathcal{MA}$ is a subset of the $\mathcal{SS}$ [14].

This is illustrated in Figure 1a. The two points p_1 and p_2 lie on a maximal circle and give rise to a $\mathcal{MA}$ and $\mathcal{SS}$ point. The points p_1 and p_4 give rise to a $\mathcal{SS}$ point.

To calculate these sets, the following procedure can be used, see Figure 1b: Let a circle with unknown location be tangent to the shape at two points. Then its center can be found by using the normalvectors at these points: it is located at the position of each point minus the radius of the circle times the normal vector at each point.

To find these two points, the location of the center and the radius, do the following: Given two vectors p_i and p_j (Figure 1b, with $i = 1$ and $j = 2$) pointing at two locations at the shape, construct the difference vector $p_i - p_j$.

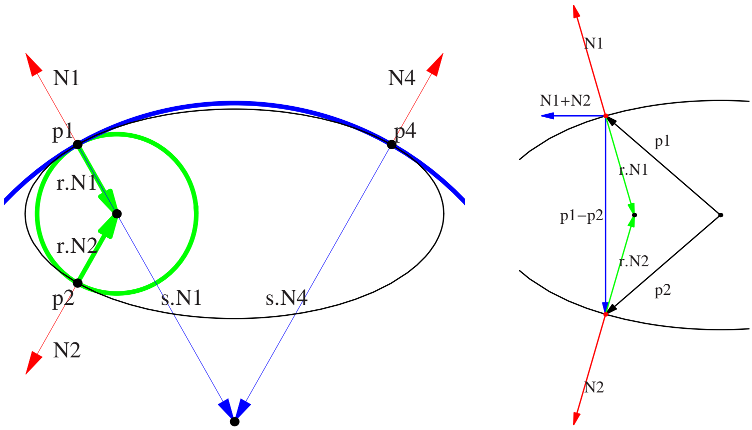

Fig. 1. a) Point p_1 contributes to two tangent circles and thus two $\mathcal{SS}$ points. Only the inner circle contributes to the $\mathcal{MA}$. b) Deriving the Medial Axis and Symmetry Set geometrically. See text for details.

Given the two unit normal vectors N_i and N_j at these locations, construct the vector $N_i + N_j$. If the two constructed vectors are non-zero and perpendicular,

$$(p_i - p_j).(N_i \pm N_j) = 0, \tag{1}$$

the two locations give rise to a tangent circle. The radius r and the center of the circle are given by

$$p_i - rN_i = p_j \pm rN_j \tag{2}$$

and for the $\mathcal{MA}$ one only has to make sure that the circle is maximal. In the remainder of this paper we focus on the $\mathcal{SS}$.

2.2 Classification of Points on the Symmetry Set

It has been shown by Bruce et al. [12] that only five distict types of points can occur for the $\mathcal{SS}$, and by Giblin et al. [13,14] that they are inherited by the $\mathcal{MA}$.

- An A_1^2 point is the "common" midpoint of a circle tangent at two distinct points of the shape.
- An A_1A_2 point is the midpoint of a circle tangent at two distinct points of the shape but located at the evolute.
- An $A_1^2A_1^2$ point is the midpoint of two circles tangent at two pairs of distinct points of the shape with different radii.
- An A_1^3 point is the midpoint of one circle tangent at three distinct points of the shape.
- An A_3 point is the midpoint of a circle located at the evolute and tangent at the point of the shape with the local extremal curvature.

2.3 Properties of the Symmetry Set

Since the $\mathcal{SS}$ is defined locally, global properties of it are not widely investigated and difficult to derive. Banchoff and Giblin [15] have proven an invariant to hold for the number of A_3, A_1A_2, and A_1^3 points, both for the continuous case as the piecewise one. These numbers hold if the shape changes in such a way that the $\mathcal{SS}$ changes significantly. At these changes, called transitions [16], a so-called non-generic event for a static $\mathcal{SS}$ occurs, for instance the presence of a circle tangent to four points of the shape. Sometimes the number of A_3, A_1A_2, and A_1^3 points changes when the $\mathcal{SS}$ goes through a transition. For the $\mathcal{MA}$ part it implies e.g. the birth of a new branch of the skeleton. A list of possible transitions, derived from [16] is given in section 3.3.

