Eigenmode Determination of Arbitrary Electromagnetic Waveguide

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Motivation

A waveguide guides waves. A common type for microwave communication is a metal *pipe* of constant cross section. The electromagnetic field propagation can only be solved analytically for certain canonical cross sections: Rectangular, circular, coaxial, elliptical. For this reason, these waveguides are almost exclusively used in practice. It is therefore desirable to be able to model arbitrary cross section waveguides accurately and efficiently. Additionally many electromagnetic problems can be modeled with waveguides; e.g. a grid of shaped holes in a metal plate can be modeled as small waveguides.

Model

The waveguide cross section is assumed to lie in the xy-plane and be infinite along the z-direction. This means that the z-dependence on the field will be simple:

$$\mathbf{E}(x,y,z) = \mathbf{E}(x,y)e^{-j\beta z}$$
$$\mathbf{H}(x,y,z) = \mathbf{H}(x,y)e^{-j\beta z}.$$

Time harmonic fields with a suppressed time factor of $e^{j\omega t}$ are assumed. Under this condition one can show that Maxwells equations are satisfied for field configurations with $H_z = 0$ (TM modes), if the following scalar Helmholtz equation is satisfied

$$\nabla_t^2 E_z + k_c^2 E_z = 0 \qquad (1)$$

where

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the transverse gradient operator. Since E_z is tangential to the perfectly conducting walls of the waveguide, it must be zero at this interface. Thus, we have a zero Dirichlet boundary condition.
The physical interpretation of the eigenvalue, k_c^2, is

$$k_c^2 = k^2 - \beta^2$$
$$\Rightarrow \beta = \sqrt{k^2 - k_c^2}.$$

If k_c is larger than the wavenumber, k, the solution will decay exponentially along z. Thus, k_c is called the cutoff wavenumber and the associated cutoff frequency is

$$f_c = \frac{k_c c}{2\pi}$$

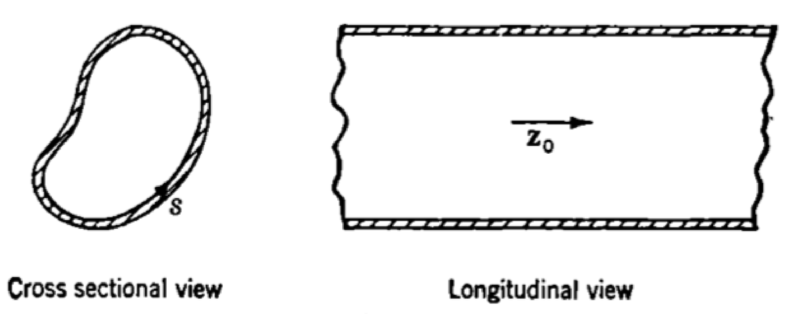

Fig. 1·1.—Uniform waveguide of arbitrary cross section.

Figure 1: Arbitrary waveguide. Source: Marcuvitz 1986.

Week Formulation

We multiply (1) by a test function v which is zero on the boundary and integrate over the domain

$$\iint_\Omega \left((E_z)_{xx} + (E_z)_{yy} + k_c^2 E_z \right) v dxdy = 0.$$

Integrating by parts

$$\int_\Gamma (E_z)_x v d\Gamma + \int_\Gamma (E_z)_y v d\Gamma - \iint_\Omega \left[(E_z)_x v_x + (E_z)_y v_y \right] d\Omega + k_c^2 \iint_\Omega E_z v d\Omega = 0,$$

and using the property of the test functions yields

$$\iint_\Omega \left[(E_z)_x v_x + (E_z)_y v_y \right] d\Omega - k_c^2 \iint_\Omega E_z v d\Omega = 0.$$

Choosing linear basis functions and the same test functions (Galerkin) on discrete grid points on the interior of the waveguide, we can define the elements of two matrices

$$\mathbf{A}_{i,j} = \iint_\Omega \left[(N_i)_x (N_j)_x + (N_i)_y (N_j)_y \right] d\Omega$$
$$\mathbf{B}_{i,j} = \iint_\Omega N_i N_j d\Omega$$

and solve the generalized eigenvalue problem

$$\mathbf{A}E_z = k_c^2 \mathbf{B} E_z,$$

thus obtaining cutoff wave numbers and associated field distributions of the waveguide modes.

Rectangular Waveguide

Analytical solution for TM modes in a rectangular waveguide:

$$k_c^2 = \pi^2 \frac{m^2 \frac{b}{a} + n^2 \frac{a}{b}}{ab}$$
$$E_z \propto \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)$$

The FEM solution for the four modes with the lowest eigenvalues are shown in Figure 2. The plotted fields and cutoff frequencies correspond well with the analytical TM_{11}, TM_{21}, TM_{31}, and TM_{12} modes.

