Spectral/hp-FEM for the Helmholtz, Poisson and Laplace Equations in 1D/2D

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Spectral/hp-FEM

Linear interpolants are easy to use and implement, but offer only algebraic h-refinement convergence. One option is to use piecewise P-order polynomials as basis functions for the approximating solution. This approach opens a new avenue of convergence, namely the p-type, where the order P is increased.

There are two interchangeable ways for representing an approximation locally on any triangular element: Modal representation by Proriol's orthonormal basis for the P-order polynomial space

$$\phi_m(r,s) = \sqrt{2}(1-s)^j P_i^{(0,0)}\Big(2\frac{1+r}{1-s}-1\Big)P_j^{(2i+1,0)}(s)$$

on the triangle T given by $-1 \le r,s$ and $r+s \le 0$ where the index is $m = j+(P+1)i+1-\frac{1}{2}i(i-1)$ with $0 \le i,j$ and $i+j \le P$ being valid subindices. Nodal representation by Lagrange polynomials

$$\hat{u}(r,s) = \sum_{k=1}^{M_P} \hat{u}_k N_k^{(n)}(r,s), \quad (r,s) \in T$$

defined by $M_P = \frac{1}{2}(P+1)(P+2)$ interpolation nodes $\{(r_n,s_n)\}_{n=1}^{M_P}$ adequately placed on the triangle T.

The cardinal property of the interpolating functions establishes a linear relationship between these forms. It is given by the generalized Vandermonde matrix

$$(\mathcal{V})_{n,m} = \phi_m(r_n,s_n)$$

with derivatives defined by

$$(\mathcal{V}_r)_{n,m} = \phi_{m,r}(r_n,s_n)$$
$$(\mathcal{V}_s)_{n,m} = \phi_{m,s}(r_n,s_n)$$

To take this from T to any triangular element E_n, one applies the transfinite interpolation formula

$$\begin{bmatrix}x\\y\end{bmatrix} = -\frac{r+s}{2}\begin{bmatrix}x_1\\y_1\end{bmatrix} + \frac{1+r}{2}\begin{bmatrix}x_2\\y_2\end{bmatrix} + \frac{1+s}{2}\begin{bmatrix}x_3\\y_3\end{bmatrix}$$

Spectral elements

Elemental contributions can be computed exactly by exploiting orthonormality

$$\int_{E_n} N_i^{(n)} N_j^{(n)}\, d\mu = J_n((\mathcal{V}\mathcal{V}^T)^{-1})_{i,j}$$
$$\int_{E_n} (N_i^{(n)})_x (N_j^{(n)})_x\, d\mu = J_n(\mathcal{D}_x^T(\mathcal{V}\mathcal{V}^T)^{-1}\mathcal{D}_x)_{i,j}$$
$$\int_{E_n} (N_i^{(n)})_y (N_j^{(n)})_y\, d\mu = J_n(\mathcal{D}_y^T(\mathcal{V}\mathcal{V}^T)^{-1}\mathcal{D}_y)_{i,j}$$

where the matrices $\mathcal{D}_x$ and $\mathcal{D}_y$ are

$$\mathcal{D}_x = r_x\mathcal{V}_r\mathcal{V}^{-1} + s_x\mathcal{V}_s\mathcal{V}^{-1}$$
$$\mathcal{D}_y = r_y\mathcal{V}_r\mathcal{V}^{-1} + s_y\mathcal{V}_s\mathcal{V}^{-1}$$

and J_n is the Jacobian of the transform $T \to E_n$.

Strong and weak formulations

The Helmholtz, Poisson and Laplace equations can be approached via the general strong form

$$(\lambda_1 u_x)_x + (\lambda_2 u_y)_y - ku = -q \quad \text{in} \quad \Omega$$
$$u = f \quad \text{on} \quad \partial\Omega$$

The weak form of this is to find u such that

$$\int_\Omega (\lambda_1 u_x v_x + \lambda_2 u_y v_y) + kuv\, d\mu = \int_\Omega qv\, d\mu$$

is satisfied for any test function $v \in H_0^1(\Omega)$.