3 A Linear Data Structure for the Symmetry Set

One of the main advantages of the $\mathcal{SS}$ is the possibility to represent it as a linear data structure. In general, such a structure is faster to query (according to e.g. [10,11]) than graph structures - the result of methods based on the $\mathcal{MA}$. The fact that the $\mathcal{SS}$ is a larger, more complicated set than the $\mathcal{MA}$ turns out to be advantageous in generating a simpler data structure. In this section the structure is described, together with the stability issues. Examples are given in section 4. Details on the implementations are given in [17].

3.1 Construction of the Data Structure

The data structure contains the elements described in the previous sections: The $\mathcal{SS}$, its special points and the evolute. They are combined in the following way (see also [17]) for an *arbitrary* planar shape:

1. Parameterize the shape.
2. Get the order of the cusps of the evolute by following the parameterization.
3. Find for the $\mathcal{SS}$ the A_3 points: they form the end of individual branches.
4. Relate each cusp of the evolute to an A_3 point.
5. Link the cusps that are on the same branch of the $\mathcal{SS}$.
6. Augment the links with labels, related to the other special points that take place when traveling from one cusp point to the other along the $\mathcal{SS}$-branch.
7. Assign the same label to different branches if an events involves the different branches: the crossings at A_1^3 (three identical labels) and A_1^2/A_1^2 (two identical labels) points. The latter can be left out, since they occur due to projection.
8. Insert moth branches (explained below) between two times two cusps as void cusps.
9. Done.

Moth branches [16] are $\mathcal{SS}$ branches without A_3 points. They contain four A_1A_2 points, that are located on the evolute. Each point is connected by the $\mathcal{SS}$ branch to two other points along the moth.

The data structure thus contains the A_3 points in order, links between pairs of them and augments along the links. Alternatively, one can think of a construction of a set of strings (the links), where each string contains the special points of the $\mathcal{SS}$ along the branch represented by the string. Figure 3 gives an example of the $\mathcal{SS}$ and evolute of a shape, and the derived datastructure.

3.2 Modified Data Structure

Since the introduction of void cusps due to moth branches violates the idea of using only the A_3 points as nodes, a modified structure can be used as well. In this structure the nodes contain A_3 and A_1A_2 points. These can be lined up easily, since the A_1A_2 points are located on the evolute between A_3 points. The linked connections made (strings) are now the subbranches of the $\mathcal{SS}$. The augmentation now only consists of the crossings of subbranches (either at A_1^3, or at both A_1^3 and A_1^2/A_1^2). This is shown in the bottom row of Figure 3.

3.3 Transitions

In this section the known transitions of the $\mathcal{SS}$ [16] in relation to the proposed data structure is presented. The similar thing has been done in the work on comparision of different Shock Graphs, yielding meaningful possible changes of the SG [13,4,5,6,7].

- At an A_1^4 transition a collision of A_1^3 points appears. Before and after the transition six lines, four A_1^3 points and three A_1^2/A_1^2 occur. The result on the $\mathcal{MA}$ is a reordering of the connection of two connected Y-parts of the skeleton. For the $\mathcal{SS}$, however, the Y-parts are the visible parts of $\mathcal{SS}$ branches going through A_1^3 points. So for the $\mathcal{SS}$ representation nothing changes.
- At an A_1A_3 transition, a cusp of the evolute (and thus an endpart of a $\mathcal{SS}$ branch including a A_3 point) intersects a branch of the $\mathcal{SS}$ and an A_1^3 point as well as two A_1A_2 points are created or annihilated. The A_1^3 point lies on the A_3 containing branch, while the other branch contains a "triangle" with the A_1^3 and the A_1A_2's as cornerpoints: the strings $A_3[1]-a$ and b change to $A_3[1]-A_1^2/A_1^2[1]-A_1^3[1]-a$ and $b_1-A_1^3[1]-A_1A_2[1]-A_1^2/A_1^2[1]-A_1A_2[2]-A_1^3[1]-b_2$, vice versa.
- The A_4 transition corresponds to creation or annihilation of a swallowtail structure of the evolute and the creation or annihilation of the enclosed $\mathcal{SS}$ branch with two A_3 and two A_1A_2 points: the string $A_3[1]-A_1^2/A_1^2[1]-A_1A_2[1]-A_1A_2[2]-A_1^2/A_1^2[1]-A_3[2]$.
- At an $A_1^2A_2$ transition two non-intersecting A_1A_2-containing branches meet a third $\mathcal{SS}$ branch at the evolute, creating two times three different branches intersecting at two A_1^3 points. Or the inverse transition occurs: the strings a, $b_1-A_1A_2[1]-b_2$ and $c_1-A_1A_2[2]-c_2$ become $a_1-A_1^3[1]-A_1^3[2]-a_3$, $b_1-A_1^3[1]-A_1A_2[1]-A_1^2/A_1^2[1]-A_1^3[2]-b_2$ and $c_1-A_1^3[1]-A_1^2/A_1^2[1]-A_1A_2[2]-A_1^3[2]-c_2$, vice versa.

