A Hydrbid-Spectral Method to Solve the Fully Nonlinear Potential Flow Problem for Water Waves

Mathias Klahn

Technical University of Denmark

Physical Model

We consider the horizontally periodic motion of an incompressible and irrotational fluid moving in the (x,z)-plane. At any given time t the fluid is bounded from below by the impermeable seabed, $-h(x)$, and from above by its free surface, $\eta(x,t)$. A sketch of the system can be seen in Figure 1.

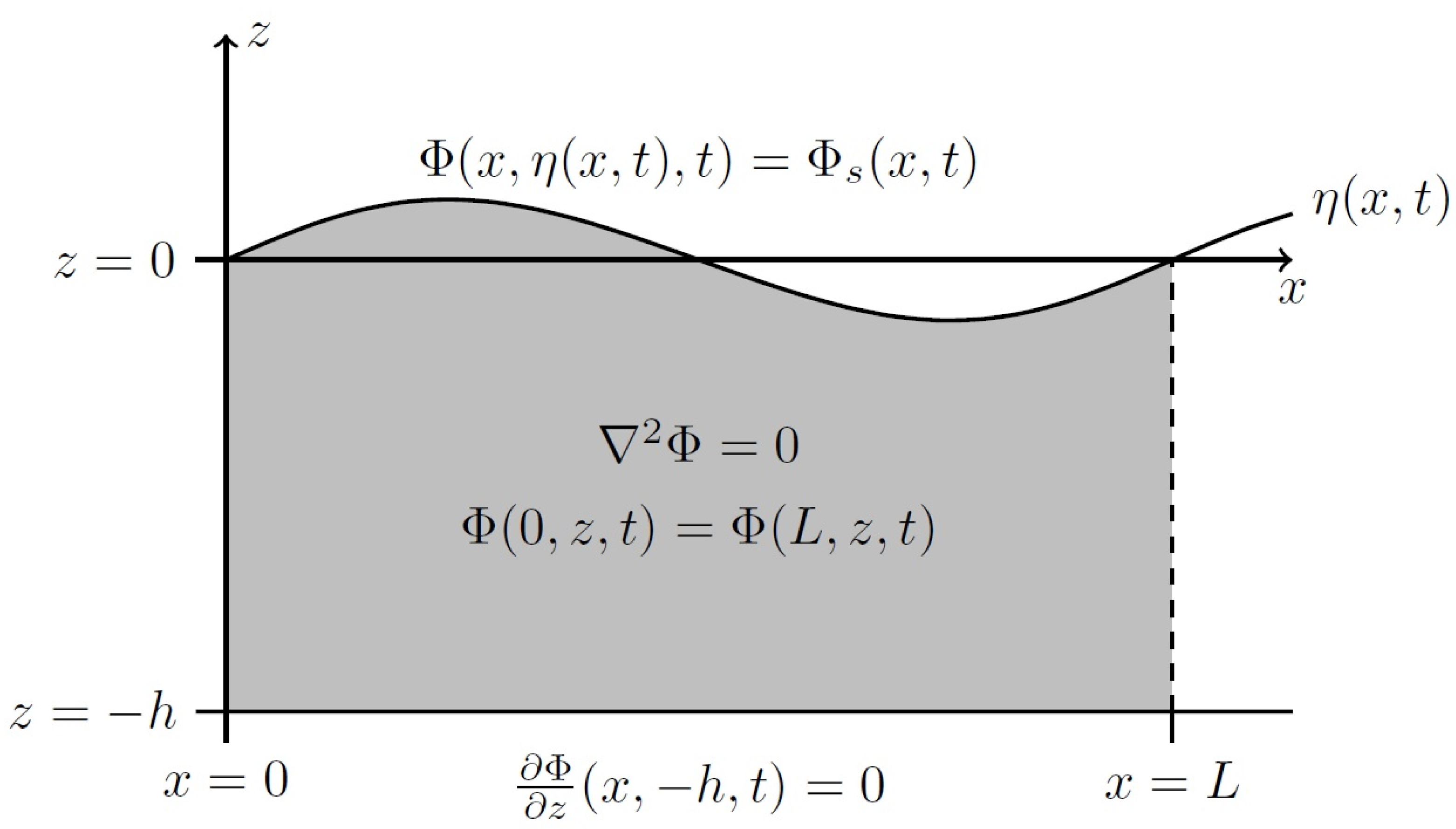

Figure 1: An illustration of the physical system for a flat seabed, $h(x)=h$.