Convergence rates

We test h and p-type rates on two trial problems: In 1D on $u''(x) = e^{4x}$ in $(-1,1)$ with $u(\pm 1) = 0$, and in 2D on a Poisson equation BVP given by

$$u_{xx} + u_{yy} = -q \quad \text{in} \quad (0, \tfrac{2}{3}\pi)^2$$
$$u(x,y) = \cos(x^2+y^2) \quad \text{on} \quad \partial(0, \tfrac{2}{3}\pi)^2$$

with $q(x,y) = 4\sin(x^2+y^2) + 4(x^2+y^2)\cos(x^2+y^2)$. Successive h-refinement is by mesh equipartitioning.

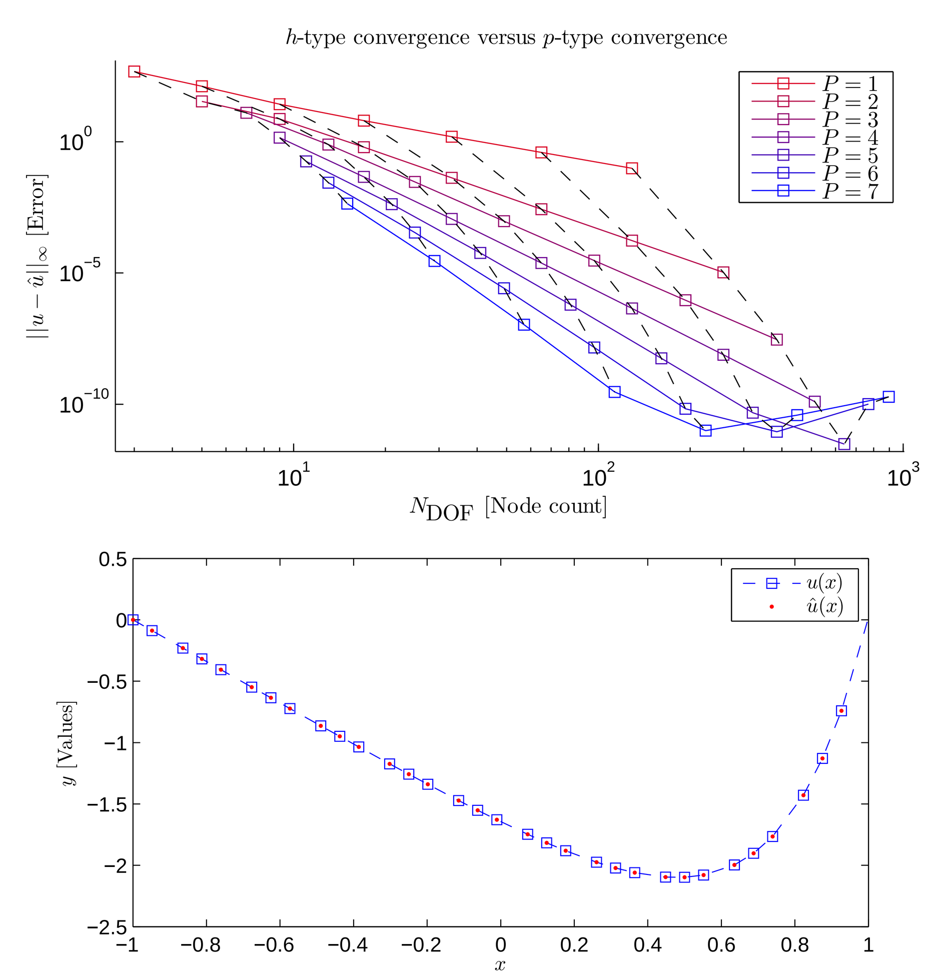

Figure 1: Convergence by equipartitioning the interval $[-1,1]$. Sample plot: Order $P=3$ with 2^4 elements.

