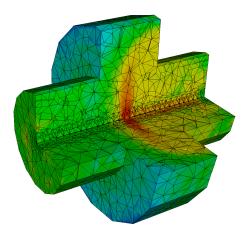
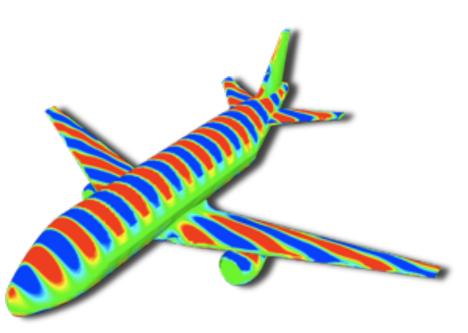


## DG-FEM for PDE's Lecture 3

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Tuesday, August 18, 2009

### A brief overview of what's to come

- Lecture I: Introduction and DG-FEM in ID
- Lecture 2: Implementation and numerical aspects
- Lecture 3: Insight through theory
- Lecture 4: Nonlinear problems
- Lecture 5: Extension to two spatial dimensions
- Lecture 6: Introduction to mesh generation
- Lecture 7: Higher order/Global problems
- Lecture 8: 3D and advanced topics

- $\checkmark$  Let's briefly recall what we know
- Why high order methods ?
- ✓ Part I:
  - Constructing fluxes for linear systems
     Approximation theory on the interval
- ✓ Part II:
  - Convergence and error estimates
  - ✓ Dispersive properties
  - ✓ Discrete stability and how to overcome

We already know a lot about the basic DG-FEM

- Stability is provided by carefully choosing the numerical flux.
- Accuracy appear to be given by the local solution representation.
- We can utilize major advances on monotone schemes to design fluxes.
- The scheme generalizes with very few changes to very general problems -- multidimensional systems of conservation laws.

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- Stability is provided by carefully choosing the numerical flux.
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At least in principle -- but what can we actually prove ?

### Why high-order accuracy ?

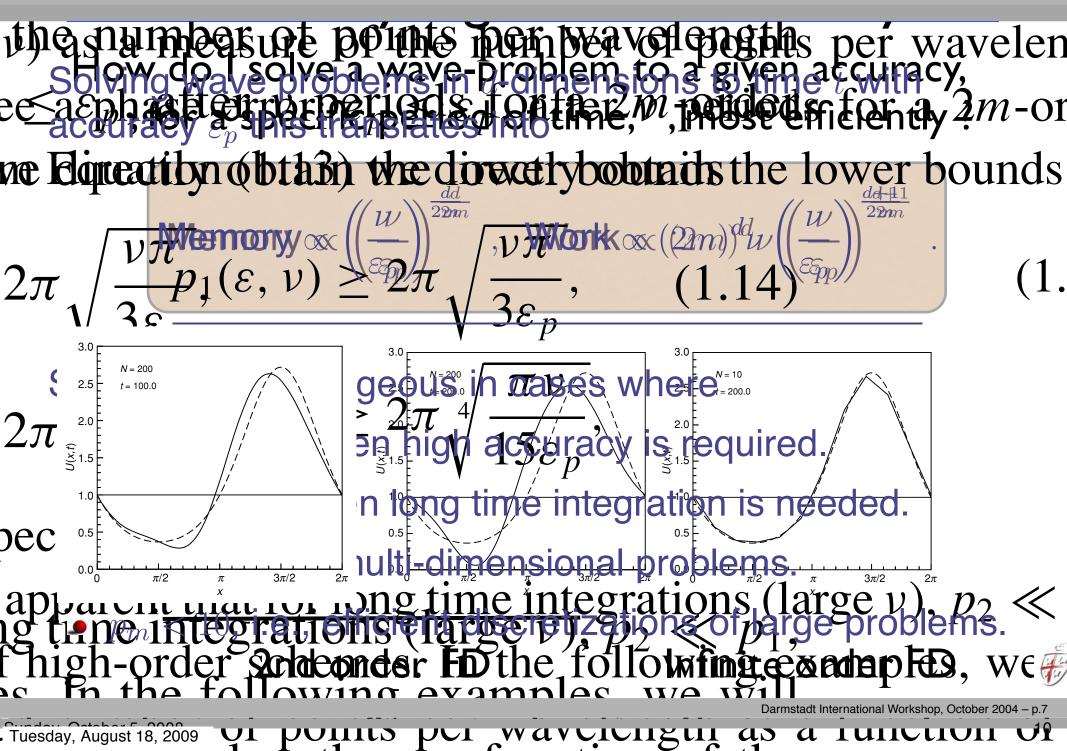
Let us just make sure we understand why high-order accuracy/methods is a good idea

General concerns/criticism:

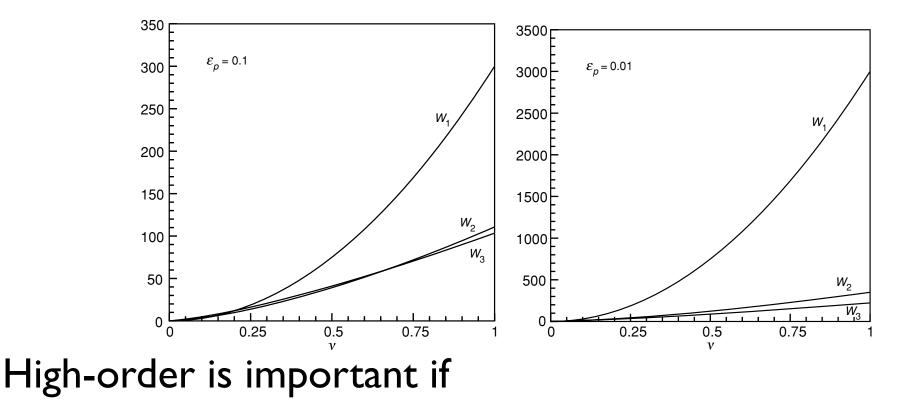
High-order accuracy is not needed for real appl.
The methods are not robust/flexible
They only work for smooth problems
They are hard to do in complex geometries
They are too expensive

After having worked on these methods for 15 years, I have heard them all

# ... high-order cont'

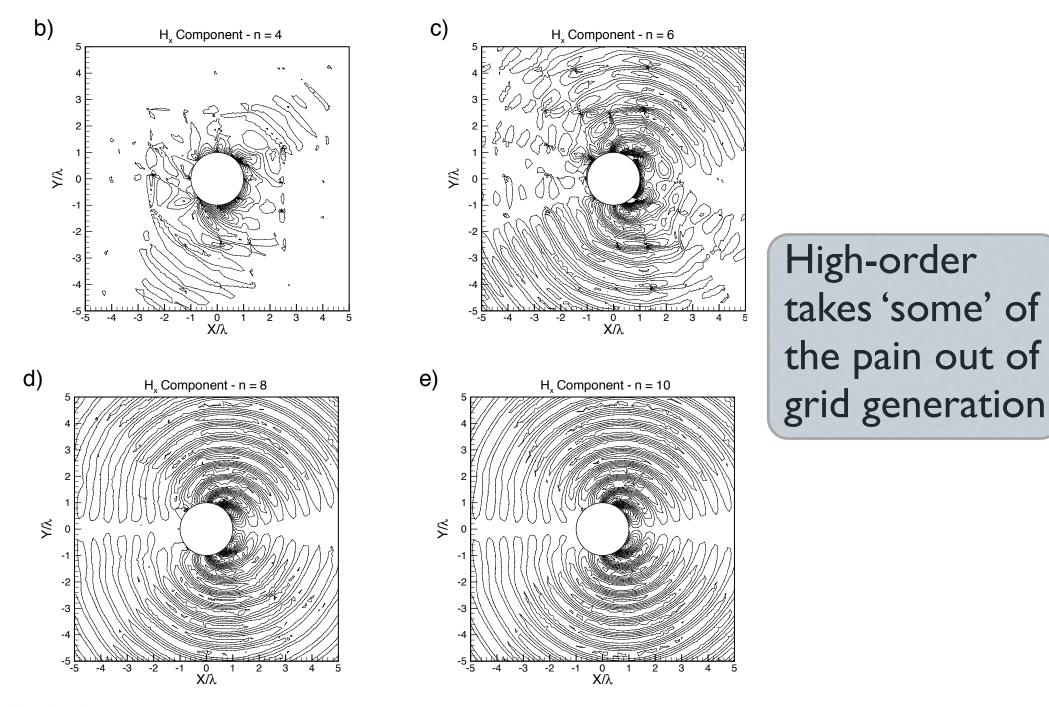


### Why high-order accuracy ?



- High accuracy is required and it increasingly is !
  Long time integration is needed
- High-dimensional problems (3D) are considered
- Memory restrictions become a bottleneck

### Addedbeenefitofhighordersupport



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### But first a bit more on fluxes But first a bit more on fluxes

Let us briefly look a little more carefully at linear systems

$$\mathcal{Q}(\boldsymbol{x})\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \mathcal{F} = \mathcal{Q}(\boldsymbol{x})\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{F}_1}{\partial x} + \frac{\partial \boldsymbol{F}_2}{\partial y} = 0,$$

$$\mathcal{F} = [\mathbf{F}_1, \mathbf{F}_2] = [\mathcal{A}_1(\mathbf{x})\mathbf{u}, \mathcal{A}_2(\mathbf{x})\mathbf{u}].$$

