# Sparse Grids

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## 1 Introduction

A common obstacle when solving PDE's using ordinary full grid discretization techniques is the curse of dimensionalty. That is the number of points needed to discretize a $d$-dimensional space becomes $O(N^d)$. Because of storage requirements and computatiuonal power, it is only possible to consider PDE's in three or four dimensions using conventional methods. If we consider a piecewise linear basis we will obtain an accuracy of $O(N^{-2})$ for full grids. In addition a sparse grid with $O(N \log(N)^{d-1})$ grid points yields an accuracy of $O(N^{-2} \log(N)^{d-1})$ thus breaks the curse of dimensionalty for a cost on accuracy. Besides being used for PDE's sparse grids are also used for interpolation, numerical integration, high dimensional SDE's and data mining.

| | $d = 2$ | | $d = 3$ | | $d = 4$ | |
|---|---|---|---|---|---|---|
| | n = 10 | n = 20 | n = 10 | n = 20 | n = 10 | n = 20 |
| $V_n$ | $1.05 \cdot 10^6$ | $1.10 \cdot 10^{12}$ | $1.07 \cdot 10^9$ | $1.15 \cdot 10^{18}$ | $1.10 \cdot 10^{12}$ | $1.21 \cdot 10^{24}$ |
| $\hat{V}_n$ | 9217 | $1.99 \cdot 10^7$ | 47103 | $2 \cdot 10^8$ | $1.78 \cdot 10^5$ | $1.41 \cdot 10^9$ |

**Table 1:** Dimensions of full grid, $V_n$, and sparse grid, $\hat{V}_n$, for different values of dimension, $d$, and level, $n$.[1]

## 2 Construction of Hierarchical Bases

We will consider the construction of sparse grids in 1D using piecewise linear basis functions given by the hat function.

$$\phi(x) = \begin{cases} 1 - |x| & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

The idea is to consider the hat function on various levels. As we increase the levels the hat function will change by dilation and translation such that the support decreases. An example of this is seen in Figure 1.
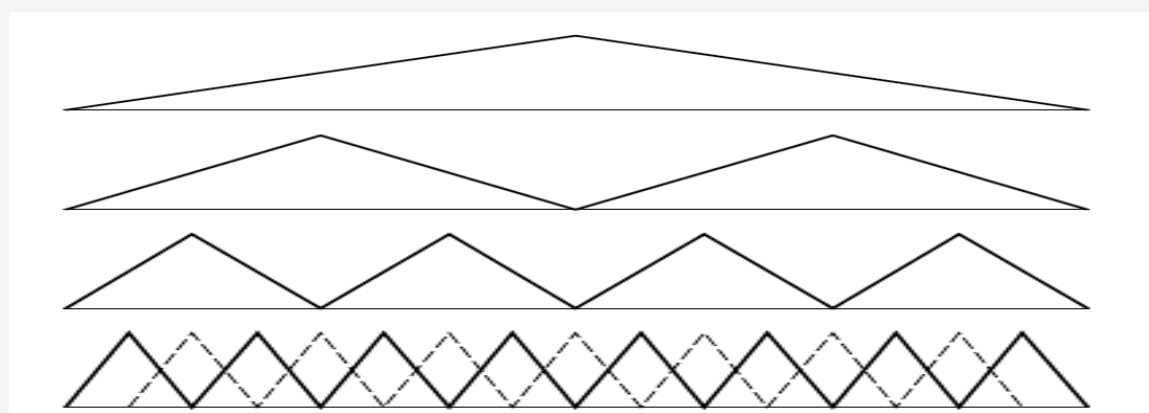


**Figure 1:** Piecewise linear hierarchical basis (solid) vs. nodal point basis (dashed).[1]

When constructing a sparse grid we consider several levels of basis function below a given tolerance, $n$. The basis functions for a given level,

$\mathbf{l} \in \mathbb{R}^d$, is given as

$$W_{\mathbf{l}} = \text{span}\{\phi_{\mathbf{l},\mathbf{i}} : 1 \leq \mathbf{i} \leq 2^{\mathbf{l}} - 1, i_j \text{ odd } \forall \ 1 \leq j \leq d\}$$

