DTU Compute
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Optimal Heating of a Metal Strip

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Introduction

The aim of this project is to apply a PDE solver to solve a PDE-constrained optimization problem. For a metal strip illustrated in figure 1 which is heated with a controlled temperature $T_c(x)$ so there is a change in temperature distribution all over the metal strip.

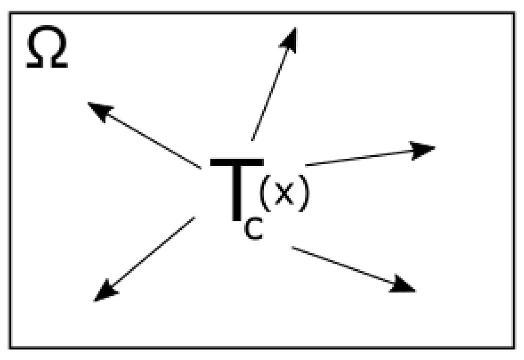

Figure 1: *The metalstrip Ω heated with temperature $T_c(x)$*

The actual temperature of the strip is given by $T(x)$. The purpose is to see how close we can get to a desired temperature $T_d(x)$ in the metal strip. The problem can be formulated as

$$\min_{T_c} J(T, T_c) = \frac{1}{2}\int_\Omega (T(x) - T_d(x))^2 dx + \frac{\alpha}{2}\int_\Omega T_c(x)^2 dx$$

subject to:

$$-\Delta T = T_c \quad \text{in } \Omega \qquad (1)$$
$$T = 0 \quad \text{on } \Gamma \qquad (2)$$
$$T_a \leq T_c \leq T_b \qquad (3)$$

This is known as the cost function which can also be written as

$$\min_{T_c} \frac{1}{2}||T(x) - T_d(x)||^2_{L^2(\Omega} + \frac{\alpha}{2}||T_c(x)||^2_{L^2(\Omega}$$

In our project we have tested two different settings:

$$T_d = \begin{cases} 1, & (x,y) \in [0, \frac{1}{2}]^2 \\ 0, & \text{elsewhere} \end{cases}, \quad T_d = \begin{cases} 1, & (x,y) \in [\frac{1}{4}, \frac{3}{4}]^2 \\ 0, & \text{elsewhere} \end{cases}$$

Deriving the weak formulation

To derive the weak formulation of the problem we define an arbitrary function v which satisfies $v = 0$ on Γ.

$$\begin{aligned} & -\Delta T - T_c = 0 \\ \Leftrightarrow & (-\Delta T - T_c)v = 0 \\ \Leftrightarrow & \int_\Omega (T_{xx} + T_{yy} - T_c)v ds = 0 \\ \Leftrightarrow & \int_\Omega -T_{xx} v ds + \int_\Omega -T_{yy} v ds + \int_\Omega -T_c v ds = 0 \end{aligned}$$

If we split this equation up and look at the first integral we can rewrite it as

$$\begin{aligned} \int_\Omega -T_{xx} v ds &= \int_\Gamma -T_x \cdot v \cdot n d\sigma - \int_\Omega T_x v_x ds \\ &= \int_\Omega T_x v_x ds \end{aligned}$$

And the same applies for $\int_\Omega -T_{yy} ds$, therefore the problem can be formulated as

$$\int_\Omega (T_x v_x + T_y v_y) ds + \int_\Omega -T_c v ds = 0 \qquad (4)$$
$$\Leftrightarrow A(T, v) + B(T_c, v) = F(v) \qquad (5)$$

Discretization

To discretize the problem we assume that T_c is a constant on each element Ω_e where we define e_i, $i = 1, .., m$ with the property

$$e_i = \begin{cases} 1, & i \in \Omega_e \\ 0, & i \notin \Omega_e \end{cases}$$

We now define the following anzats functions, where φ_i are piecewise linear basis functions

$$T(x) = \sum_{i=1}^{n} T_i \varphi_i(x)$$
$$T_c(x) = \sum_{i=1}^{m} T_{c,i} e_i(x)$$

Where T_i and $T_{c,i}$ are the unknown. We can now discretize equation 4 by substituting the arbitrary function v for φ and get the following

$$\int_\Omega T_x \varphi_{x,i} + T_y \varphi_{y,i} ds + \int_\Omega -T_c \varphi_i ds = 0$$
$$\int_\Omega T_x \varphi_{x,i} + T_y \varphi_{y,i} ds = \int_\Omega T_c \varphi_i ds$$

