Developing spectral numerical solver for the Helmholtz equation with nonconstant coefficients

02689 - Advanced Numerical Methods for Differential Equations

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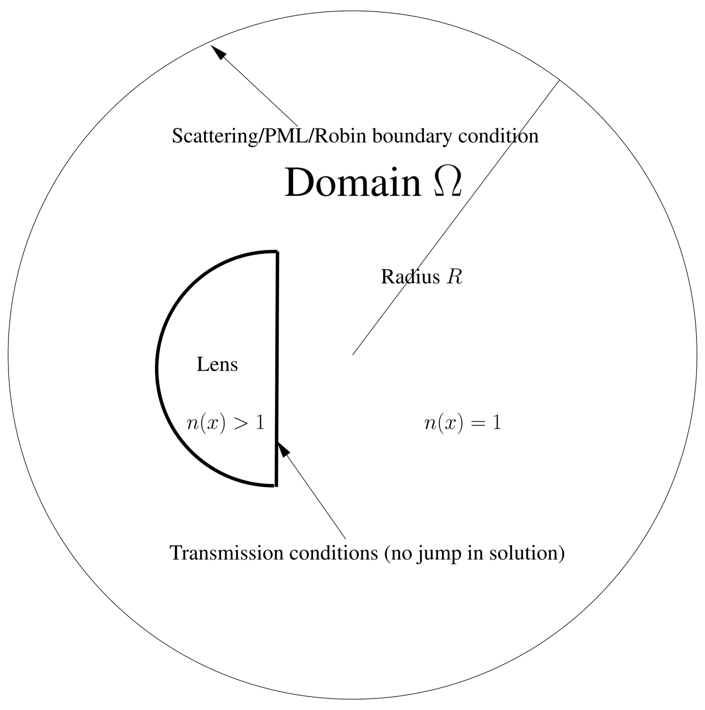
Introduction

A photonic nanojet is an optical phenomenon observed at or below micrometric scale that has not been studied extensively yet. Solving the Helmholtz equation with nonconstant coefficients is essential to the current study of photonic nanojets (PNJs), and is the first small step towards answering questions about how fine (in terms of resolution) one can generate a PNJ and how well one can control the movement of a PNJ by changing incident signal.



Spectral FEM scheme

Consider the Helmholtz equation in two dimensions with Dirichlet boundary condition:

$$\begin{aligned}\left(\Delta + k_0^2 n(\mathbf{x})^2\right) u &= 0 \text{ in } \Omega, u \in \mathbb{C} \\ u &= g \text{ on } \delta\Omega\end{aligned} \tag{1}$$

where $u = u_{incident} + u_{resulting}$ is the wave field to be solved for initially, $n(\mathbf{x})$ is a function modelling the permeability of the material in the circular domain, effectively allowing a lens to be modelled inside the domain. k_0 is the angular wave number related to the wave length λ of the signal that constitutes $u_{incident}$ by $\lambda = \frac{2\pi}{k_0}$

As for the Photonic Nanojet problem, the above problem has irrelevant boundary conditions, they should instead be minimizing reflections back into the domain to mimic an infinite domain. The simplest, and least effective, example of this would be to employ the Robin boundary condition:

$$\begin{aligned}\left(\Delta + k_0^2 n(\mathbf{x})^2\right) u &= 0 \text{ in } \Omega, u \in \mathbb{C} \\ \nabla u \cdot \mathbf{n} + i k_0 u &= g \text{ on } \delta\Omega\end{aligned} \tag{2}$$

where $\mathbf{n}$ is the outward normal vector and $i = \sqrt{-1}$ is the imaginary unit.

Domain triangulation

Since Ω is circular it is not possible to triangulate the domain perfectly using straight-edged triangles D_k which means that we are approximating Ω in a way that will never conform geometrically to a circle, but approximates it better the higher the resolution.

$$\Omega \approx \bigcup_{k=1}^{K} D_k$$

Hence the boundary $\delta\Omega$ is a piecewise linear polygon where each line segment is an outward-facing face of a triangle on the boundary of the triangulated domain. The global solution is approximated on a mesh with K elements by combining local solutions:

$$u(\mathbf{x}) \approx u_h(\mathbf{x}) = \bigoplus_{k=1}^{K} u_h^k(\mathbf{x}) \in V_h = \bigoplus_{k=1}^{K} \left\{\varphi_m(D^k)\right\}_{m=1}^{M_p}$$

where $\varphi_m(D^k)$ is a two-dimensional polynomial basis of order m defined on element D^k.