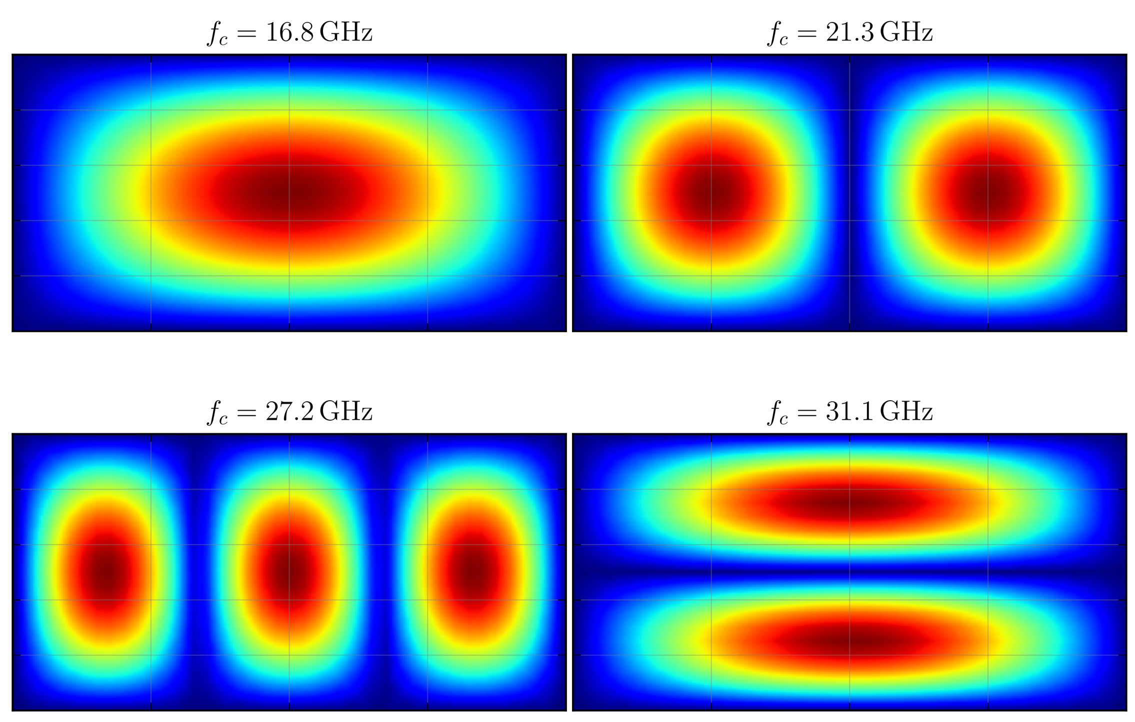

Figure 2: FEM solution for the four first modes in rectangular waveguide of dimensions $a = 20$ mm and $b = 10$ mm (see Figure 5).

Non-canonical Waveguide

Figure 3 is a picture of a non-canonical waveguide drawn and meshed with Gmsh. The geometry has been solved with the same code as the rectangular waveguide. Zero Dirichlet boundary conditions have been applied to the horseshoe-like boundary of the waveguide. The resultant cutoff frequencies and field distributions for the first four modes are shown in Figure 4. The field distributions seem realistic compared with the rectangular waveguide.

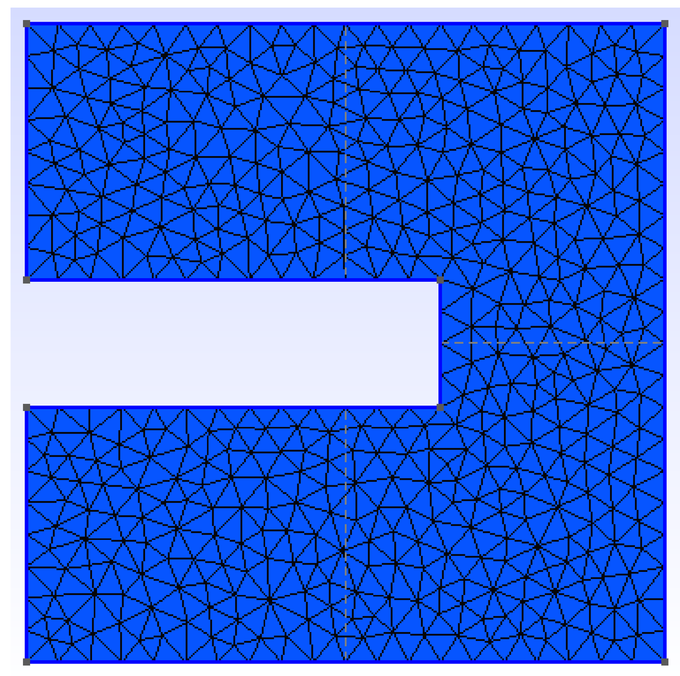

Figure 3: Waveguide cross section defined and meshed in Gmsh. The dimensions from the bottom left to top right corner are 20 by 20 mm.

