Approximating optical modes by Mindlin plate deformation

02623 The finite element method for partial differential equations

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Motivation

In any optical system (for instance a telescope) manufacturing inaccuracies, or atmospheric distortion, means the image produced is imperfect. To analyze such imperfections, light may be modelled as wavefronts. As an example, light coming from an infinitely distant point source is modelled as a plane wavefront.

More complicated wavefronts may be modelled as a sum of Zernike Modes (infinite series of polynomials). We will consider: *Tilt*, arising from e.g. a tilted sensor in a camera, *Defocus* arising e.g. from incorrect spacing and finally *Coma*, a higher order optical (Zernike) mode.

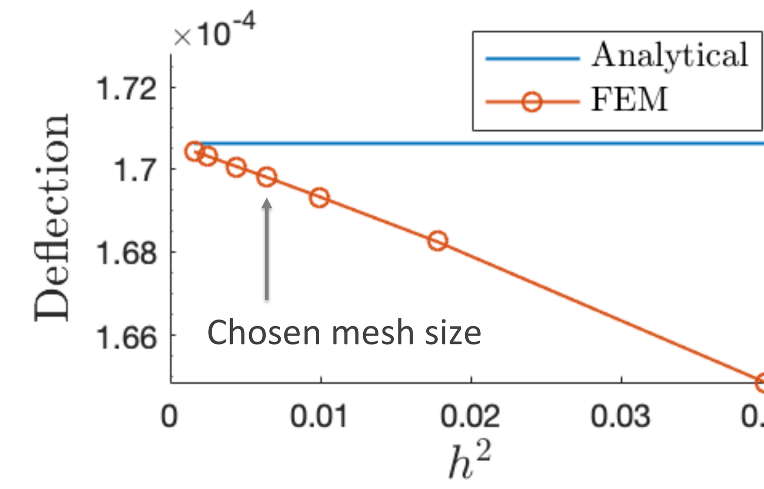
The mentioned imperfections can be corrected by reflecting light off a mirror, which is deformed so as to counteract the shape of the wavefront.

Model & Convergence

The reflector is modelled as a flat circular Reissner-Mindlin plate with clamped edges. Modelling is carried out in FEniCS, a python toolbox for solving partial differential equations using the Finite Element Method[3]. The solver is adapted from the work of Jeremy Bleyer[1].

The solver uses a mixed element formulation, where a Lagrange interpolation is used in the plate deflection and Crouzeix-Raviart interpolation is used in the plate rotation. This is necessary to avoid "shear locking" (excess shear stress).

As a first step, the solver is tested against Kirchoff-Love plate theory, which gives the same displacement for small plate thickness. A convergence study of the mesh size, where h is the maximum element height, is seen on the figure to the right



In Kirchhoff-Love plate theory, the deflection of a circular plate subjected to a uniform distributed load is:

$$w(r) = -\frac{q}{64D}(a^2 - r^2)^2$$

with load per unit area q, radius $a = 1$, and flexural rigidity D.

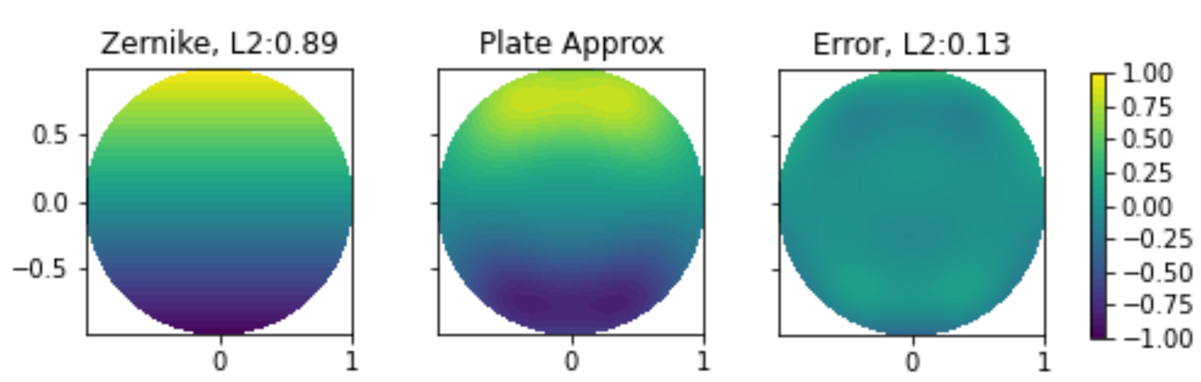


Fig 1: Zernike mode Tilt. The plots from left to right denote: the Zernike Tilt wavefront error, the displacement of the modelled reflector, the resultant wavefront when the plate displacement is subtracted from the Tilt wavefront. L2 indicates the L2 error norm. A significant reduction in wavefront error is attained.