By using the discretized definition of T and T_c we get

$$\begin{aligned} & \int_\Omega \sum_{j=1}^{n} T \varphi_{j,x} \varphi_{i,x} ds + \int_\Omega \sum_{j=1}^{n} T \varphi_{j,y} \varphi_{i,y} ds \\ = & \int_\Omega \sum_{j=1}^{n} T_c e_j \varphi_i ds \\ \Leftrightarrow & \sum_{j=1}^{n} T \int_\Omega (\varphi_{x,j}\varphi_{x,i} + \varphi_{y,j}\varphi_{y,i}) ds \\ = & \sum_{j=1}^{m} T_{c,j} \int_\Omega e_j \varphi_i ds \end{aligned}$$

By using the notation from equation 4 it can be seen that this is equal to the notation

$$\mathbb{A}T = \mathbb{B}T_c$$

Where the elements of the matrices $\mathbb{A}$ and $\mathbb{B}$ are

$$\mathbb{A}_{i,j} = A_e(\varphi_j, \varphi_i), \qquad \mathbb{B} = B_e(e_j, \varphi_i)$$

The equivalent discretized optimization problem for the temperature problem is

$$\begin{aligned} & \min \left\{ \frac{1}{2} T^T \mathbb{M} T - c^T T + \frac{\alpha}{2} T_c \mathbb{D} T_c \right\} \\ & s.t. \\ & \mathbb{A}T = \mathbb{B}T_c \\ & T_a \leq T_d \leq T_b \end{aligned}$$

Where

$$\mathbb{M} := B_e(\varphi_j, \varphi_i), \qquad \mathbb{D} := B_e(e_j, e_i),$$
$$c := B_e(T_d, \varphi_i)$$

To solve the optimization problem we state the optimality conditions

$$\begin{aligned} \mathbb{A}T &= \mathbb{B}T_c, \quad T_a \leq T_c \leq T_b \\ \mathbb{A}\lambda &= \mathbb{M}T - c \\ \mu &= -((\alpha\mathbb{D})^{-1}\mathbb{B}^T\lambda + T_c) \end{aligned}$$

Here λ and μ are Lagrange multipliers.

The algorithm

In the algortihm we calculate $\mathbb{A}$, $\mathbb{B}$, **c**, $\mathbb{D}$, $\mathbb{E}$, and $\mathbb{M}$. Then, using the convergence criteria

$$||T_c^{k+1} - T_c^k||_2 > \text{Tol}$$

for some chosen tolerance, we solve the following system and update values for λ, **T**, and T_c until convergence

$$\begin{pmatrix} 0 & \mathbb{A} & -\mathbb{B} \\ \mathbb{A} & -\mathbb{M} & 0 \\ \mathbb{E}\mathbb{B}^T & 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} \lambda \\ \mathrm{T} \\ \mathrm{T_c} \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathrm{c} \\ \chi_a^k + \chi_b^k \mathrm{T_b} \end{pmatrix}$$

Where the matrices χ_a^k and χ_b^k are calculated by finding the sets

$$\Pi_a^k = \left\{ i \in \{1, \cdots, m\} : T_{c,i}^{k-1} + \mu_i^k < T_a \right\}$$
$$\Pi_b^k = \left\{ i \in \{1, \cdots, m\} : T_{c,i}^{k-1} + \mu_i^k > T_b \right\}$$

Now we create the matrices

$$\chi_{a,ii}^k = \begin{cases} 1, & i \in \Pi_a^k \\ 0, & i \notin \Pi_a^k \end{cases}, \qquad \chi_{b,ii}^k = \begin{cases} 1, & i \in \Pi_b^k \\ 0, & i \notin \Pi_b^k \end{cases},$$

After the system is solved the results can be analysed.

Numerical results

The results computed with the algorithm can be seen on figures 2 - 4.