### Prominent examples are

- Acoustics
- Electromagnetics
- Elasticity

### In such cases we can derive exact upwind fluxes

 $\frac{1}{2} + \frac{1}{2} + \frac{1}$ 

must hold across each wave. This is also known as th Lineartisystems and flexes equence of conservation of discontinuity. To appreciate this, consider the scalar way For non-smooth coefficients, it is a little more complex  $\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} = 0, \quad x \in [a, b].$ Integrating over the interval, we have  $\frac{d}{dt} \int_{a}^{b} \frac{\lambda}{u \, dx} = -\lambda \left( \underline{u}(\underline{b}, \underline{t}) \overset{\mu}{=} u(a, t) \right) = f(a, t) - \mathbf{b}$ Then since  $f(a) = \lambda u$  on the other hand, since the wave is propared speed,  $\lambda$ , we also have  $\frac{d}{dt} \int_{a}^{b} u \, dx = -\lambda (u(b,t) - u(a,t)) = f(a,t) - f(b,t),$   $\frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{-} + (b - \lambda t)u^{+} \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx = \frac{d}{dt} \left( (\lambda t - a)u^{+} \right) \left( \frac{d}{dt} \int_{a}^{b} u \, dx \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx \right) = \lambda \left( \frac{d}{dt} \int_{a}^{b} u \, dx \right) = \lambda \left( \frac{d}{d$ 

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- derive the proper numerical upwind flux for such cases, we need to re cardine of systems and the simple up of the system of the property of the system of t
- for  $a \rightarrow x^{-}, b \rightarrow x^{+}$ seneralization to Eq. (2.20) is now straightforward. orresponding to  $A_{1}$  is entering the domain, the orresponding to  $A_{2}$  is entering the domain, the strated in Fig. 2.3.
- lowin **For** the ligenel parts steen, the semare solvers [218, 303], we that 36 2 The key ideas

 $\forall i: -\lambda_i \mathcal{Q}[u^- - u^+] + [(\Pi u)^- - (\Pi u)^+] = 0, \qquad (2.20)$ nold across cach wave. This is also known as the markine Higoniot ion and is a simple consequence of conservation of  $\hat{u}_1$  across the point of tinuity. They product hold, across reach calar wave equation wave and can be used to connect across the interface, b].

ating over the interval, we have  $\mathbf{Fig. 2.3.}$  Sketch of the characteristic wave speeds c Tuesday, August 18, 2009 boundary between two states  $u^-$  and  $u^+$ . The two interval

## Linear systems and fluxes

### So for the 3-wave problem we have 37

ming to the problem in Fig. 2.3, we have the system of equations  

$$\lambda \mathcal{Q}^{-}(\boldsymbol{u}^{*}-\boldsymbol{u}^{-}) + \left[ (\boldsymbol{\Pi}\boldsymbol{u})^{*} - (\boldsymbol{\Pi}\boldsymbol{u})^{-} \right] = 0, \qquad \lambda_{1} \qquad \lambda_{2} \qquad \lambda_{2} \qquad \lambda_{2} \qquad \lambda_{3} \qquad \lambda_{4} \qquad \lambda_{4} \qquad \lambda_{5} \qquad \lambda_{4} \qquad \lambda_{5} \qquad \lambda_{5}$$

\*,  $u^{**}$  and r the characteristic wave speeds of a through t

e can attempt to express using  $(u^-, u^{\text{T}})$  through the propromenticial spectral states in the such the set of the

2.5. Candbeisolved hear hyperbolic problem

$$\lambda_1 = -\lambda, \ \lambda_2 = 0, \ \lambda_3 = \lambda,$$

 $\frac{\partial \boldsymbol{q}}{\partial t} + \mathcal{A} \frac{\partial \boldsymbol{q}}{\partial x} = \frac{\partial}{\partial t} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} + \begin{bmatrix} \boldsymbol{a}(x) & 0 \\ 0 & \text{with } \lambda \\ 0 & \text{wave corresponding to } \lambda_3 \text{ is leaving, and } \lambda_2 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_2 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_2 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_2 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_2 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_2 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_2 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ is leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ leaving, and } \lambda_3 \text{ corresponds to } \lambda_3 \text{ leaving, and } \lambda_3 \text{ lea$ 

### An example

### Consider Maxwell's equations

$$\varepsilon(x)\frac{\partial E}{\partial t} = -\frac{\partial H}{\partial x}, \ \ \mu(x)\frac{\partial H}{\partial t} = -\frac{\partial E}{\partial x},$$

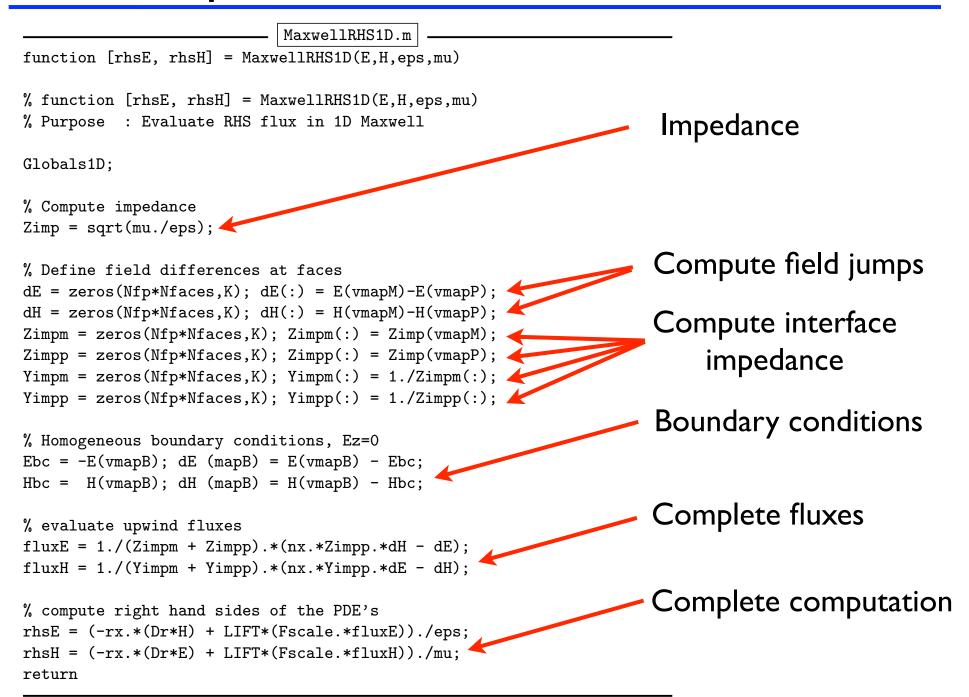
### On the DG form

$$\frac{d\boldsymbol{E}_{h}^{k}}{dt} + \frac{1}{J^{k}\varepsilon^{k}}\mathcal{D}_{r}\boldsymbol{H}_{h}^{k} = \frac{1}{J^{k}\varepsilon^{k}}\mathcal{M}^{-1}\left[\boldsymbol{\ell}^{k}(x)(H_{h}^{k}-H^{*})\right]_{x_{l}^{k}}^{x_{r}^{k}}$$
$$= \frac{1}{J^{k}\varepsilon^{k}}\mathcal{M}^{-1}\oint_{x_{l}^{k}}^{x_{r}^{k}}\hat{\boldsymbol{n}}\cdot(H_{h}^{k}-H^{*})\boldsymbol{\ell}^{k}(x) dx,$$

with the flux

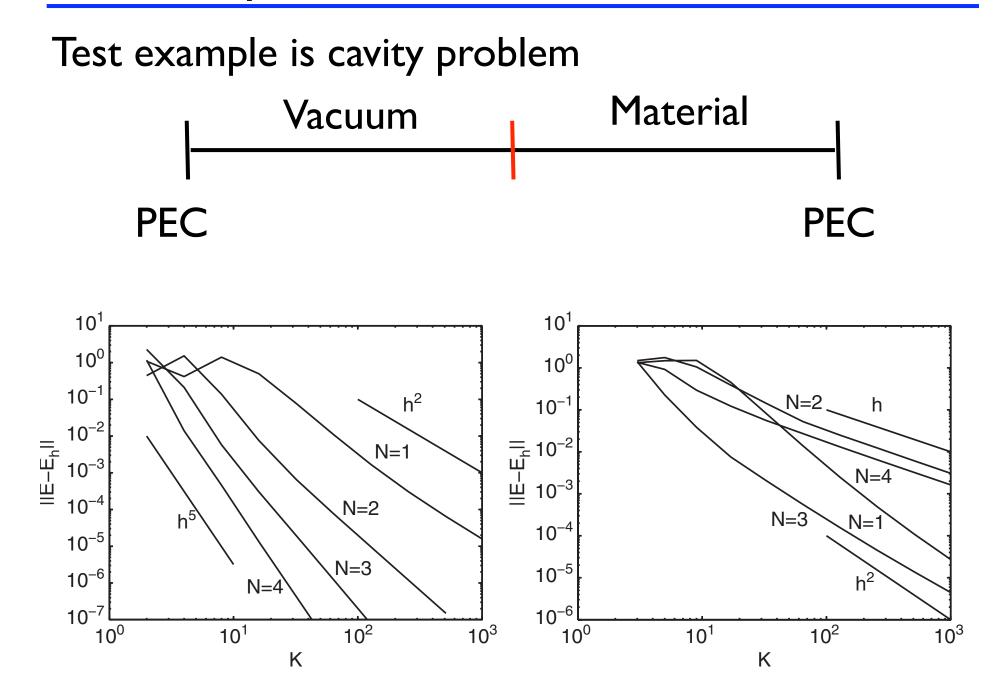
$$H^{-} - H^{*} = \frac{1}{2\{\{Z\}\}} \left( Z^{+} \llbracket H \rrbracket - \llbracket E \rrbracket \right),$$
$$E^{-} - E^{*} = \frac{1}{2\{\{Y\}\}} \left( Y^{+} \llbracket E \rrbracket - \llbracket H \rrbracket \right),$$