Formulation

Using Green's first identity to arrive at the weak formulation in which the boundary integral vanishes as the test functions ϕ are 0 on the boundary of the domain, and then splitting the global integrals into sums of element integrals:

$$\begin{aligned}&(\nabla^2 u_h, \phi) + (k_0^2 n^2 u_h, \phi) = 0 \Leftrightarrow \int_{\delta\Omega} \phi \tfrac{\mathrm{d}u_h}{\mathrm{d}\mathbf{n}} \,\mathrm{d}L - \iint_{\Omega} \nabla u_h \cdot \nabla\phi \,\mathrm{d}x\mathrm{d}y + \iint_{\Omega} k_0^2 n^2 u_h \phi \,\mathrm{d}x\mathrm{d}y = 0 \\ &\Leftrightarrow -\sum_{k=1}^{K} \underbrace{\left\{\iint_{D^k} \nabla u_h^k \cdot \nabla\phi \,\mathrm{d}x\mathrm{d}y\right\}}_{I_1^k} + \sum_{k=1}^{K} \underbrace{\left\{\iint_{D^k} k_0^2 n^2 u_h^k \phi \,\mathrm{d}x\mathrm{d}y\right\}}_{I_2^k} = 0\end{aligned} \tag{3}$$

Representation

The local solution can be represented as a series for $\mathbf{x} \in D^k$ with polynomial basis constituted by the two-dimensional Lagrange polynomials $\phi_{i,j}(\mathbf{x}) = h_i(x)h_j(y)$:

$$u_h^k(\mathbf{x}) = \sum_{i=1}^{M_P} \sum_{j=1}^{M_P} u_{h,ij}^k \phi_{i,j}(\mathbf{x}) \tag{4}$$

Helmholtz operator

The element contributions to the global integrals can be evaluated by mapping an equilateral triangle to a reference triangle in (r, s) coordinates with corners in $(-1,-1)$, $(-1,1)$ and $(1,-1)$, and mapping this to the elements of the mesh, further details in [1].

Inserting the representation (4) in the sums of contributions to the global integrals become the following:

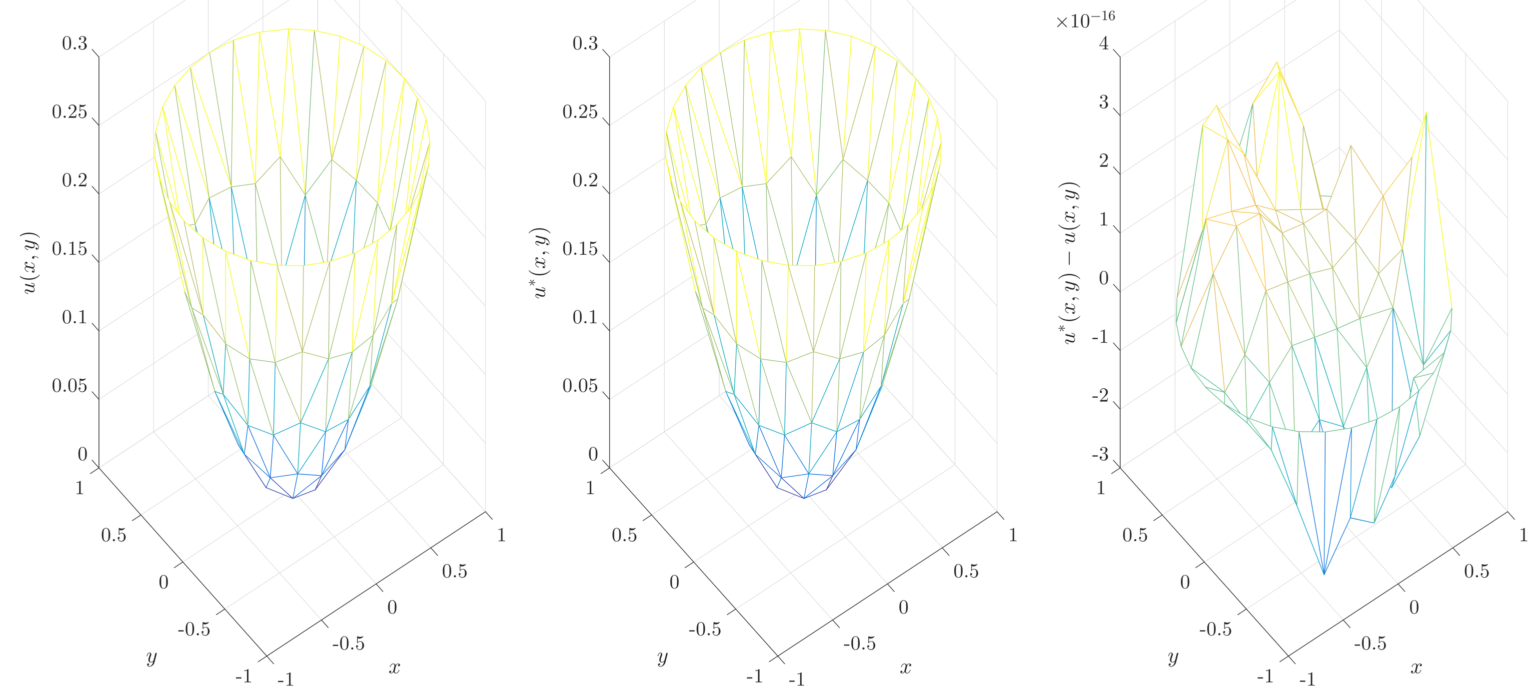
$$I_1^k = \left[-J^k D_x^T \mathcal{M} D_x - J^k D_y^T \mathcal{M} D_y\right] u_h^k \tag{5}$$