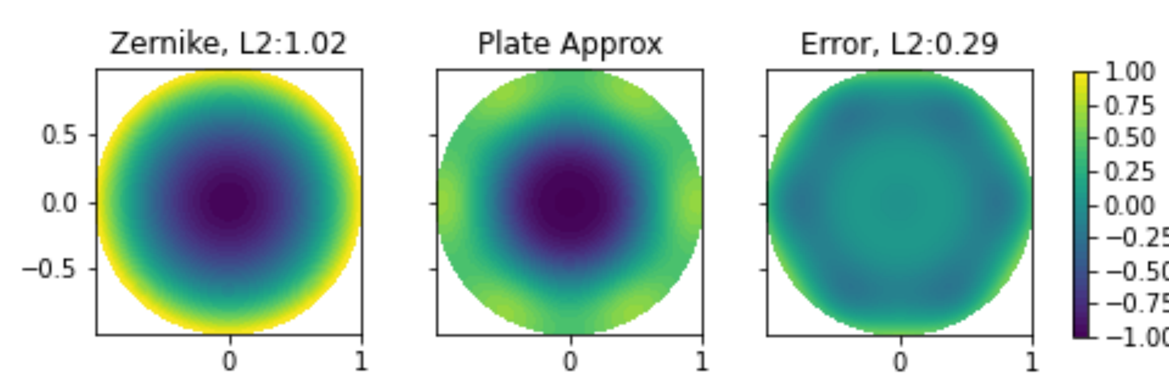


Fig 2: Zernike mode Defocus. The plots from left to right denote: the Zernike Defocus wavefront error, the displacement of the modelled reflector, the resultant wavefront when the plate displacement is subtracted from the Defocus wavefront. L2 indicates the L2 error norm. A moderate reduction in wavefront error is attained.

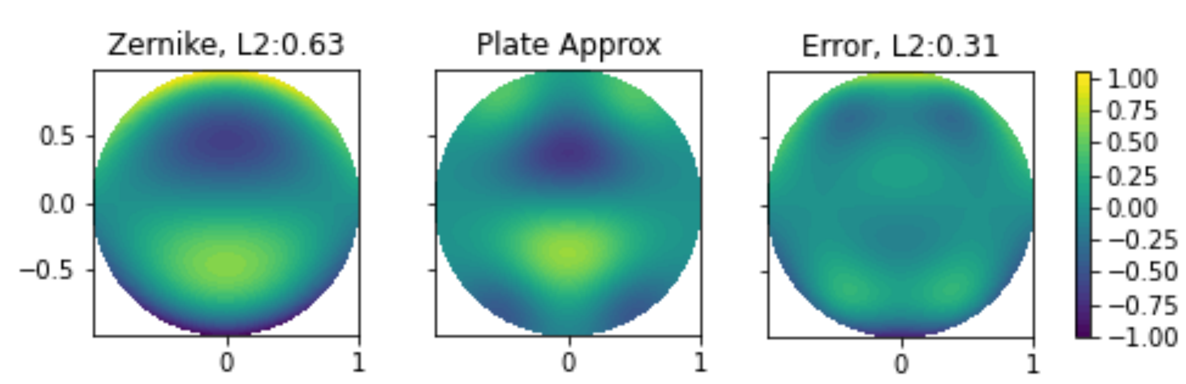


Fig 3: Zernike mode Coma. The plots from left to right denote: the Zernike Coma wavefront error, the displacement of the modelled reflector, the resultant wavefront when the plate displacement is subtracted from the Coma wavefront. L2 indicates the L2 error norm. A smaller reduction in wavefront error is attained.