- The A_2^2 moth transition describes the creation or annihilation of a $\mathcal{SS}$ branch containing only four A_1A_2 and no A_3 points. These points lie pairwise on two opposite parts of the evolute. each point is connected via the $\mathcal{SS}$ to the two points on the opposite part of the evolute: the strings $A_1A_2[1] - A_1^2/A_1^2[1] - A_1A_2[3] - A_1A_2[2] - A_1^2/A_1^2[1] - A_1A_2[4]$, if the pairs 1,2 and 3,4 are one the same part of the evolute.
- When going through an A_2^2 nib transition, two branches of the $\mathcal{SS}$, each containing an A_1A_2 point, meet and exchange a subbranch. The strings $a - A_1A_2[1] - b$ and $c - A_1A_2[2] - d$ become $a - A_1A_2[1] - c_1 - A_1^2/A_1^2[1] - c_2$ and $b_1 - A_1^2/A_1^2[1] - b_2 - A_1A_2[2] - d$.

Stability. The possible transitions as given above invoke only deletion, insertion or reordering of special points or branches on the data structure in an exact and pre-described manner. It is therefore a robust and stable description of the original shape.

Arc-annotated sequences. The structure as described above is strongly related to the so-called *arc-annotated sequences* used for RNA sequence matching and comparison. It allows the elementary edit-distance - with the insert, delete and replacement operations - as a measure of (dis)similarity between two RNA structures. The operations are directly related to the transitions as described above. For more details on this structure, the reader is referred to [10,11].

Not for the $\mathcal{MA}$. The string representation is *not* suitable for the $\mathcal{MA}$: Since the $\mathcal{MA}$ is a subset of the $\mathcal{SS}$, of the string only a subset of the A_3 points are part of the $\mathcal{MA}$ (less or equal than half of the number of points). But worse, the connections between two A_3 points can consist of unconnected segments. This is due to the fact that at the $\mathcal{SS}$ all local extrema of the curvature are taken into account, in contrast to the $\mathcal{MA}$.

4 Example: The Cubic Oval

As example shape the closed part of a cubic oval is taken, which is implicitly given by $f(x, y; a, b) = 2bxy + a^2(x - x^3) - y^2 = 0$ and $x \geq 0$. Although this is a very simple shape, it clearly shows all the possible points of the $\mathcal{SS}$ and yields a data structure that can be visually verified. Complicated shapes (e.g. those from "Shape Indexing of Image Databases (SIID)" [4]) generate strings that are too complicated to discuss in detail without having seen the elementary building blocks of the string structure.

Figure 2a shows this shape for $a = 1.025$ and $b = 0.09, 0.15, 0.30$. Changing one of these parameters, one is likely to encounter one of the transitions described above. On this shape with these values for the parameters, six extrema of the curvature occur, while the curvature doesn't change sign and the shape is thus convex. Firstly the case $a = 1.025$ and $b = 0.09$ is considered [12].