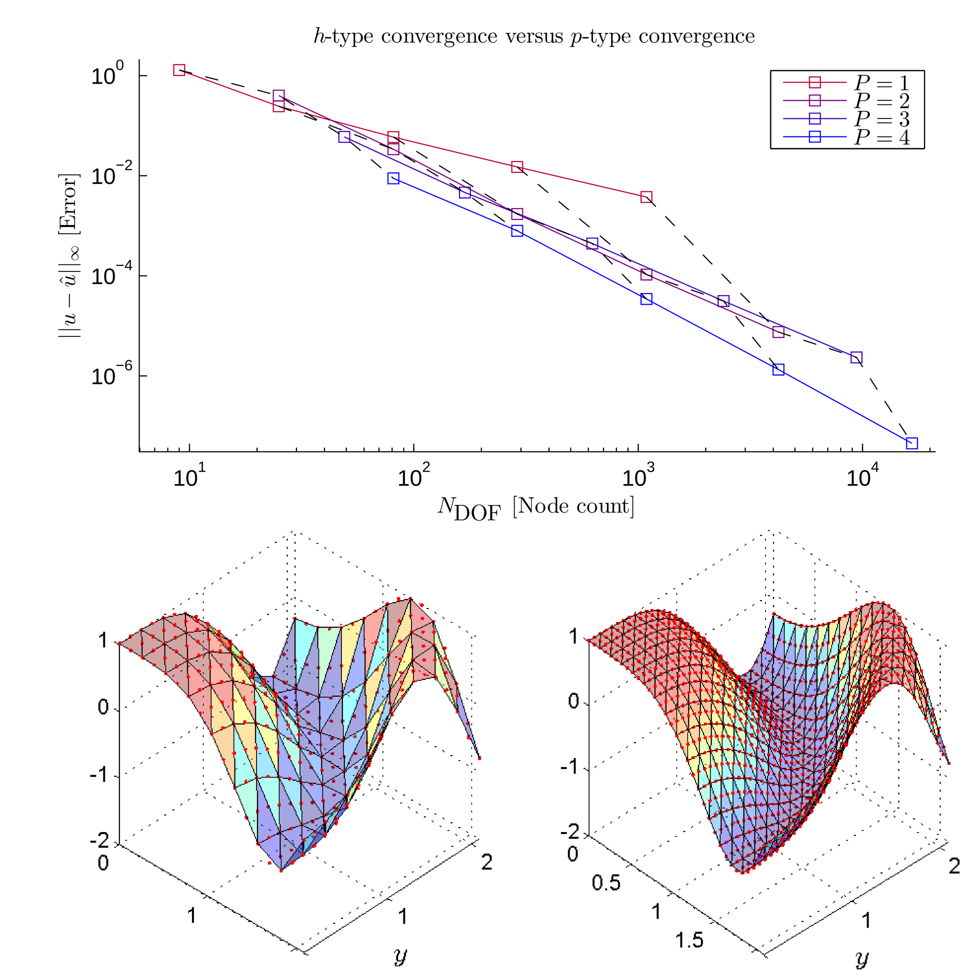

Figure 2: Convergence by equipartitioning the square $[0, \frac{2}{3}\pi]^2$. Sample plot: Order $P=2$ with 2^7 (left) and 2^9 (right) elements.

Convergence by h-refinement is algebraic with the order depending on P, whereas p-type is exponential with rate depending on h.

Flow surrounding a cylinder

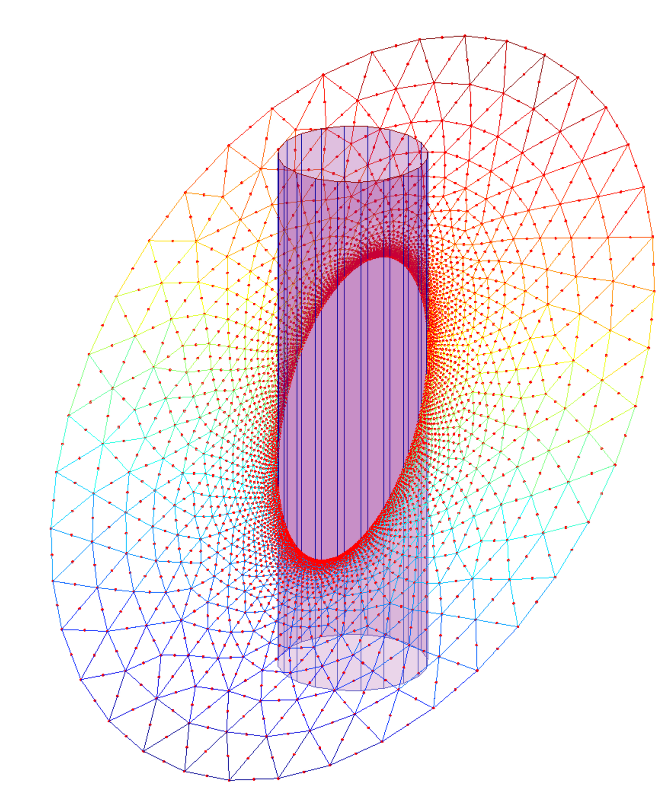

Figure 3: Weights (red) of an order $P=2$ approximation to ϕ with the vertex nodes connected. Error is shown in Figure 4.