$$I_2^k = \left[k_0^2 \mathrm{diag}(n^2) J^k \mathcal{M}\right] u_h^k \tag{6}$$

which is used in the assembly[2] of the system matrix A in the system of equations $Au_h = b$. $\mathcal{M} = (\mathcal{V}\mathcal{V}^T)^{-1}$ is the mass matrix and $\mathcal{V}$ is the Vandermonde matrix for the polynomial basis and J^k is a diagonal matrix whose elements are the Jacobian of the mapping from reference element to triangle mesh element and D_x and D_y are the one-dimensional differentiation matrices.

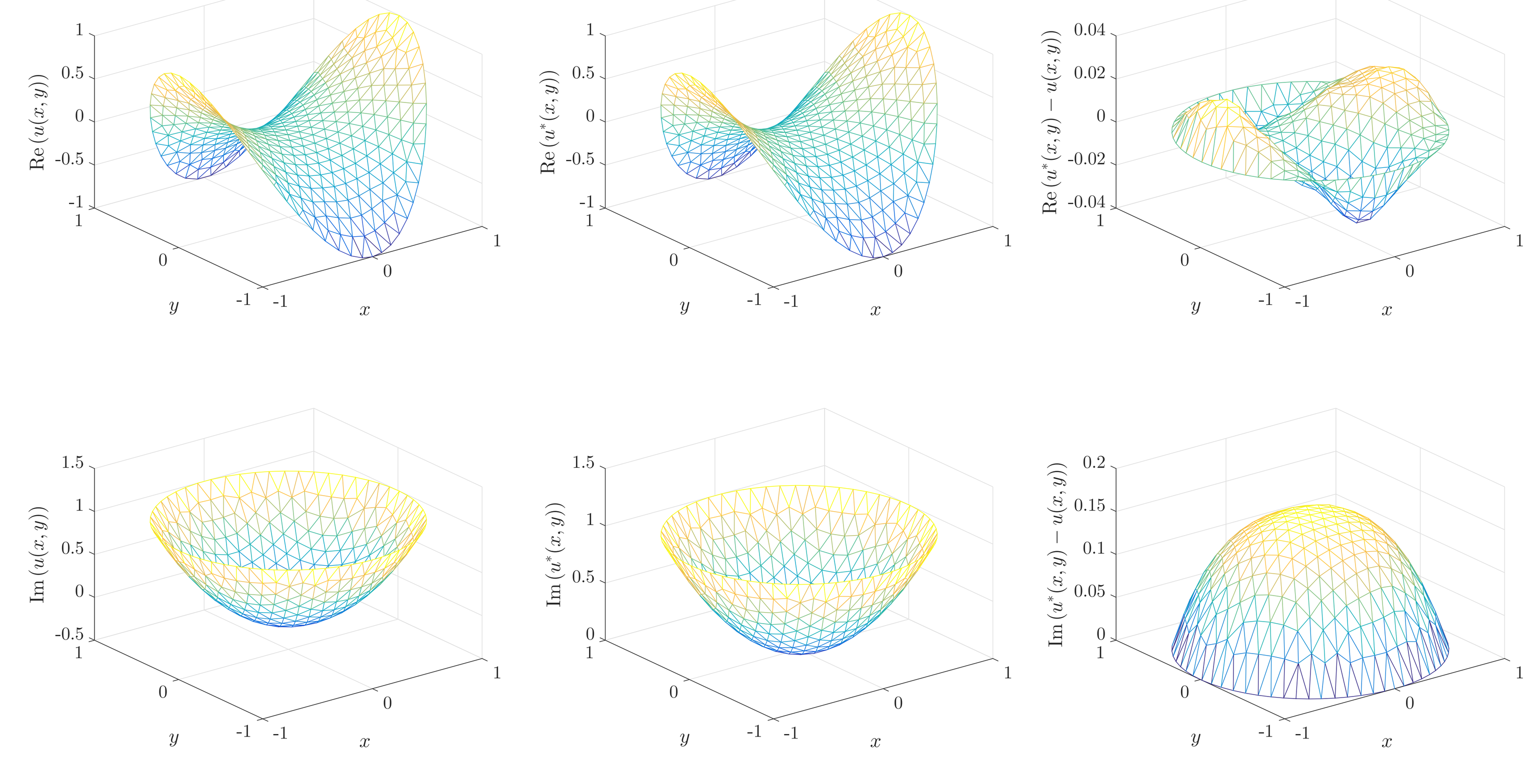
Results

The Poisson equation, whose operator consists only of I_1^k contributions is a suitable simple problem to verify parts of the solver on. The exact solution is widely known and it is, with Dirichlet boundary condition, used here to verify part of the Helmholtz operator.

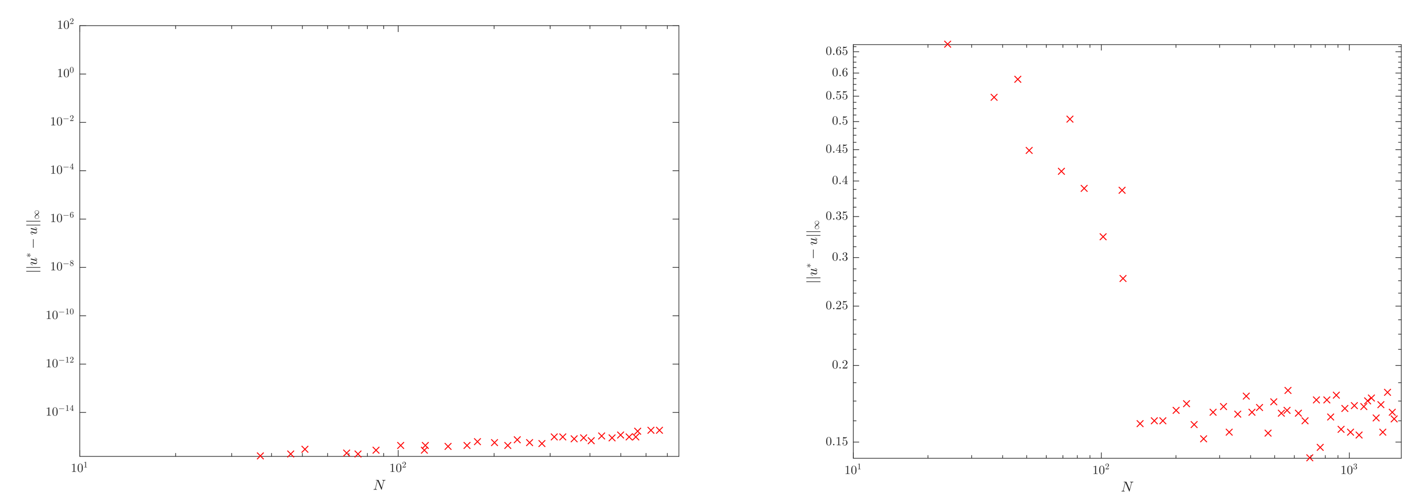


Above is seen the solution to the Poisson equation $\Delta u = a$, where $a \in \mathbb{R}$, the exact solution of which is $u^*(x,y) = \frac{a}{4}\left(x^2 + y^2\right)$, in the above $a = 1$, the mesh is a circle of radius 1 approximated by 164 triangular elements and Dirichlet boundary condition were imposed.

Using the method of manufactured solution to investigate the implementation of the Helmholtz operator the following plot emerges, confirming suspicion that there must be something wrong with the implementation. The manufactured solution is $u^*(x,y) = i(x^2 + y^2) + x^2 - y^2$, and the meshing has 661 elements in the plots below.



Convergence plots for the two problems solved show the same story. Left is the Poisson problem, right is the Helmholtz problem (1) for different resolutions yielding different amounts of global nodes N.



In the left hand plot the poisson equation is solved for different amounts of global nodes (N) yielding very small errors. In the right hand plot there is no convergence signifying a flaw in the implementation of the local integral contributions (6) for the Helmholtz operator.

References

[1] J.S. Hesthaven and T. Warburton. *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*. Texts in Applied Mathematics. Springer New York, 2007.

[2] A. P. Engsig-Karup. *The Spectral/hp-Finite Element Method for Partial Differential Equations*. 2014.

Future work

As the Helmholtz operator is not yet either derived or implemented correctly the future work should first and foremost seek to remedy this flaw. Immediately following that, the future work should seek to implement the Robin boundary condition in (2).