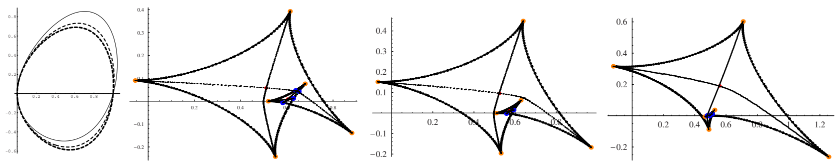

Fig. 2. a) The cubic oval for $a = 1.025$ and $b = .09$ (thick, dashed), $b = .15$ (intermediate thickness, dashed), and $b = .30$ (thin, continuous). The evolute and the $\mathcal{SS}$ of he cubic oval for a=1.025 and b) b=.09 c) b=.15 d) b=.30

4.1 Symmetry Set

The two extra extrema of the curvature, compared to the ellipse, arise from a perturbation of the shape involving an A_4 transition[1]. A direct consequence is that a new branch of the $\mathcal{SS}$ is created. In Figure 3b the complete $\mathcal{SS}$ with the evolute is visualized. The newly created branch has the shape of a swallowtail, as expected from the A_4 transition.

Since the extrema of the curvature alternate along the shape, a maximum and a minimum are created. As a prerequisite of the A_4 transition, the evolute is self-intersecting. Furthermore, the evolute contains six cusps. A direct consequence is that the new branch of the $\mathcal{SS}$ that is created, since branches always start in the cusps, must be essentially different from the two other branches, since the original branches start in cusps that both arise from either local maxima of the curvature, or minima. The newly created branch, however, has endpoints due to a minimum and a maximum of the curvature, so its behaviour must be different. Since real intersections - A_1^3 points - always involve 3 segments, a close-up is needed there.

The newly created branch introduces besides the A_3 and A_1^2/A_1^2 points the other types of special points, viz. the A_1^3 and A_1A_2 points, as shown in Figure 3a. The points are marked with dots on top of them. There are six A_3 points on the cusps of the evolute, four A_1A_2 points on the evolute close to the selfintersection part, three A_1^2/A_1^2 points and one A_1^3 point. The latter can be seen in more detail in the close-up in Figure 3b. It is close to two A_1A_2 points and an A_1^2/A_1^2 point.

4.2 Data Structure

To obtain the data structure, the first cusp of the evolute (A_3 of the $\mathcal{SS}$), is the one in the middle at the bottom. The others are taken clockwise. Then the $\mathcal{SS}$ consists of the branches $1-3$, $2-4$, and $5-6$. Branch $1-3$ intersects $2-4$ at the first A_1^2/A_1^2 point. The close-up of the branch $4-5$, Figure 3 toprow right, gives insight in the behaviour around this part of the $\mathcal{SS}$.

The branches $2-4$ and $5-6$ each contain two A_1A_2 points. Both branches intersect at an A_1^3 point, which is close to the two A_1A_2 points of branch $2-4$:

[1] For $b = 0$ and $a = .5$ an egg-shape is obtained with a $\mathcal{SS}$ similar to the ellipse, although the vertical branch is curved.

