Finite Element Method applied to the Bending Beam

Project in course 02623

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Introduction

A vital factor in building acoustics due to the issue of structure-borne sound, structural vibrations constitute an important research area in acoustics.

Sound is mainly transmitted as bending waves in the structural elements and as such, the basic understanding of the topic lays on bending beam and plate theory.

A simple 1D problem of transversal vibration of a beam is presented here. The problem is formulated and modelled using the Finite Element Method to determine the deflections.

Figure 1: Distributed load on beam (from [3], [4])

Problem

The governing PDE balances the internal and external forces:

$$EI\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}\right)+m'\frac{\partial^2 w}{\partial t^2}=q$$

q is the distributed load on the beam, *m'* the mass per unit length, and *EI* the beinding stiffness. *w* is the deflection to be determined.

FEM

The weak form of the PDE requires the FEM basis functions to have smooth 2nd order derivatives. Cubic Hermite splines are suggested in litterature [1][3].

They approximate the desired function on an element as a linear combination of Hermite polynomials (shown left) and the corresponding displacement *u* and slope *β* at the nodes, which are the 2 degrees of freedom.

$$N_1(\xi)=\frac{1}{4}(1-\xi)^2(2+\xi)$$

$$M_1(\xi)=\frac{l_e}{8}(1-\xi)^2(1+\xi)$$

$$N_2(\xi)=\frac{1}{4}(1+\xi)^2(2-\xi)$$

$$M_2(\xi)=\frac{l_e}{8}(1+\xi)^2(\xi-1)$$

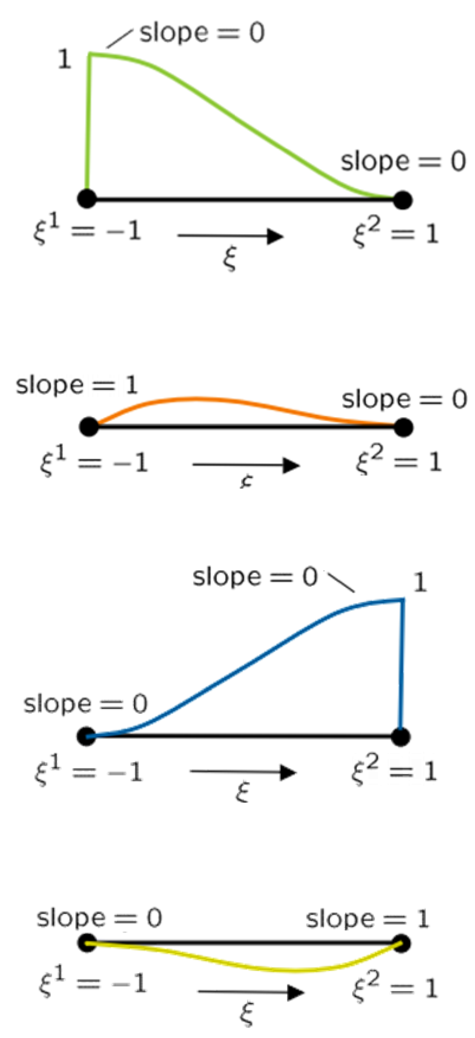

Figure 2: Hermite polynomials [3]

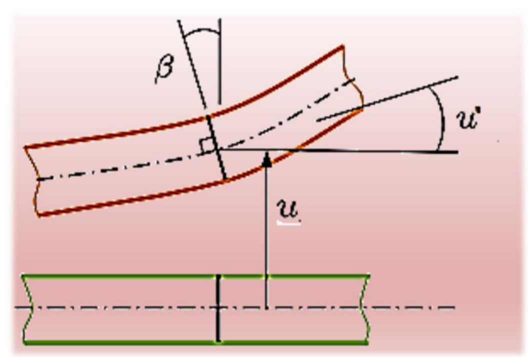

Figure 3: Bending – displacement u and slope $\beta = u'$ [3]