### An example



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### An example



e they are *d*-dimensional simplexes. The the local inner product and  $L^2(\mathbb{D}^{k})$  horm  $h(x,t) = \sum \hat{u}_n^k(t)\psi_n(x) = \sum_{i=1}^{k} u_h^k(x_i^k,t)\ell_i^k(x)$ Legenets and nodal elements i=1 $(u, v) \stackrel{\text{Here}}{\Box_k} \stackrel{\text{Here}}{=} (we u v dx, introduced (two) complementary expressions for the local solution$ apterA2 wekeesing at the model formation discharter between a besis of the global terms and the model formation of the sister of the standing of the standard Lagrange polynomial,  $\ell_i^k(x)$ . The connection between these two forms is the (u, v) the expansion coefficients  $\hat{u}_{u}^{k}$ . We return to a discussion of these choices in n Lets' normal coefficients accurately to the endetaile of these representation The global solution u(x,t) is then assumed to be approximated by the ) reflects that  $\mathbf{D}_{\mathbf{v}}$  is  $\mathbf{C}_{\mathbf{v}}$  in  $\mathbf{C}_{\mathbf{v}}$  is a bit careless, as we do not careful **Recarr** will note that this notation is a bit careless, as we do not ss what exactly happens at the overlapping interfaces. However, a more il definition ages not add anything essential at this point and we will use lotation to reflect that the global solution is obtained by combining the eavisbhotidistinguisbfthe two demains hunders needed. We assume the local solution to be be another local; information as well as an ormation from the neighent along an intersection between two elements. Often we will refer n of these intersection  $x_i^k$  in  $x_a^k$  : even that as the  $\hat{u}_n^k$  te  $\psi$  of  $x_i^k$  is  $x_i^k$ ,  $t)\ell_i^k(x)$ . bods by discuss here, we will have two or  $\frac{n}{n}$  derived solutions or  $\frac{1}{n}$  dund-ons at the same physical location along the trace of the element of the basis in the same physical location along the trace of the element of the basis in the data basis in t

a consequentes of antiquertess of and parametrical anterparametric wear the details of the individual node position that is important; for example, one could Garza Del Cox in a tig indiget is not a problem in the interpolation. Considering the corresponding nodal distribution in Fig. 3.3 for these values of  $\alpha$ , this is perhaps not surprising. when simplify matters, introduce a provide the mapping  $[T]^T$ . We este details of the individual to be position that is impartant for the Debesque of which interval to be position to be applied on the providence of the debesque of which interval to be the position of the We have  $\underline{\mathbf{a}}_{i}$  are the Legendre-Gauss-Lobarto and rature points. A central component of this up (1) strug (1) on Star Phane lishes the connections It suggests that it is reasonable to seek  $\xi_i^{i=1}$  such that the denominate where  $\xi_i \equiv r_i$  are the Legendre-Gauss-Lobatto quadrature points. A central determinant of the tegendre-Gauss-Lobatto quadrature points. A central Fonethis one dimensional case, the solution to this problem is Bela are fully, choosing the set of the set of the set of the points,  $r_i$ , we have ensured that  $\mathcal{V}_i$  is a well-conditioned object and that the resulting interpolation is well behaved. A script for initializing  $\mathcal{V}$  is given in By carefully the polate of the polation of the notation These are closely related to the normalized Legendre polynomia known as the Legendre-Gauss-Lobatto (LGL) quadrature points. Tuesday, August 18, 2009

1• A 1 1

A SECONDATOORATED PROFILE CONTRACTOR ms denned on 12; orte periode a statistic of the provision of the provision of the statistic of the statisti rken norms, denneg over 12h as sums of N° elements D/; need to be introduced. Irkantie Blaggeprog of grand and signation of the contractive of the tion Is we inside the definition of the second second the dxned, eyerweh defined has state protent found to bacing the domain tions were beind b as well as the broken norms  $\|u\|_{\Omega}^{2} = \int_{\Omega} \frac{dx}{dx} \int_{K} \frac{dx}{dx} \int$ norms norms In a similar way,  $\sqrt{e^1}$  define the associated Sobolev norms In a similar way, we define the associated Sobolev norms In a similar way, we define the associated Sobolev norms  $M_{1} = \frac{1}{2} \frac{1}{2}$  $\|\overline{u}\|_{\Omega,q}^{2} \|\overline{u}\|_{Q,q}^{2} \|u\|_{Q,q}^{2} \|u\|_{Q,$ efive the charge of the second secon where the second state of  $k = |u|_{\Omega,q,h}^2 = \sum |u|_{\mathsf{D}^k,q}^2, \quad |u|_{\mathsf{D}^k,q}^{2\alpha} = \sum ||u|_{\mathsf{D}^k}^2.$ nulti-index of log k=1  $|\alpha|=q$ dex of length d. For the one-dimensional case, most in this chapter, this norm is simply the L-norm of the q-th is chapter. At his prorm is simply the  $L^2$ -norm of the q-th 4.2 Briefly on convergence Tuesday, August 18, 2009

WOn GOINDIEINEELWO forms is through the second of these representations en one of these representations. is timelato be approximated by the epolyn erlapping interfac ldress WESE areful defagiance sport and mighing eschtial het contraction whet ween the is notative expansion coefficients,  $u_n$  we have chosen one of the dimension of the scheme that we have chosen one of the dimension of the scheme that we have chosen one of edianton har for a second with a second to be a tersections in an elem we discuss here, we will have two or more solutions or bound-cuss here, we will have two or more solutions or bound-it the same chricestion is a maximate sense is lement  $(x,t) \simeq u_h(x,t)$ ime physical location along the trace of the element  $(x,t) \simeq u_h(x,t)$ K

We have observed improved accuracy in two ways

k = 1

- Increase K/decrease h
- Increase N

une that all left onto have length h is the  $x^k = \frac{x^k}{2} \frac{b}{x^k}$ ano 7 En the begin by considering the standard interval and introduce the new variable  $v(r) \equiv u(hr) \overline{r} = u(hr)$ that is, v is defined on the standard (the transformed of the transfo Ne, considers expansions as advantage of using a local orthonormal basis u in this case, the thermanized effective polynomials 1, 1, and  $\overline{x}^k =$ [-h, h]. We discussed in Chapter 3 the advantage of using a location though basis  $v_{\overline{h}}$  in this case, the present conductory prints  $v(r)P_{n}(r)dr$ n = 0Here,  $P_n(r)$  are the classic begin die polynomials of order n. A key property **Theorem 4.1**. Assume that  $v \in H^p(t)$  and that  $v_p$  represents a polynomial of these polynomials is that they satisfy a singular Sturm-Liouville problem projection of order N. Then Here,  $P_n(r)$  are the classify Legendre property point als of order n. A key property off these polynomials is drhat they satisfy a singular Sturm-Liouville problem  $\begin{array}{c} where \\ \text{Let us consider the basic question of how well 1} \end{array}$  $\frac{d}{dr}\left(1 - r \frac{\tilde{p} - \tilde{p}}{r} - \tilde{P}_{N} + q \tilde{n} + 1\right) \tilde{P}_{n} = 0.$   $v_{h}(r) = \sum \hat{v}_{n} \tilde{P}_{n}(r),$ and  $0 \leq q \leq p$ . Let us consider the basic question of how well

Tuesday, August 18, 2009  $v \in L^2(\mathbb{I})$  Note that for simplicity of the notation and to conform

# $\begin{array}{l} \left\| v - v_h \right\|_{l,0}^2 \leq \frac{(N+1-\sigma)!}{(N+1+\sigma)!} \sum_{n=N+1}^{\infty} |\tilde{v}_n|^2 \frac{(n+\sigma)!}{(n-\sigma)!}, \end{array} \right.$

which, combined with Lemma 4.2 and  $\sigma = \min(N + 1, p)$ , gives the result. A sharper result can be obtained by using result.

**Lemma 4.4.** If  $\psi \in H^p(\mathbb{A}), p \geq 1$  then

$$\| v_{\mathcal{V}}^{(q_{\ell})} - v_{\mathcal{H}_{h}}^{(q_{\ell})} \|_{\mathbf{H}, \mathcal{H}} \leq \left[ \frac{(N + 1 - \mathcal{H})!}{(N + 1 + \mathcal{H} - 4q_{\ell})!} \right]^{1/2} |v_{\mathcal{H}}|_{\mathbf{H}, \mathcal{H}},$$

where  $\sigma \equiv \min((N + 1, p))$  and  $q \leq p$ .