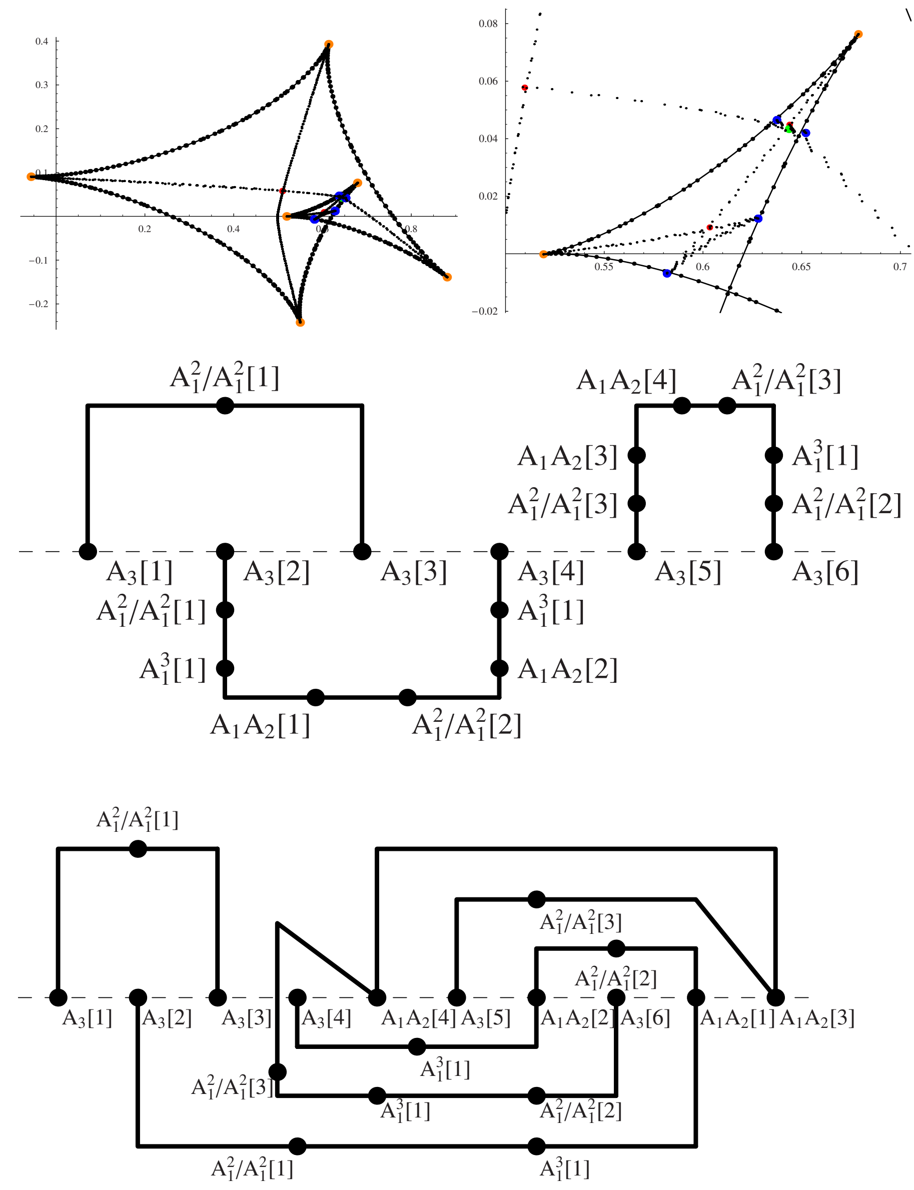

Fig. 3. Top: a) The evolute and the $\mathcal{SS}$ of the oval for $b = 0.09$. The contour with points is the evolute. The point $A_3[1]$, a cusp point of the evolute and an endpoint of an $\mathcal{SS}$ branch, is located at the bottom in the middle, while $A_3[3]$ is located at the top in the middle. b) Close-up, showing the A_1^3 point and the branch $A_3(5)$ (bottom left) - $A_3(6)$ (top right). Middle: String representation of the $\mathcal{SS}$. Bottom: Modified string representation.

At this point two subbranches of branch $2-4$ (the ones combing the A_3's with the A_1A_2's) and branch $4-5$ intersect. Just above this point, branch $5-6$ intersects a subbranches of branch $2-4$ (the one combing the A_1A_2's) in the second A_1^2/A_1^2 point. Finally, two subbranches of branch $2-4$ (the ones combing the A_3's with the A_1A_2's) intersect at the third A_1^2/A_1^2 point.

So the data structure is given by the string and the links of Figure 3, middle row. The modified data structure is given by the string and the links of Figure 3, bottom row. The latter representation clearly decreases the number of augments, but increases the number of points along the string, and thus the number of links. The difference along the string between A_3 points and A_1A_2 points is due to the number of links starting and ending at a point. An A_3 point has one link, an A_1A_2 point two. Note that ignoring the projective A_1^2/A_1^2 points, the second data structure contains only A_1^3 points as augments.

The dashed horizontal line is in fact the evolute of the shape for both representations. It is cut between the points $A_3[1]$ and $A_1A_2[3]$. Therefore the representation is independent of starting point, since the two ends of the string are connected (and thus forming the evolute). It can be cut almost everywhere (albeit not at A_3 and A_1A_2 points).