Here $\mathbf{l}$ is a multi-index with the specified level in each dimension. Now, the full grid and the sparse grid are constructed as follows

$$V_n = \bigoplus_{|\mathbf{l}|_\infty \leq n} W_{\mathbf{l}} \quad , \quad \hat{V}_n = \bigoplus_{|\mathbf{l}|_1 \leq n+d-1} W_{\mathbf{l}}$$
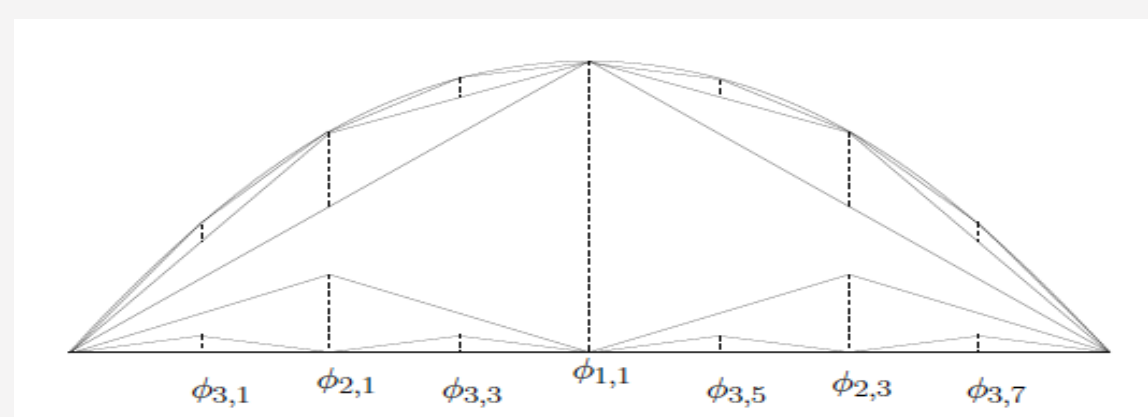


**Figure 2:** Hierarchical interpolation of a parabola with bases of level $n = 3$.[2]

The coefficients of the hierarchical bases in finer grid points can be seen to decrease very fast, and the idea behind sparse grid construction is to omit these points with small coefficients.

## 3 Sparse grid in 2D

In order to create the sparse grids we use the toolbox *Sparse grids Matlab kit*[3] which uses a linear combination of coarse full grids to create the sparse grid as illustrated in Figure 3
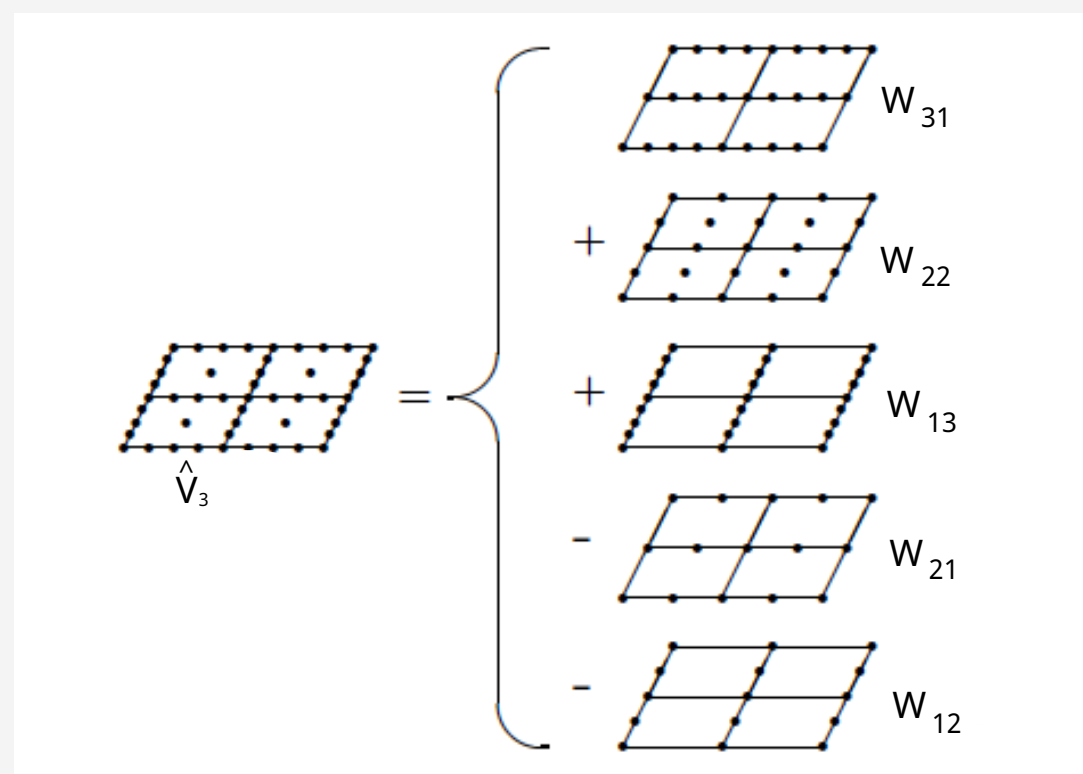