The proof follows the one above and is omitted. Note that in the limit **Note that** in the limit **Note that** in the limit of **N>p** we recover

 $\begin{aligned} \|v^{(q)} - v^{(q)}_{h}\|_{l,0} &\leq N^{2q-p} |v|_{l,p}, \\ \text{in agreement with Theorem 4.1.} & N^{2q-p} |v|_{l,p}, \end{aligned}$ 

# A minor issues arises -- these results are based on projections and we are using interpolations ?

ed ins et en Brockate is ister to be brock in the tisters deticate beganal polynon end of the second of the secon er ussen here for the property of the property of the sential to er ussen here to be the property of the prope eciate it. Conside is based on  $N_{\rm wth}$ ectate it. Consider the structure of the section o appreciate it.  $\underbrace{\tilde{v}_n \tilde{P}_n(r_i)}_{(r_i)} + \underbrace{\sum_{n=N_i \in \mathbb{N}} \tilde{v}_n \tilde{P}_n(r_i)}_{\tilde{v}_n P_n(r_i), \tilde{v}_n P_n(r_i), \tilde{v}_n P_n(r_i), \tilde{v}_n \tilde{P}_n(r_i), \tilde{v}_n \tilde{v}_n \tilde{P}_n(r_i), \tilde{v}_n \tilde{v}_n$  $\tilde{v}_{\boldsymbol{x}}\tilde{P}_n(r_i),$ The interpolation of the period of the period of the property we have  $\tilde{v}_{n}$  and  $\tilde{v}_{n}$  $(\mathcal{V}\hat{\boldsymbol{v}})_{\boldsymbol{v}} \underbrace{\tilde{\boldsymbol{v}}_{n}}_{\boldsymbol{v}} \underbrace{\tilde{\boldsymbol{v}}_{n}} \underbrace{\tilde{\boldsymbol{v}}_{n}}_{\boldsymbol{v}} \underbrace{\tilde{\boldsymbol{v}}_{n}} \underbrace{\tilde{\boldsymbol{v}}}_{n}} \underbrace{\tilde{\boldsymbol{v}}_{n}} \underbrace{\tilde{\boldsymbol{v}}_{n}} \underbrace{\tilde{\boldsymbol{v}}}_{n}} \underbrace{\tilde{\boldsymbol{v}}_{n}} \underbrace{\tilde{\boldsymbol{v}}} \underbrace{\tilde{\boldsymbol{v}}}_{n}} \underbrace{\tilde{\boldsymbol{v}}} \underbrace{\tilde{\boldsymbol$ is jim involucion to be cover involucion to the expression of the second cover where  $\hat{v} = \hat{v} = \hat{v} + \hat{v}$ whicomsider the Slies Now, consider the  $\tilde{v}_{h}(r)$   $\tilde{F}_{h}(r)$   $\tilde{P}_{h}(r)$   $\tilde{P}_{h}(r)$   $\tilde{P}_{h}(r)$   $\tilde{F}_{h}(r)$   $\tilde{F}$ Tuesday, August 18, 2009

n = N + 1

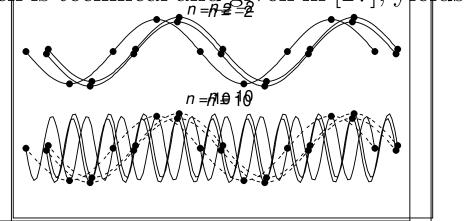
Approximation the property he right-hand side,

 $\begin{aligned} \mathbf{Consider} & \tilde{\mathbf{P}}^{T}(\mathbf{r}) \mathcal{V}^{-1} \tilde{\mathbf{P}}_{n}(\mathbf{r}) = \sum_{\substack{n=N+1 \\ n=N+1}}^{\infty} \tilde{v}_{n} \left( \tilde{\mathbf{P}}^{T}(\mathbf{r}) \mathcal{V}^{-1} \tilde{P}_{n}(\mathbf{r}) \right), \\ & \tilde{\mathbf{P}}^{T}(\mathbf{r}) \mathcal{V}^{-1} \sum_{\substack{n=N+1 \\ n=N+1 \\ v \in \mathbf{r}}} \tilde{v}_{n} \tilde{P}_{n}(\mathbf{r}) = \sum_{\substack{n=N+1 \\ v \in \mathbf{r}}} \tilde{v}_{n} \left( \tilde{\mathbf{P}}^{T}(\mathbf{r}) \mathcal{V}^{-1} \tilde{P}_{n}(\mathbf{r}) \right), \\ & \text{which is allowed provided} \left| v^{\mathsf{P} \in \mathbf{r}} \mathcal{P}_{n} \mathcal{P}_{n}(\mathbf{r}) = \sum_{\substack{n=N+1 \\ v \in \mathbf{r}}} \tilde{v}_{n} \left( \tilde{\mathbf{P}}^{T}(\mathbf{r}) \mathcal{V}^{-1} \tilde{P}_{n}(\mathbf{r}) \right), \\ & \text{which is allowed provided} \left| v^{\mathsf{P} \in \mathbf{r}} \mathcal{P}_{n} \mathcal{P}_{n}(\mathbf{r}) = \sum_{\substack{n=N+1 \\ v \in \mathbf{r}}} \tilde{v}_{n} \left( \tilde{\mathbf{P}}^{T}(\mathbf{r}) \mathcal{V}^{-1} \tilde{P}_{n}(\mathbf{r}) \right), \end{aligned} \right. \end{aligned}$ 

$$\tilde{\boldsymbol{P}}^{T}\overline{(r)}\mathcal{V}^{-1}\tilde{P}_{n}(\boldsymbol{r}) = \sum_{l=0}^{N} \tilde{p}_{l}\tilde{P}_{l}(r), \quad \mathcal{V}\tilde{\boldsymbol{p}} = \tilde{P}_{n}(\boldsymbol{r}),$$
2.2-Discrete-trigonometric piologaboniantsials

**Cause by eighter pointer point interpolation of the prior of the pri** 

### Caused by the grid



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Figlisur. 24 Multiustration and as line in the thread way  $n = 10^{-10}$ 

### Approximation theory

### This has a some impact on the accuracy $\frac{1}{4}$ Insight through theory

**Theorem 4.5.** Assume that  $v \in H^p(I)$ ,  $p \geq \frac{1}{2}$ , and that  $v_h$  represents a polynomial interpolation of order N. Then

$$\|v = v_{h}\|_{1,q} \leq N^{2q = p + 1/2} |v|_{1,p},$$

where  $0 \leq q \leq p$ :

Note in particular that the order of convergence can be up to one order Towalsonaccountator ethe, cell size we have t.

**Theorem 4.5.** Let summet that the set of the product of the prod

for the first, example, that As, +1, p).

 $v(r) = \exp(\sin(\pi r)),$ 

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# $\begin{array}{c} & \|u - u_h\|_{\mathsf{D}^k}^2 \leq h^{1-2q} \|v - v_h\|_{\mathsf{I},\sigma}^2 \leq h^{1-2q} |v|_{\mathsf{I},\sigma}^2 = h^{2\sigma-2q} |u|_{\mathsf{D}^k,\sigma}^2, \end{array} \\ & \\ \end{array}$

where we have used Lemma 4.4 and defined  $\sigma = \min(N+1, p)$ . Summing over all elements yields the result.

For a more general grid where the element length is variable, it is natural to **Combining**, everything, we have the general besult his with

Theorem 4.7 yields the main approximation result:

**Theorem 4.8.** Assume that  $u \in H^p(\mathsf{D}^k)$ , p > 1/2, and that  $u_h$  represents a picewise polynomial interpolation of order N. Then

$$\begin{aligned} \|u - u_h\|_{\Omega,q,h} &\leq C \frac{h^{\sigma - q}}{N^{p - 2q - 1/2}} \|u\|_{\Omega,\sigma,h}, \\ \|u - u_h\|_{\Omega,q,h} &\leq C \frac{N^{p - 2q - 1/2}}{N^{p - 2q - 1/2}} \|u\|_{\Omega,\sigma,h}, \end{aligned}$$

for  $0 \le q \le \sigma$ , and  $\sigma = \min(N+1, p)$ . for  $0 \le q \le \sigma$ , and  $\sigma = \min(N+1, p)$ .

This result gives the essence of the approximation properties; that is, it shows  $\mathcal{M}_{k}^{\text{clearly}} \mathcal{M}_{k}^{\text{clearly}} \mathcal{M}_$ 

4.4 Stability

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Fluxes:

✓ For linear systems, we can derive exact upwind fluxes using Rankine-Hugonoit conditions.

#### Accuracy:

Legendre polynomials are the right basis
 Local accuracy depends on elementwise smoothness
 Aliasing appears due to the grid but is under control
 For smooth problems, we have a spectral method
 Convergence can be recovered in two ways
 Increase N
 Decrease h

Convergence of the solution at all times ?

### Lecture 3

- ✓ Let's briefly recall what we know
- ✓ Why high-order methods ?
- ✓ Part I:
  - Constructing fluxes for linear systems
  - Approximation theory on the interval
- ✓ Part II:
  - Convergence and error estimates
  - Dispersive properties
  - $\checkmark$  Discrete stability and how to overcome

### Et 20 EDEREYT VIOLUTEVENES CERTERE La 2 Briel von gonvergenge La su Let ets us as neiden the energianely bareshelisy system A THE DEPENDENCE DOUDDARY COL is diagonizable and POSTERED AND THE EXISTS CONSTANTS EN AND A such that at the southed at the sout The second secon pickeeseiseopolynomiae, hy hydrigatistics the sengilikeiseke elseneme dunter h The semiconset of u = T(u(x,t)) du = T(u(x,t)) dt = T(u(x,t)) dt dt = T(u(x,t)) dt Tuesday, August 18, 2009

 $-u + \mathcal{L}_h u = I (u | U, l)$ 

### $\boldsymbol{\varepsilon} \in \boldsymbol{\varepsilon}(\boldsymbol{x}, \boldsymbol{t}) \stackrel{t}{=} \boldsymbol{H}(\boldsymbol{x}, \boldsymbol{t}), \boldsymbol{t})$

### is mathinal to the Really segencies in a parson of the second segments of the parson of the second s

$$\forall t \in [0, T]:: \lim_{\substack{d \in T \to \infty \\ d \to \infty}} \| \varepsilon(t) \|_{\mathcal{G}, h} \Rightarrow 0:$$

The have introduced the notion of degrees of freedom (def)<sup>effect</sup> effect the the presence can be achieved either by decreasing the cell size, by increasing on vergence can be achieved either by decreasing the cell size, by increasing on vergence can be achieved either by decreasing the cell size, by increasing the order of the approximation. W or by doing both simultaneously, known is provided to the approximation. W or by doing both simultaneously, known hp-contregence.