4.3 Transitions

When b is increased the shape modifies according to Figure 2a. The symmetry set changes also when b is increased, as shown in Figure 2b-d. At two stages a "significant" change takes place: one of the aforementioned transitions. In the following sections the resulting symmetry sets and data structures after the transitions are given. Note that it is non-generic to encounter exactly a situation at which a transition occurs, it is only clear that transitions have been traversed.

Annihilation of the A_1^3 point. When b increases to 0.15, branch $4-5$ releases branch $2-6$, see the top row of Figure 4, annihilating the involved special points. This is a typical example of an A_1A_3 transition. The data structures are now given by the string and the links of Figure 4, middle and bottom row.

Creation of an A_1^3 point. When b increases further to $b = 0.30$, again an A_1A_3 transition occurs, this time the other way round. Now branch $4-5$ intersects branch $1-3$, creating the necessary involved special points, see the top row of Figure 5. The data structures are now given by the string and the links of Figure 5, middle and bottom row.

Stability. One can verify that the structure obtained for $b = 0.09$ and $b = 0.30$ are not identical up to rotational invariance, due to the ordering of the cusps of the evolute. With respect to the $\mathcal{MA}$ representation, the first A_1^3 point does not contribute to the $\mathcal{MA}$, since the $\mathcal{MA}$ consists of the connected component with the smallest radius. For $b = 0.09$, this is only a curve (the vertical oriented