Static case

Ar first, the static case is approached:

$$EI\frac{\partial^4 w}{\partial x^4}=q$$

The weak formulation of this PDE is arrived at by multiplying by a function v and subsequently integrating over the domain:

$$EI\int_0^L w''v''\mathrm{d}x=\int_0^L qv\mathrm{d}x$$

Dynamic case

The weak formulation for the dynamic case is obtained by applying the same procedure to the full problem PDE:

$$EI\int_0^L w''v''dx+m'\int_0^L \ddot{w}vdx=\int_0^L qvdx$$

Discretisation

Each node is defined by 2 DOF. This is reflected by the use of 2 kinds of global basis functions in each node, therefore, the discretization of *w* is defined as a linear combination of those:

$$w_i=u_iN_i(x)+\beta_iM_i(x)$$

And *v* is substituted by the vector of basis functions:

$$v=\begin{bmatrix}N_j(x)\\M_j(x)\end{bmatrix}$$

Then the equation splits in 2 lines:

$$EI\left(u_i\int_0^L N_i''N_j''dx+\beta_i\int_0^L M_i''N_j''dx\right)=q_i\int_0^L N_jdx$$

$$EI\left(u_i\int_0^L N_i''M_j''dx+\beta_i\int_0^L M_i''M_j''dx\right)=q_i\int_0^L M_jdx$$

Each integral leads to the coefficients that form the matrix A and the right-hand side vector b. A stiffness matrix $K^{(i)}$ and a vector f^{i} can be defined per element from the local basis functions

$$k_{r,s}^{(i)}=\int_{x_i}^{x_{i+1}}(\mathbf{N}_r^{(i)})''(\mathbf{N}_s^{(i)})''dx, r,s=1..4$$

$$f_r^{(i)}=\int_{x_i}^{x_{i+1}}\mathbf{N}_r^{(i)}dx, r=1..4$$

with

$$\mathbf{N}=[N_1M_1N_2M_2]$$

Which can be used to obtain the total system matrix A and total right-hand side vector b by overlapping.

$$K^{(i)}=\frac{1}{h^3}\begin{bmatrix}12&6h&-12&6h\\6h&4h^2&-6h&2h^2\\-12&-6h&12&-6h\\6h&2h^2&-6h&4h^2\end{bmatrix}\quad f^{(i)}=\frac{h}{2}\begin{bmatrix}1\\\frac{h}{6}\\1\\-\frac{h}{6}\end{bmatrix}$$

Results

The deflection determined by the FEM analysis is plotted for the static and time-dependent cases and two types of boundary conditions.

Clamped | Simply supported

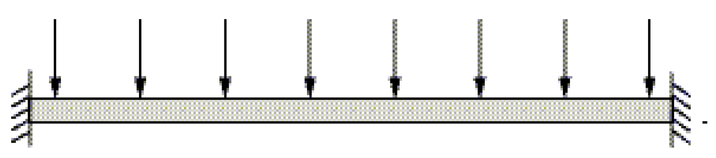

Figure 4: Clamped [4]

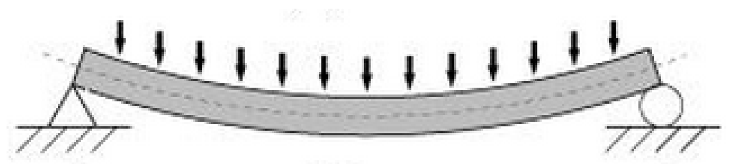

Figure 5: Simple support [5]