*hp*-convergence. Proving convergence directly is, however, complicated due to the need to Proving convergence directly is, however, complicated due to the ne ove it for all time. Fortunately, there is shortcut. Consider the error equation cove it for all time. Fortunately, there is shortcut. Consider the error equation

Conversence and a sthat have mind and consistency 77 7

Let US constitution the contact of the second solution is and consistency 777by orthogonal polynomials and consistency  $\varepsilon \mathcal{L}(w, t)$ . 4.3 Approxidations by With the exact solution with the exact solution of the first 7777 $\mathcal{T}((\mathbf{u}(\mathbf{x},t)))$  $\varepsilon_l + \mu_{\varepsilon_h} \varepsilon =$ with the exact ophilipp (-C, t) = (0) = = + e with solution of size (0) = = + e  $\stackrel{\text{\tiny +e}}{=} \underbrace{ \begin{array}{c} \mathcal{F}_{h} \\ \mathcal{F}_{h}$ where we have some set to we have some set to be the solution of the solution establing over Hinselfand and the state of t where we shave stupped sectors for the explicit the perpendence of the formula  $\mathcal{T}(u(s)) ds$ . grating over the state of the s  $\Omega, l$ ds $\exp \left( \mathcal{L}_h(s-t) \right) \mathcal{T}(\boldsymbol{u}(s)) ds \|_{\mathcal{L}_h} \leq t \int \sup_{\boldsymbol{u} \in \mathcal{T}_h} \left( \exp \left( \mathcal{L}_h(s-t) \right) \|_{\Omega,h} \|\mathcal{T}(\boldsymbol{u}(s))\|_{\Omega,h} ds, \\ \exp \left( \mathcal{L}_h(s-t) \right) \mathcal{T}(\boldsymbol{u}(s)) \|_{\Omega,h} ds, \\ \operatorname{exp} \left( \mathcal{L}_h(s) \right) \mathcal{T}(\boldsymbol{u}(s)) \|_{\Omega,h} ds, \\ \operatorname{exp} \left( \mathcal{L}_h(s-t) \right) \mathcal$ It studies to easily consistency is to easily the sense that  $(\mathcal{L}_h(s-t)) \|_{\Omega,h} \|\mathcal{T}(\boldsymbol{u}(s))\|_{\Omega,h}$ Tuesday, August 18, 2009 to ensure

 $\begin{aligned} & \left\| \begin{pmatrix} t \\ \Omega, h \\$ Selution if the second free second se

 $\frac{1}{100} = \frac{1}{100} = \frac{1}$ (4 :1)

If  $f \to \infty$  dof  $f \to \infty$  is crete approximation.  $f \to \infty$  is crete approximation.

stating so sock convergence in the sense that 

suthisneuhalfnofithe celebrated equivalence theorem by Lassing Richtigen gestsal an atural lingproach tobestablish convergence focus on understanding consistency eso the The second of th

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Recall

# s, in **Convergence and that**onsistency. In Chapter lished stability for the simple scalar problem

# 4.5 Error estimates and error boundedness 4.5 Error estimates and error boundedness 4 Insight through theory $2^{4.5}$ Error boundedness

by using an energy method project stability dayse can straightforwardly claim convergence for a number of different linear problems; for example

$$\frac{1}{2} \frac{\frac{d}{dt}}{\frac{d}{dt}} \frac{d}{dt} \frac{d}{dt$$

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which This generatizes tability in fact, we managed to prove that diagonal ensuring that we established tability of Systems when upwinding per-ensuring that we established tability of the stability of the equivalence theorem, Stability end to be a stability of the stability of the equivalence theorem, Stability end the stability of the stability of the equivalence theorem, differences ential sings only time he had i ceco fut be fit the Consider the same wall defindi-Combining this with the accuracy analysis yields

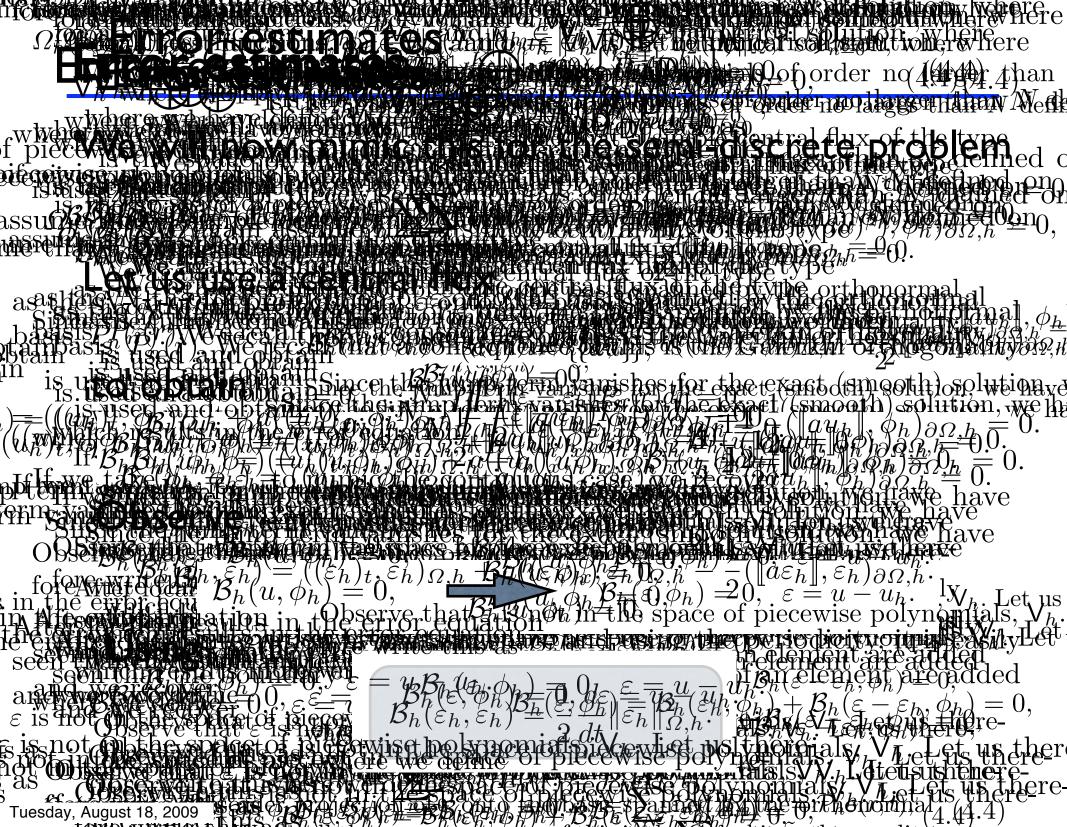
$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathcal{A} \frac{\partial \boldsymbol{u}}{\partial x} = 0_{hhN} \\ \| \boldsymbol{u} - \boldsymbol{u}_{h} \|_{\boldsymbol{\Omega},h} \leq \frac{\partial \boldsymbol{u}}{N p^{5/3/2}} \| \boldsymbol{u} \|_{\boldsymbol{\Omega},hp,h},$$

where  $\mathcal{A}$  is an  $m \times m$  diagonizable matrix with purely real eigenvalues (i.e., a hyper but a west of the second of the second state of the second and initiarebaltitorisince provident. In single statistic for this Could 60 based on central fluxes as  $||u(T) - u_h(T)||_{\Omega,h} \le h^{N+1}(C_1 + TC_2).$ 

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.1.

The propression in a pre-more detail and the here  $\mathcal{U}_{\mathcal{O}}$  and the propression of the propresect of the propression of the propression of th COVER MUSIC CONTRACTOR OF CONT OPERATE AND A PROPERTY OF A CONTRACT OF A CO  $u_1$  his (0) so Here is that we if solve Eq. (4.3) with two different initial conditions.  $u_1$  and  $u_2$  (0),  $2v_1$  (1),  $2v_2$  (1),  $2v_1$  (1),  $2v_1$  (1),  $2v_1$  (1),  $2v_2$  (1),  $2v_1$  (1),  $2v_2$  (1),  $2v_1$  (1),  $2v_1$ where we have defined the arrhy of the first the distribution of (t) When the this is the contract of the second 



The Contract of the standard to according to the standard to accord unlity [59]