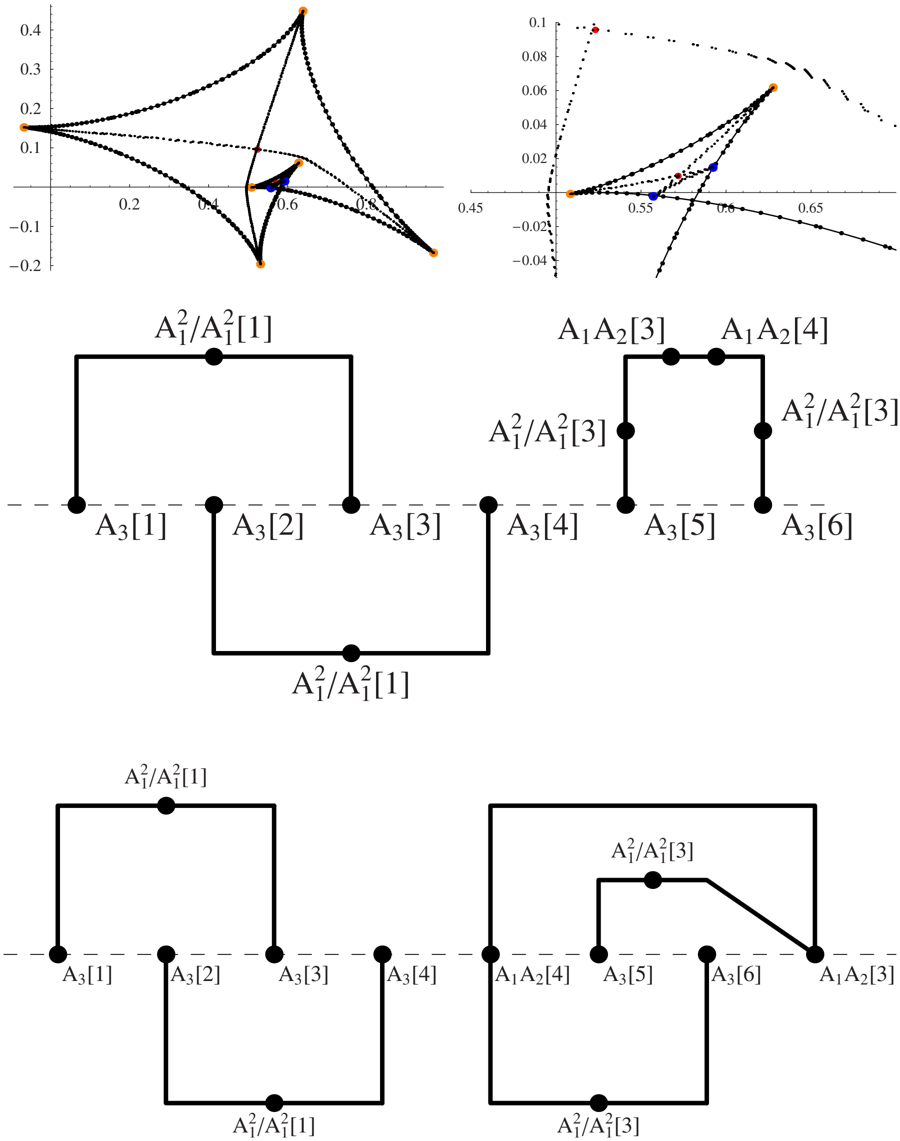

Fig. 4. Top: a) The evolute and the $\mathcal{SS}$ of the oval for $b = 0.15$. The contour with points is the evolute. The point $A_3[1]$, a cusp point of the evolute and an endpoint of an $\mathcal{SS}$ branch, is located at the bottom in the middle, while $A_3[3]$ is located at the top in the middle. b) Close-up, showing that the A_1^3 point has disappeared from the branch $A_3(5)$ (middle left) - $A_3(6)$ (top right). Middle: String representation of the $\mathcal{SS}$. Bottom: Modified string representation.