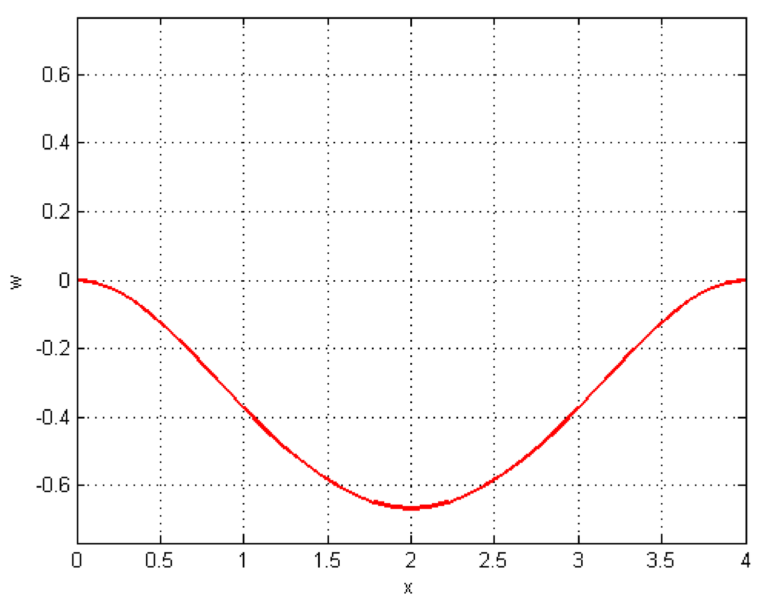

Figure 6: Deflection for clamped support – static case

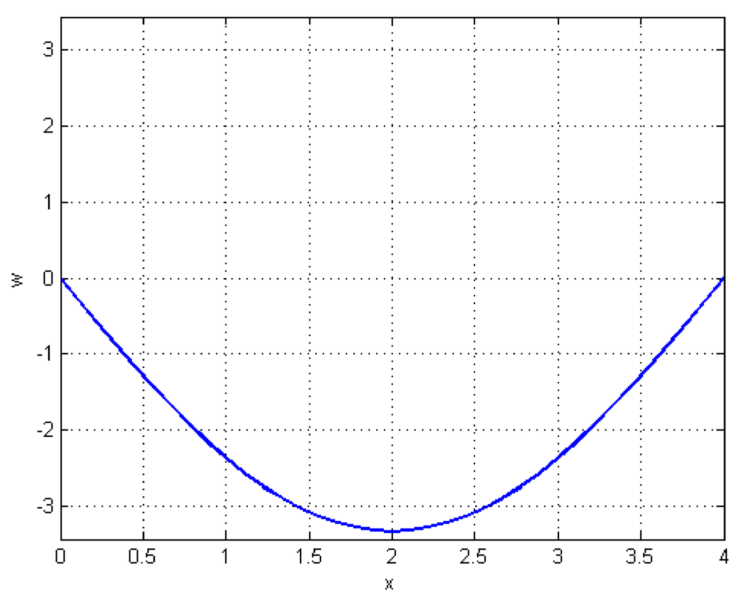

Figure 7: Deflection for simple support – static case.