 $= \underbrace{ \begin{array}{c} \mathcal{L}(\mathcal{N}) \mathcal{B}_{h}(\mathcal{P} \wedge \mathcal{U} - u, \varepsilon_{h}), \\ \mathcal{L}(\mathcal{P} \wedge \mathcal{D}) \mathcal{B}_{h}(\mathcal{P} \wedge \mathcal{U} - u, \varepsilon_{h}), \\ \mathcal{L}(\mathcal{P} \wedge \mathcal{D}) \mathcal{B}_{h}(\mathcal{P} \wedge \mathcal{D}) \mathcal{B}_{h}(\mathcal{D}) \mathcal{B}_{h}(\mathcal{D}), \\ \mathcal{L}(\mathcal{D} \wedge \mathcal{D}) \mathcal{B}_{h}(\mathcal{D}), \\ \mathcal{L}(\mathcal{D} \wedge \mathcal{D}) \mathcal{B}_{h}(\mathcal{D}) \mathcal{B}_{h}(\mathcal{D}), \\ \mathcal{L}(\mathcal{D} \wedge \mathcal{D}) \mathcal{B}_{h}(\mathcal{D}) \mathcal{B}_{h}(\mathcal{D}), \\ \mathcal{L}(\mathcal{D} \wedge \mathcal{D}) \mathcal{D}_{h}(\mathcal{D}), \\ \\ \mathcal{L}(\mathcal{D} \wedge \mathcal{D}) \mathcal{D}_{h}(\mathcal{D}), \\ \\ \mathcal{L}(\mathcal{D} \wedge \mathcal{D}) \mathcal{D}, \\ \\ \\ \mathcal{L}(\mathcal{D} \wedge \mathcal{D}) \mathcal{D}, \\ \\ \mathcal{L}(\mathcal{$ Now consider  $\mathcal{B}_h(w - \mathcal{P}_N u \in \mathcal{E}_h)$  $\mathcal{P}_N u - u, \varepsilon_h),$  $\overline{2} \, \overline{d} t$  $\| \mathcal{U} \|_{\mathcal{U}}^{2} \|_{\mathcal{U}}^$ We then have We then have **one proves**  $(whith some \mathcal{P}_{N}(u - u_{h})) \stackrel{u_{h}}{=} (u - u_{h}) \stackrel{u_{h}}{=} (u - u_{h}) = \mathcal{P}_{N}(u - u_{h}) \stackrel{u_{h}}{=} (u - u_{h}) \stackrel{u_{h}}{=} \mathcal{P}_{N}(u - u_{h}) \stackrel{$ The man result is  $\mathcal{F}_{h,\overline{\sigma},h,\sigma} = \mathcal{F}_{h,h} u - u$  and co by using Lemma 4 http://powed.thearst with the trace mediate of the second with the trace mediate of the second with the trace mediate of the second with  $h, a \neq 1$ from which we recover an improved result of the type from which we recover an improved, result of the type  $\hat{n}^{\hat{n}} \{\{a_{q}^{\hat{n}}, b_{h}^{\hat{n}}\}\}$  $\frac{d}{dt} = \frac{d}{dt} \frac{|||_{\mathcal{B}_h}(T^2)||_h}{||_h} \leq \frac{(C|_h|_h \mathcal{C}_2 T)h|_{\mathcal{O}_h}}{||_{\mathcal{O}_h}} = \frac{d}{dt} \frac{|||_{\mathcal{B}_h}(T^2)||_h}{||_h} \leq \frac{(C|_h|_h \mathcal{C}_2 T)h|_{\mathcal{O}_h}}{||_{\mathcal{O}_h}} = \frac{d}{||_{\mathcal{O}_h}} = \frac{d}{dt} \frac{|||_{\mathcal{B}_h}(T^2)||_h}{||_h} \leq \frac{(C|_h|_h \mathcal{C}_2 T)h|_{\mathcal{O}_h}}{||_{\mathcal{O}_h}} = \frac{d}{||_{\mathcal{O}_h}} = \frac{d}{||_h} = \frac{d}{||$ fram settlerwer bar mon junipeo voe result off the type LEmma 4499. If  $u \in \mathbb{H}_{h}^{p+1} ( \mathbb{Z}^{k} ( \mathbb{C}_{1}^{p+1} \mathbb{C}_{2}^{k+1} ) )$ 

The observe full order

$$||u(T) - u_h(T)||_{\Omega,h} \le h^{N+1}(C_1 + TC_2).$$

is in fact a special case !

It only works when

- When full upwinding on all characteristic variables are used
- $\checkmark$  Proof is only valid for the linear case
- Proof relies on ID superconvergence results

In spite of this, optimal convergence is observed in many problems - why ?

#### Why exact solution $a_{dof} + \mathcal{L}_{h} = 1^{(u(u, t))},$ Why often $a_{dof} + \mathcal{L}_{h} = 0$

 $\frac{dhThu}{dt} = \frac{dt}{dt} = \frac$ A South a stability e approximation. t)) it interior is the solution of the solutio If where now introduce the error  $\mathcal{L}_h t$  is the er (4.1)MARTEON CHEORY & to A HING ARE THE WEY & th gyarantegy ophiesence history haider, resence and her yen stability and allipasseurasto seek convergence in the sense that  $u \|_{\mathbf{D}^k}^2$ This (1) 215 g suggests a Gattoral supposed to hestablish convergence, for nearupuldalaone, Firen, Tha spastocus ve confisine the several areas , obta Veokinvatiotroducadettibe nation per stagese and, freedoch, (gothet question tofat ain the singer and the singer the singer the set of the singer the set of the the order of the approximation,  $N^{0}$  or by doing both Simultaneously, known  $\mathbf{F}_{n}$  in  $\mathbf{F}_{n}$  and  $\mathbf{F}_{n}$  in the series that  $h^{-2q} | v|^{-2q} | v|^{-2q} | u|^{-2q} | u|^{$  $\begin{array}{l} \mathbf{H} = \mathbf{H} \\ \mathbf{$ enver vierdes cine hos well one can approximate an fee berich where at hese functions into denversinger sent the secons phase of the second s Tuesday, August 18, 2009  $spf \rightarrow p$ 

**Dispersive** properties the wavenumber of the initial and the form  $dt = \frac{\partial a}{\partial t} = 0$ . If we seek provide the solutions of the form  $dt = \frac{\partial a}{\partial t} = 0$ . patiany periodic solutions of the rolling being by discussing signal be gained by discussing signal be gained by discussing we and dispersive properties of the solution of th dissipative and dispersive properties of the solvence (*it*) avenumber of the initial of the **Solvence** and **Solvence** of the initial of the simple wave equation is the wavenumber of the initial of this easily receiver the simple wavenumber with the being the exact phase velocity. In purpose of the discussion in this chapter is to understand how well the discontinuous Galerkin, scheme reproduces behavior. behavior. We now assume that the computeriona Repredis Boy assume t seiousistiete Philippe A We, now assument hat the long zerat Weeponsider an a gest of the second Ve nove)assume the We consider the general basic semidiscrete local scheme  $\omega_{i}$  being the approach is the set of t along the clispersion readion with a being the purpose of the discussion of this chapter is the purpose of the discontinuous Gelerkin scheme reproduces this  $(a_{\mathbf{f}})$ velocity. well the €K<sub>v</sub>\*iS  $e_0$ The stand of the second provide the second provide the second stand  $a \gg 100$  s to second stand sta hewere that the computational domain is split into equidistant be now assume that the computational domain is split into equidistant s. D. all of length h. For k(k, t) the top exp[i(ln sector)], 2 we recover ents. D. all of length h. For k(k, t) the top exp[i(ln sector)], 2 we recover level of length h. For k(k, t) the top exp[i(ln sector)], 2 we recover the relationship where the the the the transfer of the transfer 

$$+ \underbrace{\begin{array}{c}a(2-\alpha)\\T} \\ + \underbrace{\begin{array}{c}2-\alpha\\T}\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T}\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T} \\ + \underbrace{\begin{array}{c}2-\alpha\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T} \\ + \underbrace{\begin{array}{c}2-\alpha\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T} \\ + \underbrace{\begin{array}{c}2-\alpha\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T} \\ + \underbrace{\begin{array}{c}2-\alpha\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T} \\ + \underbrace{\begin{array}{c}2-\alpha\\T} \\ + \underbrace{\begin{array}{c}2-\alpha\\T\end{array} \\ + \underbrace{\begin{array}{c}2-\alpha\\T} \\ + \underbrace{\begin{array}{c}2-\alpha}T\end{array} \\$$

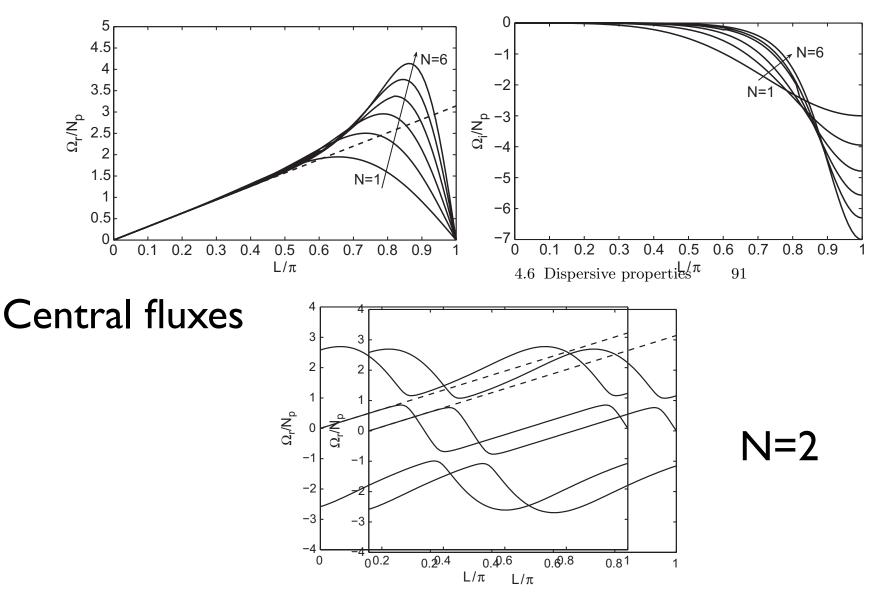
ed as a generalized eigenvalue problem This is recognized as a generalized eigenvalue problem This  $\mathbf{F}_{N} = \mathbf{e}_{N} \mathbf{e}_{$  $\begin{array}{l} 2S \stackrel{\frown}{\rightarrow} \alpha e_{NF}(e_{N}^{*}) \stackrel{\downarrow}{\rightarrow} \exp(iL(N+1))e_{0}^{T}) \\ SSee \\ = \exp(2iE_{N}^{*})e_{N}(e_{N}^{*})e$  $(2-\alpha)\boldsymbol{e}_0 \begin{vmatrix} 2\boldsymbol{s}_2 & \boldsymbol{s}_2 & \boldsymbol{s}_2 \\ \boldsymbol{e}_0 & \boldsymbol{s}_2 & \boldsymbol{s}_2 \\ \boldsymbol{e}_0 & \boldsymbol{e}_1 \\ \boldsymbol{e}_0 & \boldsymbol{e}$ 

lized things as We have normalized things as  $L = \frac{\mathbf{W} \mathbf{h} \mathbf{e}}{N+1} = \frac{2\pi}{\lambda} \frac{h}{L} = \frac{h}{N+1} \frac{h}{1} = \frac{2\pi}{\lambda} \frac{h}{N+1} = \frac{2\pi}{\lambda} \frac{h}{N+1}$ 

where  $p = \frac{\lambda}{h/(N+1)} = \frac{p}{p} = \frac{p}{h (N+1)} \frac{-p}{p} = \frac{p}{h (N+1)} \frac{p}{p} = \frac{p}{h (N+1)} \frac{p}{1}$ 

the number of degrees of freedom per wavelength. The min-is a measure of the number of tegers of the decompense wavelengthe The min-uls wine asure his the party of digit of solar to the new wavelength. The minnigree voideneitona nervavaveer uls valme a sure hot. the laarhoer tot in the mean in the laarhoer tot in up the all instants in the sure to a tot im up the all instants in a sure to a tot Techenize that T = 12/(777+1) is now the expression dispersion relation and igeographic the dispersion relation and by solving the enditive problem for  $T = r_1 r_n$  one recovers the disper-the solving the enditive problem for  $T = r_1 r_n$  one recovers the disper-sion relation of the numerical scheme, with  $\Omega_r$  representing an approximation wave propagates





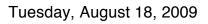


Fig. 4.2. Numerical dispersion relations for the linear advection operator with a purely central flux. The dashed line represents the exact case and the solid lines

ary conditions. A number of concluding conjectures made in that work have subjective subjective property for the subjective following, we will outline the key results of this analysis.