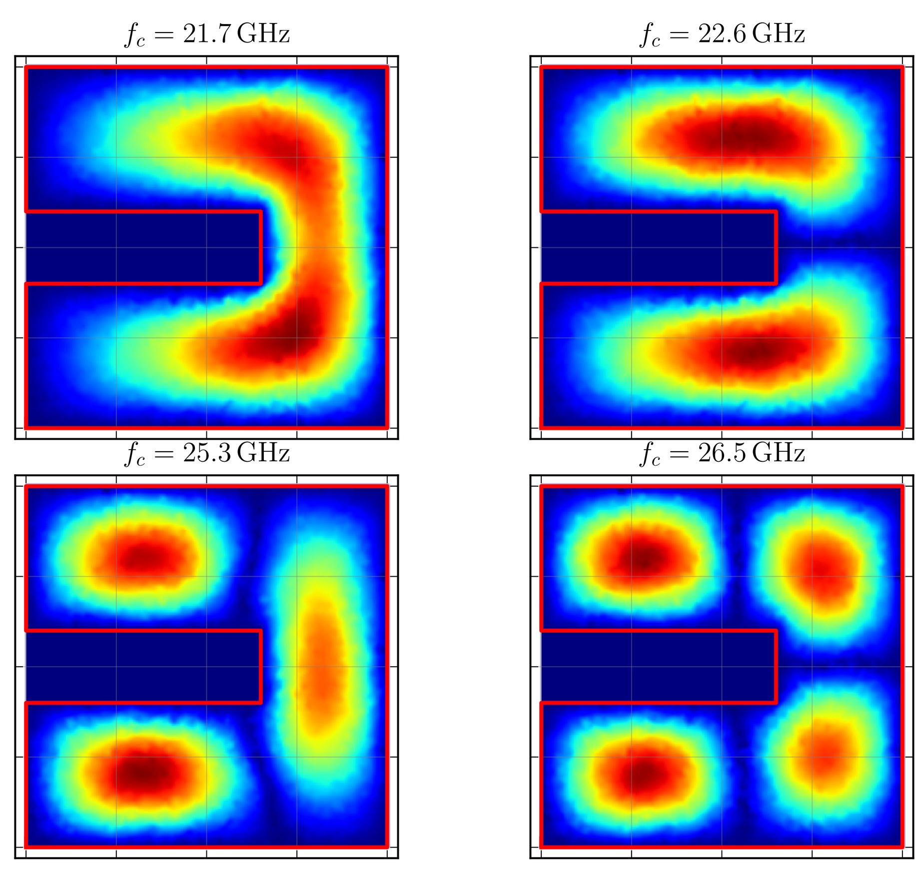

Figure 4: First four eigenmodes for the geometry in Figure 3 (Note that the used mesh was finer).

Conclusion

The finite element method has proven effective for solving the eigenvalue problem of a electromagnetic waveguide.

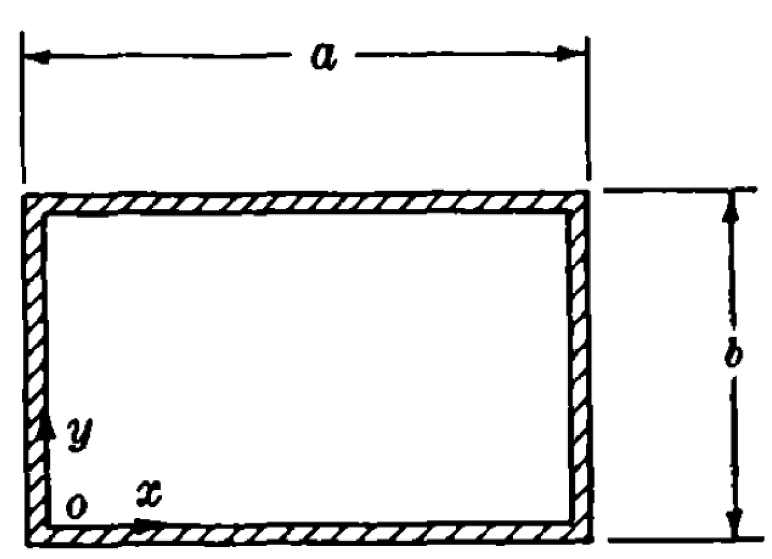

Fig. 2·1.—Rectangular waveguide cross section.

Figure 5: Geometry of a rectangular waveguide. Source: Marcuvitz 1986.

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