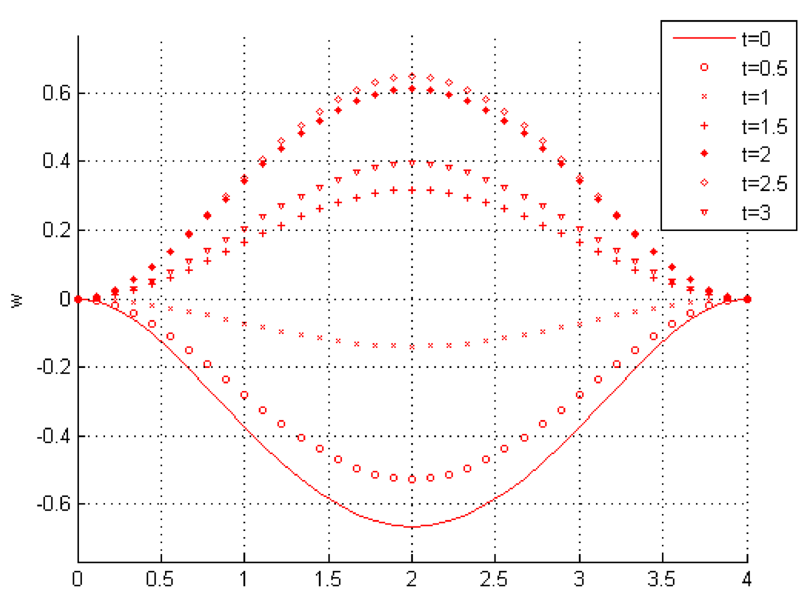

Figure 8: Deflection for clamped support – time-dependent case

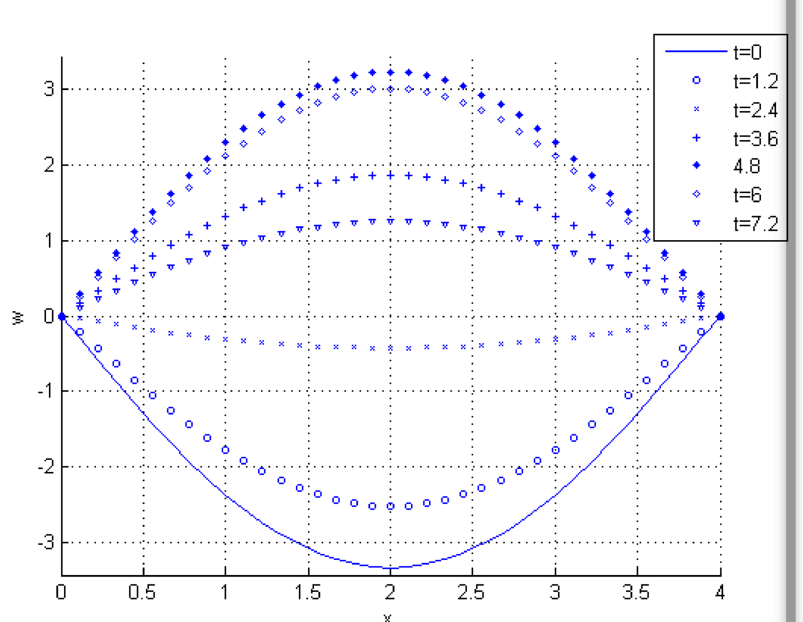

Figure 9: Deflection for simple support – time-dependent case.

Time-dependency discretisation

The seocond integral contains the time derivative of *w*. When the same discretization of *w* is applied and *v* is substituted by the vector of basis functions, this integral splits in 2 rows as with the stiffness matrix:

$$m'\left(u_i\int_0^L N_iN_jdx+\beta_i\int_0^L M_iN_jdx\right)$$

$$m'\left(u_i\int_0^L N_iM_jdx+\beta_i\int_0^L M_iM_jdx\right)$$

Each integral leads to coefficients that form the mass matrix M. The mass matrix per element $M^{(i)}$ can then be defined from the local basis functions.

$$M^{(i)}=\frac{h}{2}\begin{bmatrix}\frac{26}{35}&\frac{11}{105}h&\frac{9}{35}&-\frac{13}{210}h\\\frac{11}{105}h&\frac{2}{105}h^2&\frac{13}{210}h&-\frac{1}{70}h^2\\\frac{9}{35}&\frac{13}{210}h&\frac{26}{35}&-\frac{11}{105}h\\-\frac{13}{210}h&-\frac{1}{70}h^2&-\frac{11}{105}h&\frac{2}{105}h^2\end{bmatrix}$$

Newmark method

The use of truncated Taylor's series provides an approach to approximate *u* and its first time derivative:

$$u(t)=u(t-\Delta t)+[\Delta t]\dot{u}(t-\Delta t)+\frac{[\Delta t]^2}{2}\ddot{u}(t-\Delta t)+\beta[\Delta t]^3\dddot{u}$$

$$\dot{u}(t)=\dot{u}(t-\Delta t)+[\Delta t]\ddot{u}(t-\Delta t)+\gamma[\Delta t]^2\dddot{u}$$

Then, the acceleration is assumed to be linear within a time step:

$$\dddot{u}=\frac{\ddot{u}(t)-\ddot{u}(t-\Delta t)}{\Delta t}$$

For an equation with the following form:

$$\mathbf{M}\ddot{u}(t)+\mathbf{C}\dot{u}(t)+\mathbf{K}u(t)=\mathbf{F}(t)$$

Substituting the previous approximations, and introducing the coefficients b_n, which relate β, γ and Δt:

$$(b_1\mathbf{M}+b_4\mathbf{C}+\mathbf{K})u(t)=\mathbf{F}(t)+$$
$$+\mathbf{M}(b_1u(t-\Delta t)-b_2\dot{u}(t-\Delta t)-b_3\ddot{u}(t-\Delta t))+$$
$$+\mathbf{C}(b_4u(t-\Delta t)-b_5\dot{u}(t-\Delta t)-b_6\ddot{u}(t-\Delta t))$$

And so *u(t)* can be calculated.

References:

[1] A. J. M. Ferreira, *MATLAB Codes for Finite Element Analysis. Solids and Structures*, Springer 2009

[2] Edward L. Wilson, *Three Dimensional Static and Dynamic Analysis of Structures*, CSI Computers and Structues, Inc.

[3] Dr. Fehmi Cirak, *Finite Element Formulation for Beams*, hand-outs, http://www-g.eng.cam.ac.uk/csml/teaching/4d9/4D9_handout2.pdf (accessed 19.1.2014)

[4] ANSYS *Release 11.0 Documentation for Ansys,* http://www.kxcad.net/ansys/ANSYS/ansyshelp/hlp_g_cou3_strthermhar.html (accessed 22.1.2014)

[5] Wikipedia, Direct integration of a beam, http://en.wikipedia.org/wiki/Direct_integration_of_a_beam (accessed 22.1.2014)