There are some analytic results available (upwind) the dispersive error [9

4 Insight through theory 92 and the dissipative error is  $|\mathcal{R}(\tilde{h}) - \mathcal{R}(\tilde{h})| \simeq \frac{1}{2^2} \left[ \frac{N!}{(2N+1)!} \right]^2 (\tilde{h})^{2N+3},$ 

$$\left|\mathcal{I}(\tilde{l}|\mathcal{I})|\tilde{l}_{\mathcal{I}}|\tilde{l}_{\mathcal{I}}|_{2}^{2} = \left[\frac{1}{2}\left[\frac{N!}{(2N(2N_{1})!)}^{N}\right]^{2} - (1)^{(n-1)^{N}}(2N_{1})^{2N+2}, \frac{1}{(2N(2N_{1})!)}^{2N+2}\right]^{2} - (1)^{(n-1)^{N}}(2N_{1})^{2N+2}, \frac{1}{(2N(2N_{1})!)}^{2N+2}, \frac{1}{(2N(2N_{1})!)}^{2N+2},$$

as was also conjectured in [179]. Recall that l is the exact wavenumber and  $\tilde{l}$  reflects the numerical wavenumber (recall  $\ell h = 12$ ). For the nondissipative central flux (i.e.,  $\alpha = \rho_N^{-1} \pm h \frac{\exp(ilh) - \exp(ilh)}{\exp(ilh)}$ ,

 $\rho_N \simeq \left\{ \begin{array}{l} 2N + \tilde{l}h \\ lh - o(lh)^{1/3} < 2N + 1 \end{array} \begin{array}{l} \frac{lh}{2} \left[ \frac{N!}{2N + 1} \right]^2 \left[ \frac{N!}{2N + 1} \right]^2 \text{fo-c-lh}h \\ \frac{lh}{2N + 1} \left[ \frac{N!}{2N + 1} \right]^2 \\ \frac{N!}{2N + 1} \left[ \frac{N!}{2N + 1} \right]^2 \\ \frac{N!}{2N + 1} \left[ \frac{N!}{2N + 1} \right]^2 \\ \frac{N!}{2N + 1} \left[ \frac{N!}{2N + 1} \right]^2 \\ \frac{N!}{2N + 1} \left[ \frac{N!}{2N + 1} \right]^2 \\ \frac{N!}{2N + 1} \left[ \frac{N!}{2N + 1} \right]^2 \\ \frac{N!}{2N + 1} \left[ \frac{N!}{2N + 1} \right]^2 \\ \frac{N!}{2N + 1} \left[ \frac{N!}{2N + 1} \right]^2 \\ \frac{N!}{2N + 1} \left[ \frac{N!}{2N + 1} \right]^2 \\ \frac{N!}{2N + 1} \left[ \frac{N!}{2N + 1} \right]^2 \left[ \frac{N!}{2N + 1} \right]^2$ illustrating an order reduction for odd values of N. Note that, in all cases, the coefficient in front of the (lh) decreases rapidly with N, emphasizing the Convergence afor values of  $N^h$ :  $= 2\pi p^{-1}$ ; to understand  $p^p \ge \pi^{\pi}$  why one very Tuesday, August 18, 2009  $\mathcal{O}(h^{N+1})$  error in the computational experiments even if the

## Discrete stability

So far we have not done anything to discretize time.

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad \frac{d \boldsymbol{u}_h}{dt} + \mathcal{L}_h \boldsymbol{u}_h = 0.$$

We shall consider the use of ERK methods

$$\begin{aligned} \boldsymbol{k}^{(1)} &= \mathcal{L}_{h} \left( \boldsymbol{u}_{h}^{n}, t^{n} \right), \\ \boldsymbol{k}^{(2)} &= \mathcal{L}_{h} \left( \boldsymbol{u}_{h}^{n} + \frac{1}{2} \Delta t \boldsymbol{k}^{(1)}, t^{n} + \frac{1}{2} \Delta t \right), \\ \boldsymbol{k}^{(3)} &= \mathcal{L}_{h} \left( \boldsymbol{u}_{h}^{n} + \frac{1}{2} \Delta t \boldsymbol{k}^{(2)}, t^{n} + \frac{1}{2} \Delta t \right), \\ \boldsymbol{k}^{(4)} &= \mathcal{L}_{h} \left( \boldsymbol{u}_{h}^{n} + \Delta t \boldsymbol{k}^{(3)}, t^{n} + \Delta t \right), \\ \boldsymbol{u}_{h}^{n+1} &= \boldsymbol{u}_{h}^{n} + \frac{1}{6} \Delta t \left( \boldsymbol{k}^{(1)} + 2 \boldsymbol{k}^{(2)} + 2 \boldsymbol{k}^{(3)} + \boldsymbol{k}^{(4)} \right), \end{aligned}$$

Discrete stability ow-storage version [46] of the fourth-order m

and alsopa<sup>(0)</sup>L=w<sup>n</sup>Storage form

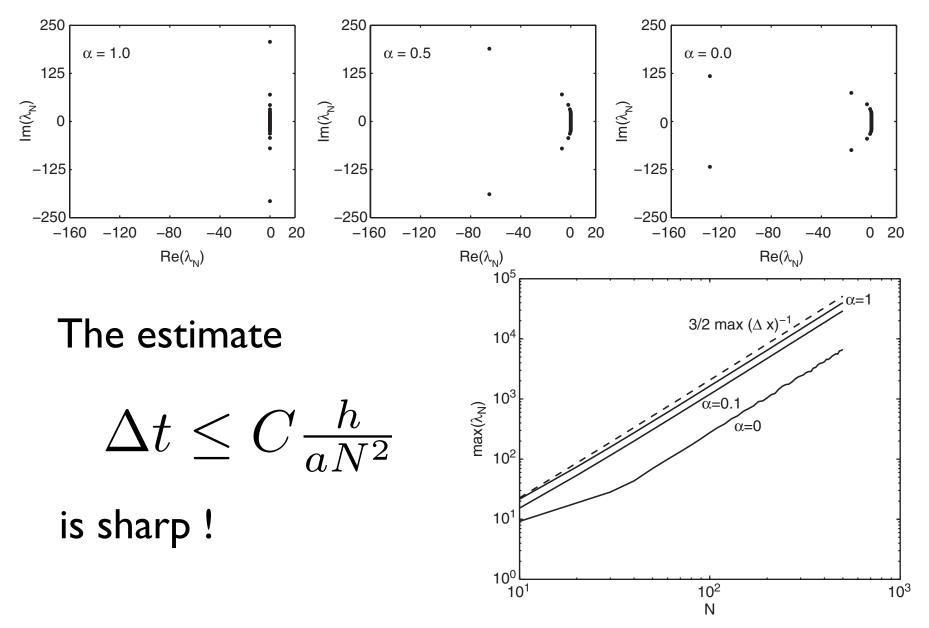
$$\begin{aligned} \boldsymbol{p}_{i}^{(0)} \in [\boldsymbol{\mathcal{U}}_{i}^{n}, \dots, 5] : & \left\{ \begin{array}{l} \boldsymbol{k}^{(i)} = a_{i}\boldsymbol{k}^{(i-1)} + \Delta t\mathcal{L}_{h}\left(\boldsymbol{p}^{(i-1)}, t^{n} + c_{i}\Delta t\right), \\ i \in [1, \dots, 5] : & \left\{ \begin{array}{l} \boldsymbol{k}^{(i)} \neq a_{i}\vec{k}^{(i)} \neq b_{i}\vec{k}^{(i)} + \Delta t\mathcal{L}_{h}^{(i)}\left(\boldsymbol{p}^{(i-1)}, t^{n} + c_{i}\Delta t\right), \\ \boldsymbol{u}_{h}^{n+1} = \boldsymbol{p}^{(5)}, \begin{array}{l} \boldsymbol{p}^{(4)} = \boldsymbol{p}^{(4-\text{Insight} \text{through theory}} \end{array} \right. \end{aligned}$$