As an illustration of both strengths and weaknesses of the hp-FEM, we consider a Laplace equation BVP corresponding to a fluid flow velocity potential ϕ surrounding an infinite cylinder

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in} \quad \Omega$$
$$\phi(x,y) = (1 + (R^2/r^2))x \quad \text{on} \quad \Gamma$$

where Ω is an annulus with inner radius $R = 0.25$ and outer boundary Γ of radius $r = 1$.

Geometry and p-convergence

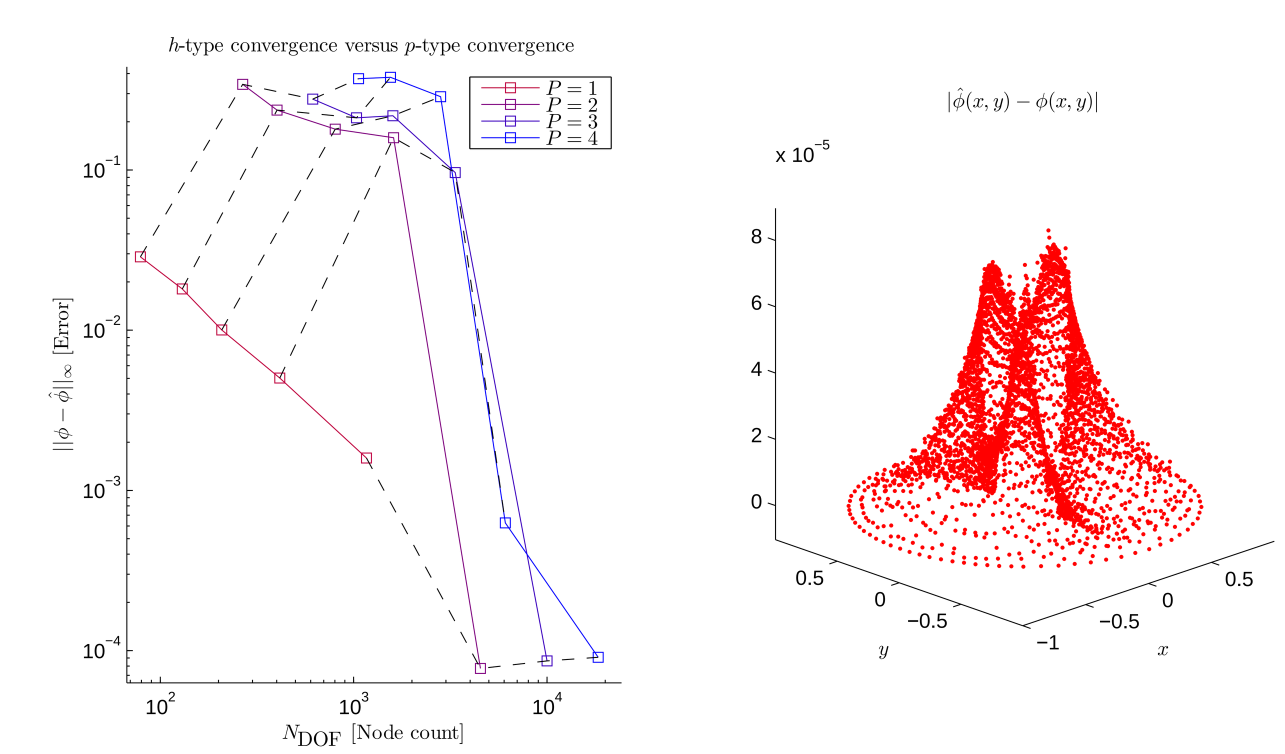

Figure 4: Convergence by mesh refinement near the cylinder. Refinements are the same for each order P.

Failure to accurately represent the BVP geometry adds error that can only be lessened by h-refinement. Increasing P adds flexibility in existing elements, but does not lessen error due to a curved geometry. Nodes are added, but the mesh remains unchanged. Thus p-convergence can not replace h-refinement. This is illustrated by the flow surrounding a cylinder, the behaviour of total error is shown in Figure 4. Increasing P without refinement causes higher error.

In general, h and p-types must be combined to have full effect.