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Solving the Linear Advection Equation with the Discontinuous Galerkin FEM

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Introduction

In DG-FEM the elements are decoupled and the solution is approximated by discontinuous polynomial functions. This means the local solution is approximated by a function only smooth on the local element and the global solution, consisting of combined local solutions, can be discontinuous. In order to enable decoupling, all the nodes are duplicated and the elements are coupled only via the so-called numerical flux, f^*. We wish to explore properties of the Discontinuous Galerkin FEM, which has experienced a big growth in interest over the last two decades.

Deriving a DG-FEM scheme

We consider the linear advection boundary value problem in 1 dimension

$$\begin{aligned} u_t + f_x = 0, x \in [0, 2\pi] \\ f_x = a u_x \\ u(x,0) = \sin(x) \\ u(0,t) = -\sin(2\pi t) \end{aligned}$$

The exact solution is

$$u(x,t) = \sin(x - 2\pi t)$$

We approximate the global solution on a mesh with K elements as some combination of local solutions

$$u(x,t) \simeq u_h(x,t) = \bigoplus_{k=1}^{K} u_h^k(x,t)$$

and require that the residual $R_h(x,t) = \frac{\partial u_h}{\partial t} + \frac{\partial a u_h}{\partial x}$ of each local solution be orthogonal to all N_p test functions:

$$\int_{D^k} R_h(x,t) v_n dx = \int_{D^k} \frac{\partial u_h}{\partial t} v_n + a \frac{\partial u_h}{\partial x} v_n dx = 0$$

which, via integration by parts, leads to the weak formulation of the BVP on each element

$$\int_{D^k} \frac{\partial u_h}{\partial t} v - a \frac{\partial v}{\partial x} u_h dx = - \int_{D^k} (\hat{\mathrm{n}}(a u_h)^* v) dx$$

By doing integration by parts again, we derive the strong formulation:

$$\int_{D^k} \frac{\partial u_h}{\partial t} v + a v \frac{\partial u_h}{\partial x} dx = \int_{D^k} (\hat{\mathrm{n}} v[(a u_h) - (a u_h)^*]) dx$$

where $(au_h)^*$ denotes the numerical flux. We take test functions from the space of interpolating Lagrange polynomials, and insert the resulting approximation of the local solution $u_h^k(x,t) = \sum_{j=1}^{N_p} u_h^k(x_j^k, t) l_j^k(x)$ into the strong formulation

$$\int_{D^k} \frac{\partial}{\partial t} (\sum_{j=1}^{N_p} u_h^k(x_j^k, t) l_j^k(x)) v +$$

$$a v \frac{\partial}{\partial x} \sum_{j=1}^{N_p} u_h^k(x_j^k, t) l_j^k(x) dx = [v(a u_h - a u^*)]_{x_l^k}^{x_r^k}$$

and choose the basis functions from the same space as the test functions

$$\int_{D^k} \frac{\partial}{\partial t} (\sum_{j=1}^{N_p} u_h^k(x_j^k, t) l_j^k(x)) l_i^k(x) +$$

$$a l_i^k(x) \frac{\partial}{\partial x} \sum_{j=1}^{N_p} u_h^k(x_j^k, t) l_j^k(x) dx = [l_i^k(x)(a u_h - a u^*)]_{x_l^k}^{x_r^k}$$

This is equivalent to

$$M^k \frac{d}{dt} u_h^k + S^k a u_h^k = (a u_h - a u^*) l^k(x_r^k) - (a u_h - a u^*) l^k(x_l^k)$$

where $M_{ij}^k = \int_{x_l^k}^{x_r^k} l_i^k(x) l_j^k(x) dx$

and $S_{ij}^k = \int_{x_l^k}^{x_r^k} l_i^k(x) \frac{dl_j^k(x)}{dx} dx$

Isolating the time derivative, we see that

$$\frac{d}{dt} u_h^k = -a (M^k)^{-1} S^k u_h^k + (M^k)^{-1} [(a u_h - a u^*) l^k(x_r^k) - (a u_h - a u^*) l^k(x_l^k)] \quad (1)$$

The matrix $(M^k)^{-1} S^k$ corresponds to a differential operator D_x, showing the connection between the continous differential equation, and the semi-discrete form we have derived.

Elementwise Operations

We introduce a mapping between x and a reference variable r, such that $r \in [-1, 1]$. The i'th Lagrange polynomial can be expressed as

$$l_i(r) = \sum_{n=1}^{N_p} (V^T)_{in}^{-1} \psi_n(r)$$

where V is the Vandermonde matrix, and $\psi_n(r)$ are orthonormal Legendre polynomials. In general,

$$\mathrm{l} = (V^T)^{-1} \psi$$

Using this, we look at the entries of the mass matrix M:

$$M_{ij}^k = \int_{x_l^k}^{x_r^k} l_i^k(x) l_j^k(x) dx = \int_{-1}^{1} l_i(r) l_j(r) dr$$

and after substituting and rearranging, we arrive at

$$M = (V V^T)^{-1}$$

Similarly, we find

$$D_r = V_r V^{-1}$$

where $V_{r,(i,j)} = \frac{d\psi_j(r_i)}{dr}$. Thus, all elementwise operations involve small, dense matrices, and there was no numerical integration, nor problem-specific analytical integration.