The time evolution of the system is governed by the kinematic and dynamic free surface conditions

$$\frac{\partial \eta}{\partial t} = w_s - u_s\frac{\partial \eta}{\partial x} \tag{1}$$

and

$$\frac{\partial \Phi_s}{\partial t} = -g\eta - \frac{1}{2}\left[\frac{\partial \Phi_s}{\partial x}\right]^2 + \frac{1}{2}w_s^2\left(1+\left[\frac{\partial \eta}{\partial x}\right]^2\right) \tag{2}$$

where g is the gravitational constant. The problem is closed by the fact that the velocity potential $\Phi(x,z)$ solves the following Laplace problem inside the fluid domain:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \tag{3a}$$

$$\frac{\partial h}{\partial x}(x)\frac{\partial \Phi}{\partial x}(x,-h(x),t)+\frac{\partial \Phi}{\partial z}(x,-h(x),t)=0 \tag{3b}$$

$$\Phi(x,\eta(x,t),t)=\Phi_s(x,t) \tag{3c}$$

$$\Phi(0,z,t)=\Phi(L,z,t) \tag{3d}$$

Numerical Method

We solve the time dependent boundary equations using a method of lines approach. The spatial part is discretized with the Fourier-Galerkin method and the temporal part is discretized using the classical 4th order Runge Kutta algorithm. To solve the Laplace problem (3) we introduce the coordinate $s(x,z) = (2z + h(x) - \eta(x))/(h(x)+\eta(x))$ and define $F(x,s(x,z)) = \Phi(x,z)$. From this definition it follows, that $F(x,s)$ solves the following modified Laplace problem:

$$\frac{\partial^2 F}{\partial x^2}+\left(\left[\frac{\partial s}{\partial x}\right]^2+\left[\frac{\partial s}{\partial z}\right]^2\right)\frac{\partial^2 F}{\partial s^2}+2\frac{\partial s}{\partial x}\frac{\partial^2 F}{\partial x\partial s}+\frac{\partial^2 s}{\partial x^2}\frac{\partial F}{\partial s}=0 \tag{4a}$$

$$(h(x)+\eta(x))\frac{\partial h}{\partial x}\frac{\partial F}{\partial x}\bigg|_{s=-1}+2\left(\left[\frac{\partial h}{\partial x}\right]^2+1\right)\frac{\partial F}{\partial s}\bigg|_{s=-1}=0 \tag{4b}$$

$$F(x,1,t)=\Phi_s(x,t) \tag{4c}$$

$$F(0,s,t)=F(L,s,t) \tag{4d}$$

To solve this we approximate $F(x,s)$ as the tensor product of a truncated Fourier series and a truncated Legendre series, i.e.

$$F(x,s)=\Sigma_{r=-N_x}^{N_x}\Sigma_{q=0}^{N_s}F_{rq}\exp\left(i\frac{2\pi r}{L}x\right)L_q(s) \tag{5}$$

where $L_q(s)$ is the q'th Legendre polynomial and use the Galerkin method in the horizontal direction and the Tau method in the vertical direction.

Verification

To test the implementation we consider a Stokes wave of steepness $H/L = 0.11$ and relative water depth $kh = 2$. We first compute the velocity potential $\Phi(x,z)$ and the result can be seen in Figure 2. To check the accuracy we construct the convergence curve of the vertical velocity at the surface, w_s, as a function of N_x and N_s. We compare w_s to the results of [1] which are exact to double precision, and the result can be seen in figure 3. It can be seen that for $N_s = 32$ the method is able to reach a relative accuracy of 10^{-13} for $N_s \geq 60$.

