Regularization Techniques for X-ray CT

Consider the following weighted least-squares problems with two different regularization terms: (i) Generalized Tikhonov regularization (*i.e.*, corresponding to the smoothness prior)

$$u_{\mathrm{GTik}} = \underset{u}{\operatorname{argmin}} \left\{ \frac{1}{2}\|Au - b\|_W^2 + \frac{\gamma}{2}\|Du\|_2^2 \right\} \tag{1}$$

and (ii) Total Variation regularization (*i.e.*, corresponding to the piecewise constant prior)

$$u_{\mathrm{TV}} = \underset{u}{\operatorname{argmin}} \left\{ \frac{1}{2}\|Au - b\|_W^2 + \gamma \sum_{i=1}^{n} \|D_i u\|_2 \right\}. \tag{2}$$

The variable $u \in \mathbb{R}^n$ represents an image of size $N \times N$ (*i.e.*, $n = N^2$), and $D_i u$ is a finite-difference approximation of the image gradient in pixel i.

1. Implement the proximal gradient method for solving the minimization problem in (1) in order to obtain the reconstruction u_{GTik}.

   Hint: In two dimensions, a complex number representation of a finite-difference approximation of the gradient can be computed as Du where D can be formed as sparse matrix in MATLAB as follows:

   ```
   % Forward-difference approx. with Neumann boundary conditions
   % for N-by-N pixel grid with pixel size h-by-h
   Dfd = spdiags([-ones(N-1,1), ones(N-1,1); 0, 1]/h, 0:1, N, N);
   D = kron(Dfd,speye(N)) + kron(j*speye(N),Dfd);
   ```

2. Implement the accelerated gradient method for minimizing a smooth approximation of the problem in (2), *i.e.*,

   $$u_{\mathrm{TV}} \approx \underset{u}{\operatorname{argmin}} \left\{ \frac{1}{2}\|Au - b\|_W^2 + \gamma \sum_{i=1}^{n} \phi_\delta(D_i u) \right\}$$

   where $\phi_\delta(x) = \sqrt{x_1^2 + x_2^2 + \delta^2}$. Show that the gradient is Lipschitz continuous and derive a Lipschitz constant (see lecture slides).

3. Use the function `phantomgallery` from *AIR Tools* to create a two-dimensional smooth phantom, *i.e.*, with `name='smooth'`, as the true image u_{true}. Simulate Poisson measurements following the procedure from yesterday's exercises and compute b. Based on A and b, calculate the reconstructions u_{GTik} and u_{TV}. Try to find the best regularization parameter γ for each problem. Compare the two "best" reconstructions obtained via (1) and (2), and observe the stair-casing artifacts in u_{TV}. (The "best" γ can be different for the two models (1) and (2).)

4. Use the function `phantomgallery` from *AIR Tools* to create a two-dimensional piece-wise constant phantom, e.g., with `name='grains'`, as the true image u_{true}. Simulate Poisson measurements and compute b. Based on A and b, calculate the reconstructions u_{GTik} and u_{TV}. Try several regularization parameters γ, and plot the error norms $\|u_{\mathrm{true}} - u_{\mathrm{TV}}\|_2$ and $\|u_{\mathrm{true}} - u_{\mathrm{GTik}}\|_2$ versus γ. Compare the "best" reconstructions from both models.