**Figure 3:** Linear combination of coarse full grids in order to create the sparse grid.[1]

Different rules exist for determining which of these subspaces to use when constructing the grid.
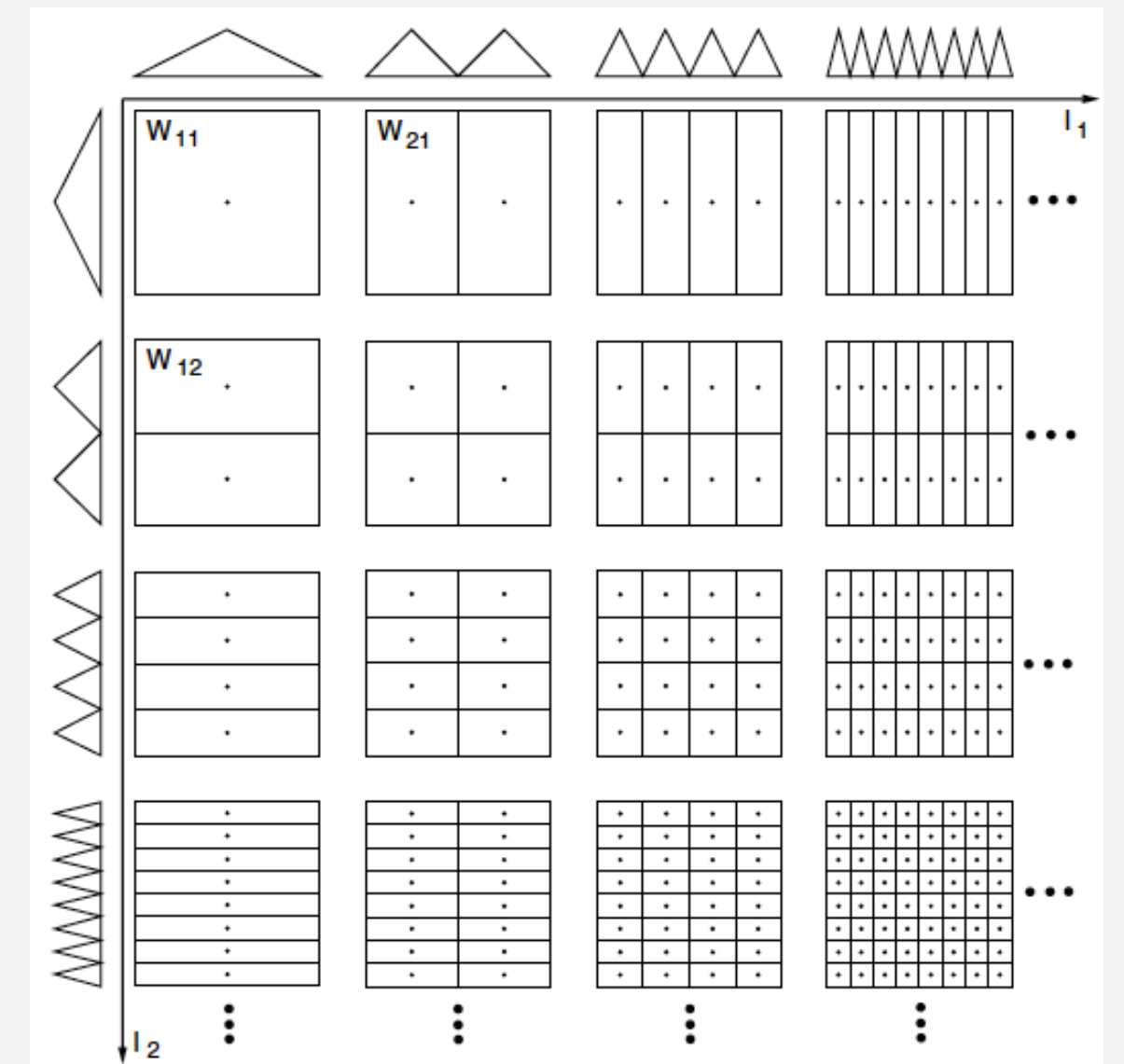


**Figure 4:** Subspaces $W_{\mathbf{l}}$ for $d = 2$.[1]

To compare a sparse grid to a full grid, we will interpolate and integrate the following functions. We have used Clenshaw-Curtis quadrature rule.