Putting the pieces together

One of the defining features of DG-FEM is that the solution is discontinous between elements. Unless we somehow impose boundary conditions on the local solutions and the test functions, the solution at the interfaces between elements is multiply defined. Also, we want to ensure numerical stability of the solutions. The way these conditions are satisfied in DG-FEM, is by using a numerical flux, f^*. We seek a numerical flux that is consistent, meaning that for two values on an interface u^- and u^+, $f^*(u^-)$ approaches $f^*(u^+)$, is unique, and mimics the dynamics, or information flow, of the PDE. An interpretation is that the numerical flux is the flux we wish to have at the interface. The flux we use is

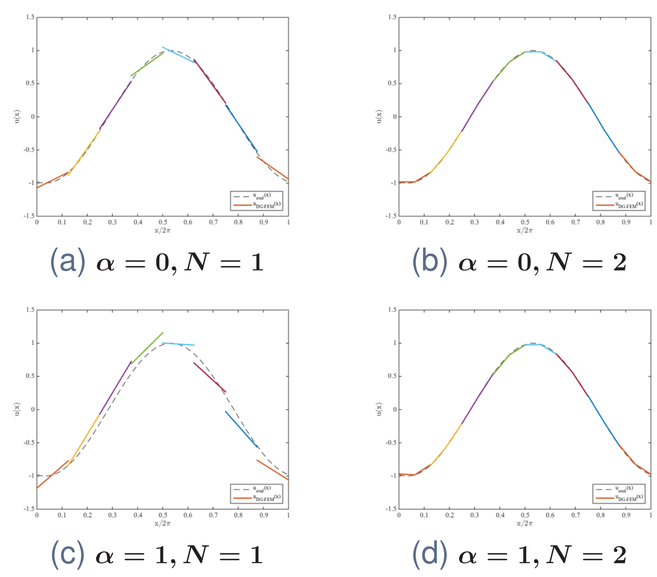
$$(au)^* = a \frac{u^- + u^+}{2} + a \frac{1 - \alpha}{2} (\hat{\mathrm{n}}^- u^- + \hat{\mathrm{n}}^+ u^+)$$

For $\alpha = 1$, we get the central flux, which is simply the average of the fluxes on the interface. For $\alpha = 0$, we get the upwind flux

$$(au)^* = a \frac{u^- + u^+}{2} + \frac{a}{2} (\hat{\mathrm{n}}^- u^- + \hat{\mathrm{n}}^+ u^+)$$

With the elementwise operations, and the flux, we have the whole right hand side (rhs) in (1). The only thing left is to use an ODE integrator like the Runge-Kutta method, to integrate the problem in time.

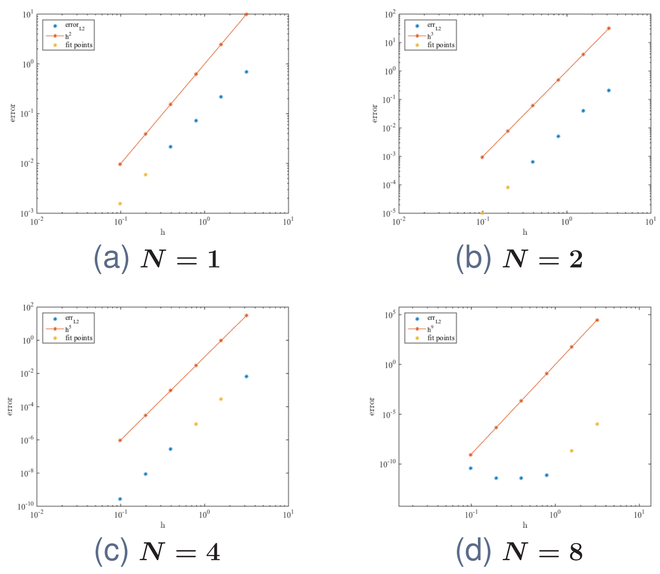
Results



(a) $\alpha = 0, N = 1$ (b) $\alpha = 0, N = 2$

(c) $\alpha = 1, N = 1$ (d) $\alpha = 1, N = 2$

Figure 1: Different fluxes and order of polynomials, $K = 8$.



(a) $N = 1$ (b) $N = 2$

(c) $N = 4$ (d) $N = 8$

Figure 2: Global L_2 error convergence for varying order of polynomials.

N\K	2	4	8	16	32	64	Conv. Rate	Th. Rate
1	6.9E-01	2.2E-01	7.3E-02	2.2E-02	6.0E-03	1.6E-03	1.93	2
2	2.1E-01	4.0E-02	5.1E-03	6.5E-04	8.2E-05	1.0E-05	3.00	3
4	6.9E-03	2.7E-04	8.8E-06	2.8E-07	8.6E-09	2.7E-10	4.97	5
8	9.7E-07	2.1E-09	7.2E-12	3.7E-12	3.9E-12	3.7E-11	8.89	9

The convergence rates are investigated by calculating the error for a variety of N and number of elements, K, and fitting appropriate points. The results listed in the Table are obtained for upwind flux and final time, $T = \pi$.

Discussion

- All operations of the rhs are local, resulting in small, dense matrices.
- Element size and polynomial order can be different in each element, allowing local change and making hp-adaptivity easy.
- No need for numerical integration, main computational work is elementwise - well suited for parallel computing
- The decoupling of elements in DG-FEM comes at the price of an increase of the total degrees of freedom compared to classical FEM. For certain problems this can become an important issue in terms of the computational work.
- For problems where flexibility and locality of the scheme is of less importance, other better suited methods might be more efficient than DG-FEM.