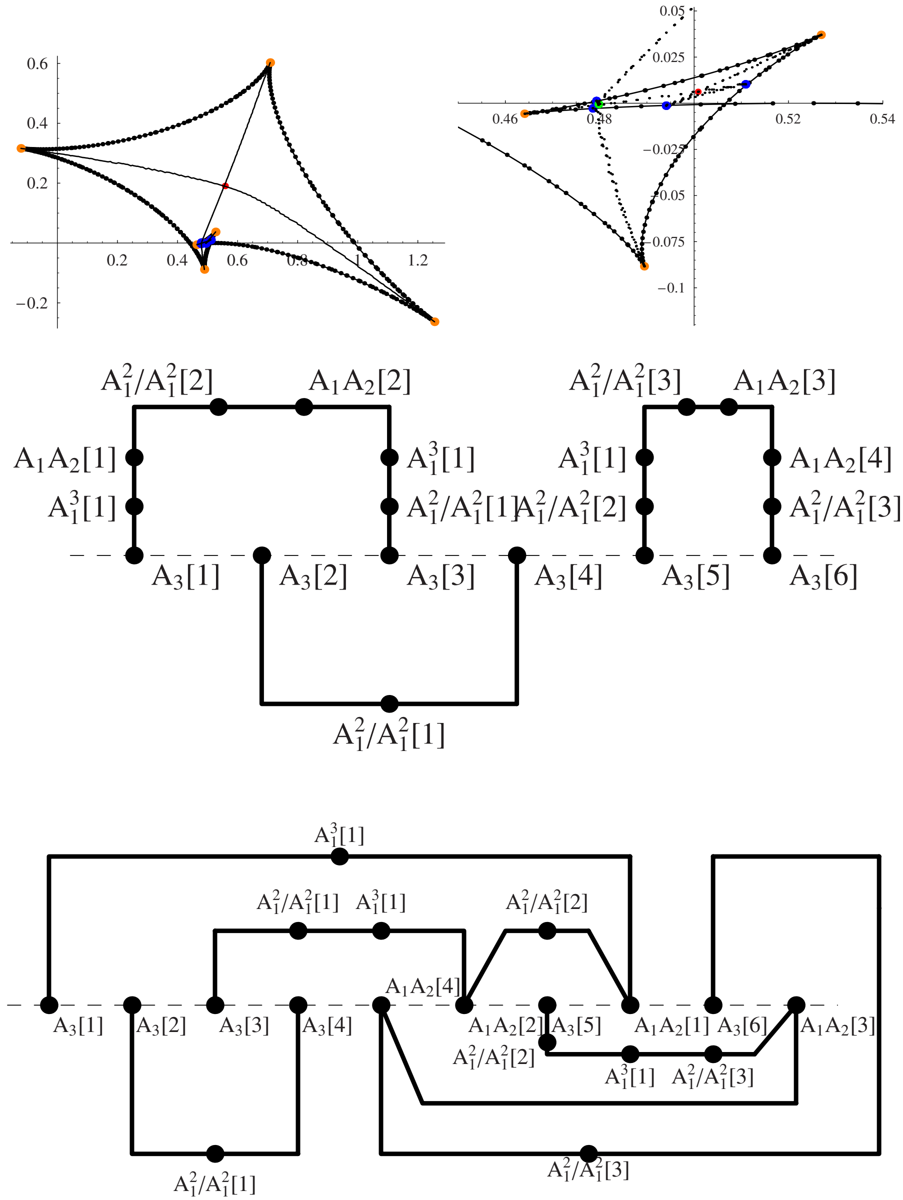

Fig. 5. Top:a) The evolute and the $\mathcal{SS}$ of the oval for $b = 0.30$. The contour with points is the evolute. The point $A_3[1]$, a cusp point of the evolute and an endpoint of an $\mathcal{SS}$ branch, is located at the bottom in the middle, while $A_3[3]$ is located at the top in the middle. b) Close-up, showing again an A_1^3 point at the branch $A_3(5)$ (middle left) - $A_3(6)$ (top right). Middle: String representation of the $\mathcal{SS}$. Bottom: Modified string representation.

one of the $\mathcal{SS}$). For $b = 0.30$, however, the swallowtail intersects this part and the $\mathcal{MA}$-skeleton now contains an extra branch, pointing from the A_1^3 point to the left. This event is known as an instability of the $\mathcal{MA}$, although it satisfies the known transitions. Regarding it as instability may come from the fact that the number of branches of the $\mathcal{MA}$ is not related to the number of extrema of the curvature of the shape, in contrast to the $\mathcal{SS}$. Changing the shape without changing the number of extrema of the curvature, the $\mathcal{MA}$ may gain or loose branches. The $\mathcal{MA}$ of Figure 3 and 4 is just $A_3[1] - A_3[3]$, but for Figure 5 it is formed by the 3 linesegments $A_3[1] - A_1^3[1]$, $A_1^3[1] - A_3[3]$, and $A_3[5] - A_1^3[1]$.

5 Conclusions

In this paper a new linear data structure representing a shape using its the symmetry set ($\mathcal{SS}$) is presented. This structure depends on the ordering of the cusps of the evolute - related to the local extrema of the curvature of the shape, as well as the A_3 points on the $\mathcal{SS}$. The A_3 points are also the endpoints of the branches of the $\mathcal{SS}$. Cusps of the evolute that are connected by the $\mathcal{SS}$ are linked in this data structure. Special points on the $\mathcal{SS}$ (where it touches the evolute, the A_1A_2 points, as well as intersection points - both real and those due to projection) are augmented on these links. A modified data structure takes all the points with evolute interaction - the A_3 and the A_1A_2 points - into account, again in order along the evolute.

Although the $\mathcal{SS}$ is a larger set than the Medial Axis ($\mathcal{MA}$) - even containing it - the representing *string* structure - related to the arc annotated sequence - is in complexity simpler than that for the $\mathcal{MA}$, which yields a *graph* structure. This allows (at least with respect to the theoretical complexity) faster algorithms for the comparison of different shapes, as well as (large) database queries. Obviously, a comparision study between these methods is needed, since lower computational complexity doesn't imply absolute faster query times.

The richer complexity of the $\mathcal{SS}$ prevents it from so-called instabilities that occur in the $\mathcal{MA}$. These "instabilities" are due to parts of the $\mathcal{SS}$ that "suddenly", i.e. due to certain well-known transitions for the $\mathcal{SS}$, become visible. Here the underlying $\mathcal{SS}$ influences the $\mathcal{MA}$, an argument for taking into account the complete $\mathcal{SS}$.

The only way to derive the $\mathcal{SS}$ is by means of a direct implementation of its geometric definition. The complexity of the obtained algorithm is quadratic in the number of points on the shape.

Examples of the data structures and the visualization methods are given on an example shape. Stability issues are discussed in relation to the known transitions, the significant changes of the $\mathcal{SS}$. Their description is translated in terms of the data structures, showing its stability and robustness. The proposed method is also applicable to real shapes, like outlines.

Obviously, still some theoretical questions with respect to the data structure and the $\mathcal{SS}$ are open. Although these questions are very interesting from both a theoretical and practical point of view, they don't influence the derivation and use of the proposed data structure in it self, but may result in a speed-up of algorithms due to advanced label-checking and verification.

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