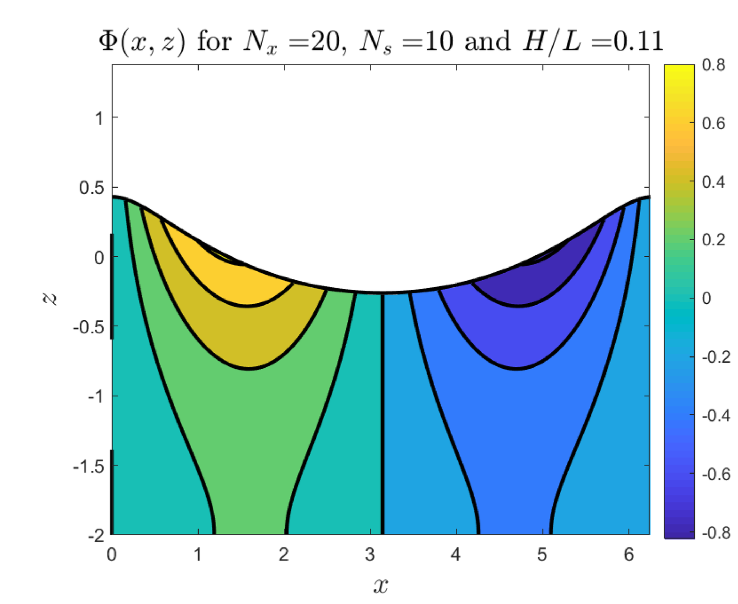

Figure 2: The potential $\Phi(x,z)$ for $N_x = 20$ and $N_s = 10$.

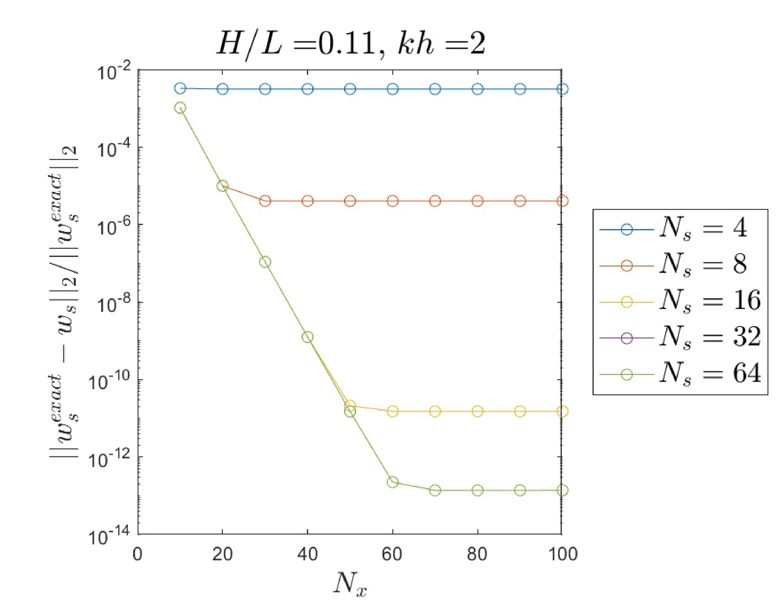

Figure 3: The convergence of w_s as a function of N_x for different values of N_s.

Numerical Examples

To illustrate the capabilities of the implemented method we give two examples of simulations it can handle. The first example is the situation in which a solitary wave runs into a wall and gets reflected. Even though the domain is periodic we can achieve the wall effect numerically by letting two identical solitary waves collide at the middle of the domain. In this way we can make sure that the flux across the middle of the domain is always zero. The time evolution of the solitary wave run up can be seen in figure 4 for a soliton of amplitude $2h$. The solitons are initialized from the solitary waves computed in [2]. The second example is the situation where linear waves propagate over a variable seabed. Because the seabed varies, the amplitude of the waves will also vary. The simulated and theoretically expected relative amplitude can be seen in figure 5. In order to perform this simulation we use a generation zone at the left end of the domain and a absorption zone at the right end of the domain - how this is done can be seen in [3].