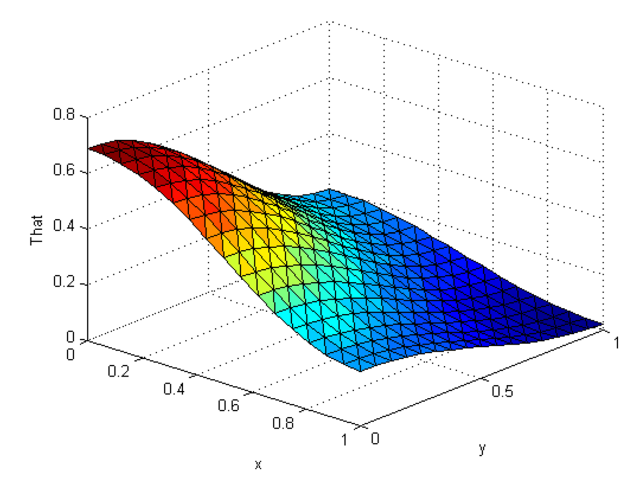

Figure 2: The results of T after 1 iteration.

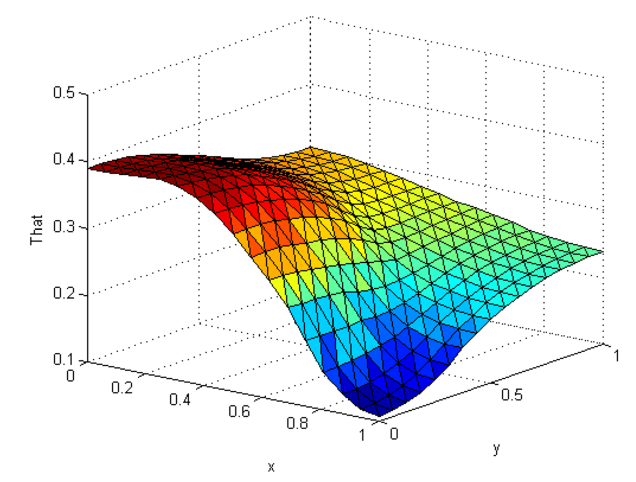

Figure 3: The results of T after 7 iterations.

In figure 2 and 3 we see that the temperature in the square $[0, 0.5]^2$ is more defined after 7 iterations, than after only 1.

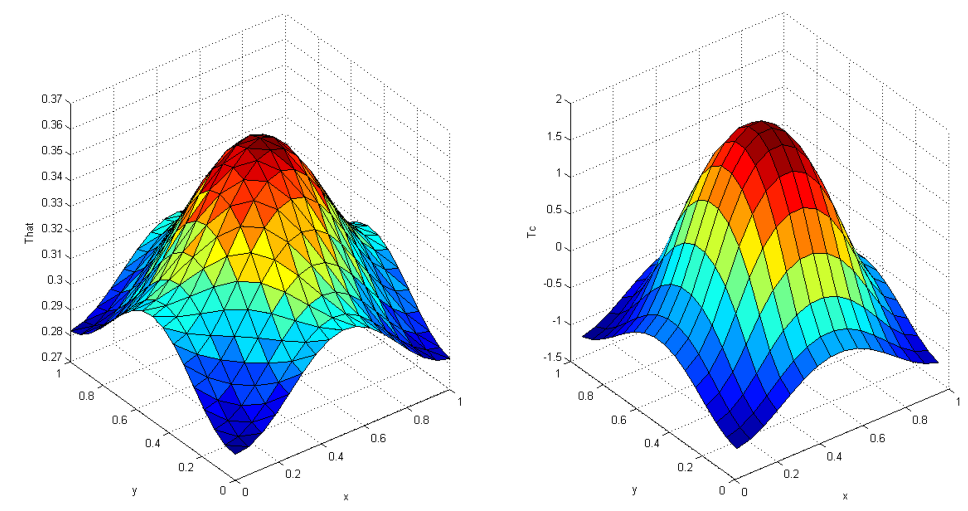

Figure 4: The results of T (left) and T_c (right), when heating at the center of the strip.

In figure 4 the behaviour of the temperature distribution is as expected, since we are looking to heat up only the center of the metal strip, though the plot does not show a distinct square shape in the center.

Conclusion and perspectives

The found solutions in figures 2 - 4 do not satisfy the boundary conditions

$$T = 0 \qquad \text{on } \Gamma$$

So the algorithm does not take them into account when finding the optimal solution.

It seems that our algorithm solves the problem. Looking at figure 4 we see that we get a metal strip with high temperatures in the center and low elsewhere. Also the plot of T_c shows that we should be applying heat to the center of the strip, while cooling around the edges. The code also does not use the convergence criteria to stop. It only stops it reaches the maximum amount of iterations. The reason for the code not working perfectly is probably that Matlab has difficulty solving the linear system of equations used to solve the problem.