$$f_1(x, y) = \cos(2\pi x) \sin(2\pi y)$$
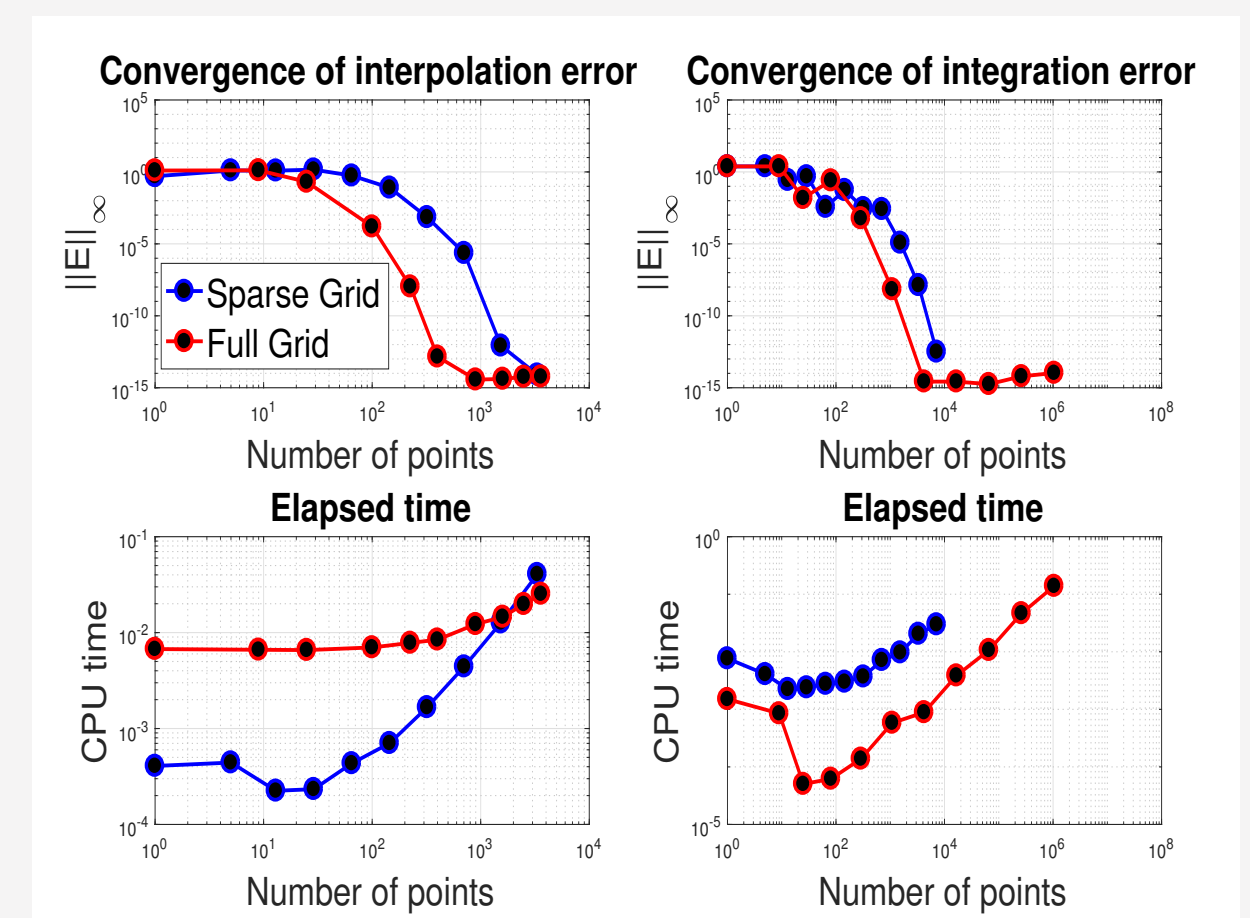$$f_2(x, y) = e^{\sin(2\pi x)} e^{\sin(2\pi y)}$$



**Figure 5:** Results for interpolating, $f_1$ (left column) and integration of $f_2$ (right column) using Sparse Grids Matlab Kit

## 4 Extensions

In order to improve the sparse grid, it is possible to use an adaptive sparse grid which concentrate points to a place of interest. The true effect of sparse grids can be a little hard to see in such low dimensions, but becomes more clear as the number of dimensions increases.

[1] Bungartz, H. & Griebel, M. (2004) Sparse grids, Acta Numerica 13, 147-269.
[2] Garcke, J. Sparse grids in a nutshell. Sparse grids and applications, volume 88 of Lecture Notes in Computational Science and Engineering, pages 57–80. Springer, 2013
[3] https://sites.google.com/view/sparse-grids-kit

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