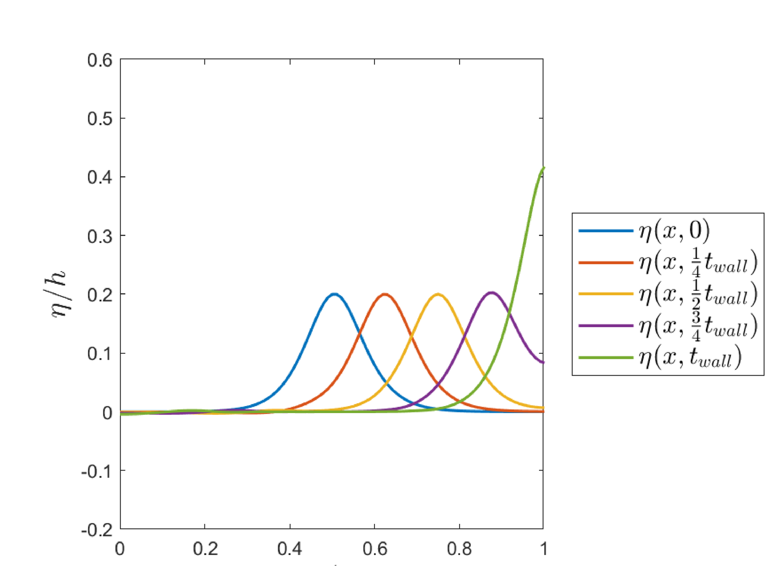

Figure 4: A solitary wave with initial amplitude $0.2h$ running into a wall at different instances of time.

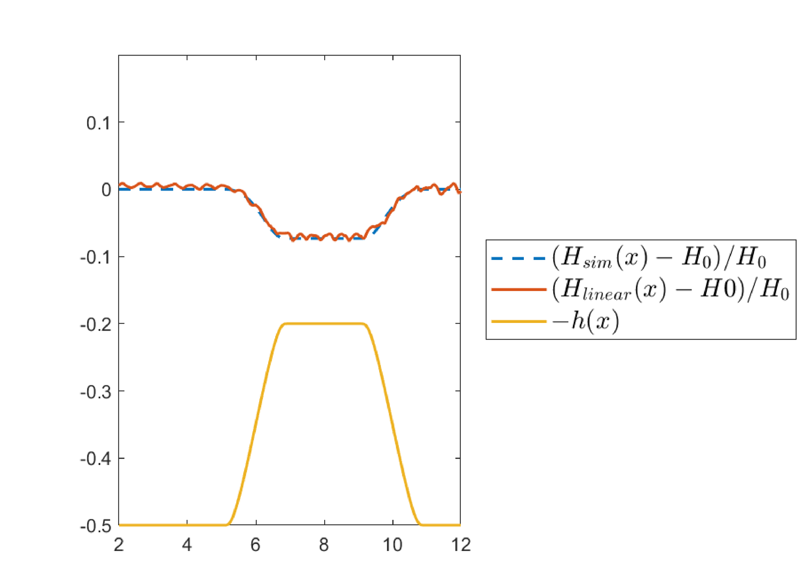

Figure 5: The simulated and expected relative wave height $(H(x)-H_0)/H_0$ for linear waves propagating over a barrier.

References

[1] CLAMOND, D. DUTYKH, D. 2018 Accurate fast computation of steady two-dimensional surface gravity waves in arbitrary depth. *J. Fluid Mech.*

[2] DUTYKH, D. CLAMOND, D. 2013 Efficient computation of steady solitary gravity waves. *Wave Motion.*

[3] ENGSIG-KARUP, A. P. et. al. 2013 Designing Scientific Applications on GPUs, Chapter: Fast hydrodynamics on heterogenous many-core hardware. *Taylor Francis.*