Strong & Weak form

The Mindlin strong form consists of two equations; force balance and moment balance, respectively[2]:

$$-\text{div}\,\boldsymbol{Q}(x,y) = f(x,y),$$
$$\textbf{div}\,\boldsymbol{M}(x,y) = \boldsymbol{Q}(x,y), \quad \forall (x,y) \in \Omega \subset R^2$$

With shear force $\boldsymbol{Q}$ and moments $\boldsymbol{M}$.
The strong form assumes (notation explained below):

$$w(x,y,z) = w(x,y,0), w(x,y)$$
$$u(x,y,z) = -z\beta_x(x,y), v(x,y,z) - z\beta_y(x,y)$$

The Mindlin weak form is given as[2]:

$$a(w,\boldsymbol{\beta};\hat{w},\hat{\boldsymbol{\beta}}) = \int_\Omega \boldsymbol{\kappa}(\hat{\boldsymbol{\beta}})^T\, \boldsymbol{D}_b\, \boldsymbol{\kappa}(\boldsymbol{\beta})d\Omega +$$
$$\int_\Omega Gt(\nabla w - \boldsymbol{\beta}) \cdot (\nabla\hat{w} - \hat{\boldsymbol{\beta}})d\Omega$$
$$l(\hat{w}) = \int_\Omega f\hat{w}\, d\Omega$$

where the operator κ is defined as:

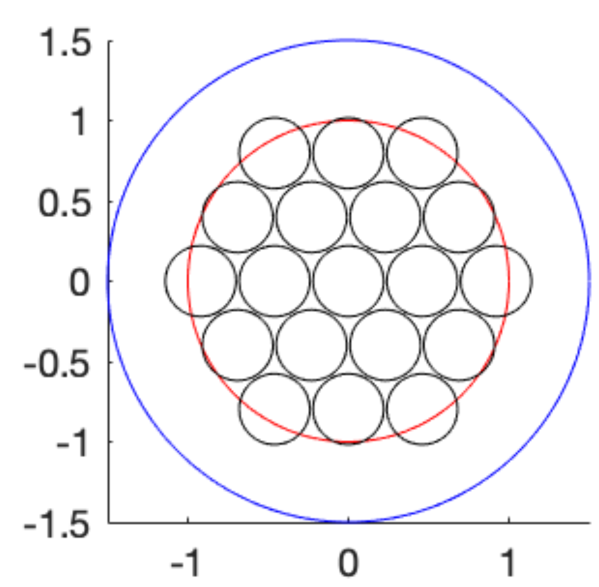
$$\kappa(\hat{\boldsymbol{\beta}})^T = \left[\frac{-\partial\beta_x}{\partial x}, \frac{-\partial\beta_y}{\partial y}, -\left(\frac{\partial\beta_x}{\partial x} + \frac{-\partial\beta_y}{\partial y}\right)\right]$$

given,

t	Thickness
G	Constant
u	Displacement in x direction
v	Displacement in y direction
$\hat{w}$	Displacement test function in z direction
w	Displacement trial function in z direction
$\hat{\beta}$	Rotation test function
β	Rotation trial function

Method

The reflector consists of a circular plate and 19 actuators. The actuators are modelled as circular subdomains, each with a uniform load (black), as shown in the figure below.



Varying the load in each subdomain enables the approximation of various optical modes. The plate has a hard clamped boundary condition (blue) (deflection and rotations zero).

To avoid a large influence from the boundary condition, the domain size has radius 1.5 (blue). The comparisons to the Zernike modes are made inside the unit domain (red). In an actual application, only this area would be reflecting.

To reach the desired shape, the problem is solved iteratively and the difference between the reflector shape and the wavefront at the center of the actuator is used to control the load applied in this location through a simple optimization procedure. After 120 iterations the problem has converged sufficiently. The computation takes approximately 45 seconds*.

Discussion & Conclusion

The performance of the modelled reflector was found to be dependent on the type of optical mode as seen in Fig 1, 2, 3. Some of the largest errors are found on edge of the unit domain. Here, the hard clamped boundary condition (BC) counteracts the plate deflection. Using a simply supported BC would reduce these errors.

The number of actuators, their shape, and relative position also clearly align with the shape of some wavefronts better than with others.

The optimization strategy used updates the actuator loads based on the point error in their center. Loads giving errors above a certain threshold are updated by a fixed amount.
A better approach would be to use an optimization metaheuristic where the objective is the L2 error norm.

*Running the simulation on a PC consuming 150 W results in a CO_2 footprint of **0.31 g**.

References

1. *Numerical Tours of Computational Mechanics with FEniCS*, Jeremyer Bleyer, 2018, https://comet-fenics.readthedocs.io
2. *9 Finite element methods for the Reissner–Mindlin plate problem*, Rak-54.3200, JN, 2016 *https://mycourses.aalto.fi/pluginfile.php/211191/mod_resource/content/2/NMSE-16-Lectures9.pdf*
3. *https://fenicsproject.org/*