The coefficient  $p^{n+1}$  needed in the LSER K are given in Table 3.2. The main ence here is that only one additional storage level is required, thus red Consider the memory usage significantly. On the other hand, this comes at the prean additional function evaluation, as the low-storage storage makes the low-st  $\mathcal{U}_{t} = \mathcal{X}_{u}$ , it would seem that the additional stage makes the low-st approach less interesting due to the additional stage makes the stable stable will ded cost. However, as we stable to the stable of the in the set of the storage RK is off allowing a larger stable timestep,  $\overline{\Delta}_{t}^{2}$ . -1 It should be emphasized that these methods are not exclusive and alternælves exist [40, Y43, 144]. In particular, for strongly nonlinear prok the added nonlinear stability of strong stability-preserving methods m advantageous, as we will discuss further4in Chapter 5. -3 Tuesday, August 18, 2009

RKJ CARCE CHARTER DE CARTER OF CONTRACTOR CONTRA The twing the materstand the scaling of s The we period the figure, "It is worthwing to understand the scaling of Control of the scaling h use have on set the well-known inequality |280|Consider ve have used the well-kandyra inequality  $[280] - \mathcal{E}]$ , where  $\mathcal{E}_{wisera}$   $\mathcal{E}_{ere}$  an atrix a with unity entries in the two diagonal corners. Where  $\mathcal{E}_{where}$  with unity entries in the two diagonal corners. he inverse  $h^2$   $h^2$   $\mathcal{L}_h \|_2^2 = \frac{h^2}{4g^2} \sup_{h \in \mathcal{H}_h} \|\mathcal{L}_h u_h\|_1^2$  $\frac{h^2}{4a^2} \frac{1}{4a^2} \frac{h^2}{4a^2} \frac{h^$  $\mathbb{R}^3 < CN^4,$ ould expect  $esults in a scaling as <math>p_{u} + (2)$   $p_{u} + (2)$ sults in a scaling and the inverse trace in equality [320]  $au_h^k$  $\sqrt{2}$  when equality [280] nd the inverse trace inequality 320 h and the inverse trace inequality 320 $\sqrt{2}\hbar k h$ and the inverse trace interplatity [320] This results in a scaling as  $\sqrt{2} \leq \frac{N+1}{\sqrt{2}} \|u_h\|_{N+1} \leq C - u_h$ ust 18, 2009 Tuesday, August 18, 2009

## Discrete stability

#### The structure also matters



The previously interested to the spacing of the considering  $(\Delta, r)$  interesting the standard of the standard toolsmant Distance State hit may main band and same as a section of the state of the section of where the property of the formula of the second consider  $\partial u^{\partial x} = 0$ , the second consider  $\partial u^{\partial x} = 0$ , the second consider  $\partial u^{\partial x} = 0$ , extension to more general canonia straightforward the ward of a billing ward of a billing ward the ward of a billing ward of a billing ward the ward of a billing ward of a billing ward the ward of a billing ward of a billing ward the ward of a billing ward the ward of a billing ward the ward of a billing ward of a billing ward the ward of a billing ward the ward of a b nd, also known as the Courant Fredrichs-Levy (CFInin  $\frac{\partial x h^k}{\partial h^k}$ ), sonable bound, also where sense leothant three dastes Lever (CFInin  $\frac{\partial x h^k}{\partial h^k}$ ), that is, we obtain the sense of a system of a system of a system. heral case of a system OxFor the general case of a system th elements of different lengths  $h^k$ , we have have known as the Courant-Friedrichs-Le heral grid with Generation Ou, Ou, Ou, Ou, Ou, Ou,  $At = At \leq Goro of a system <math>\min -(\Delta_i r)$ ,  $\begin{array}{c} \partial u & \partial u & \partial u & \partial u \\ \hline \partial t & \partial t & \hline \partial t & \hline \partial t & \partial t & \partial t \\ \Delta t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Delta t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t \\ \hline \Phi t & \Delta t & \partial t \\ \hline \Phi t & \Delta t & \partial t \\ \hline \Phi t & \Delta t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \Delta t & \partial t & \partial t & \partial t & \partial t \\ \hline \Phi t & \partial t \\ \hline \Phi t & \partial t \\ \hline \Phi t & \partial t \\ \hline \Phi t & \partial t \\ \hline \Phi t & \partial t \\ \hline \Phi t & \partial t \\ \hline \Phi t & \partial t \\ \hline \Phi t & \partial t \\ \hline \Phi t & \partial t \\ \hline \Phi t & \partial t \\ \hline \Phi t & \partial t \\ \hline \Phi t & \partial t$ sonable bonere are tricks to play to improve to play the state of this system. nd, also known as the Graphant hit error the states are stable; that is, we obtain the general sector of a state of the scaling  $\begin{array}{c} \begin{array}{c} \text{max} \\ \text{max} \\ \mathcal{A} \end{array} \text{ represent the velocities} \\ \text{the values of a set spectrum of the velocities} \\ \text{the values of } \begin{array}{c} \begin{array}{c} 1 \\ \frac{k}{2} (\Delta_i r), \\ \frac{k}{2} (\Delta_i$ the fastest craves is repeted in the faise stee of the paratometer of the second Sectext for a discussion of other methods mials, N, is increased in the CFL condition of the CFL condition of the CFL condition of the CFL condition of the condition of the condition of the condition of the created of the condition of the created of the condition of the created of the creat

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## Local time-stepping

**Problem**: Small cells, even just one, cause a very small global time-step in an explicit scheme.

A significant problem for large scale complex applications

Old idea: take only time-steps required by local restrictions.

Old problems: accuracy and stability



 $\Delta t \le C\sqrt{\varepsilon\mu}\Delta x \simeq C_1\sqrt{\varepsilon\mu}\frac{N^2}{h}$ 







Substantial recent work by Cohen, Grote, Lanteri, Piperno, Gassner, Munz etc

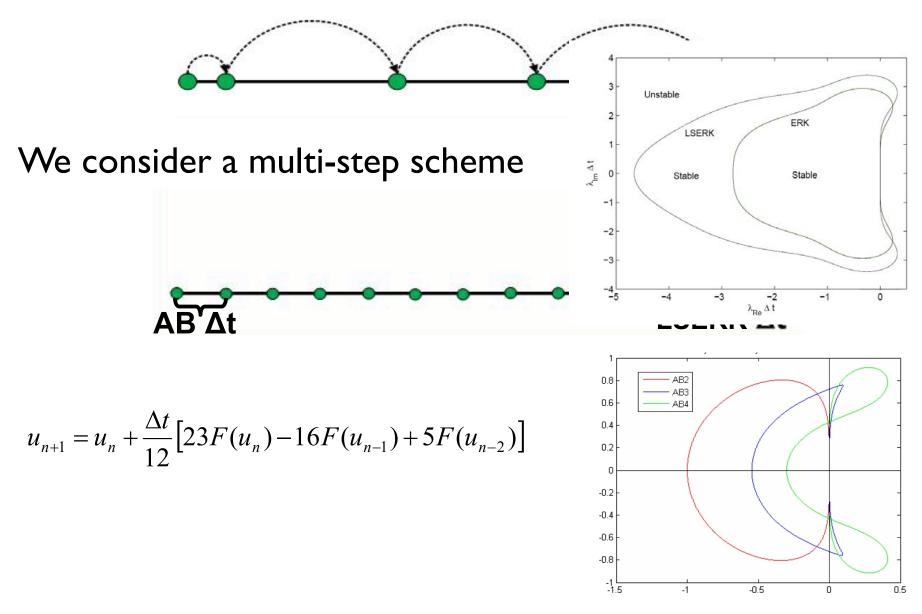
Most of the recent work is based on LF-like schemes, restricted to 2nd order in time.

Layout for **multi-rate** local time-stepping





#### Recall the ERK scheme

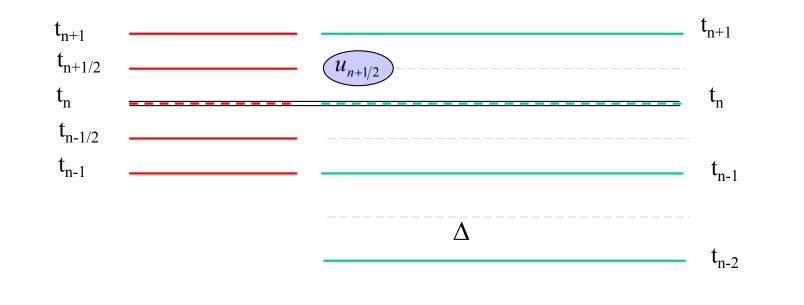


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Δ



#### Challenge: Achieving this at high-order accuracy



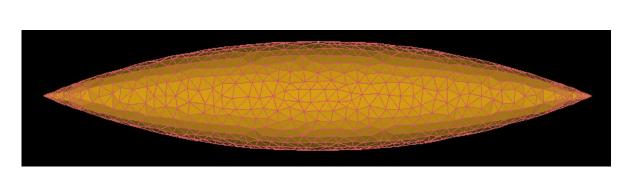
For all interior cells 
$$u_{n+1} = u_n + \frac{\Delta t}{12} [23F(u_n) - 16F(u_{n-1}) + 5F(u_{n-2})]$$

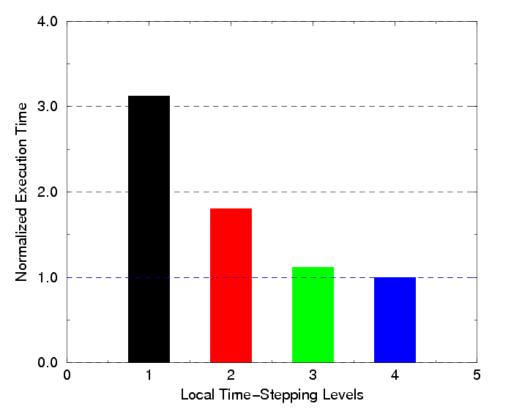
At interface cells  $u_{n+1/2} = u_n + \frac{\Delta t}{12} [17F(u_n) - 7F(u_{n-1}) + 2F(u_{n-2})]$ 

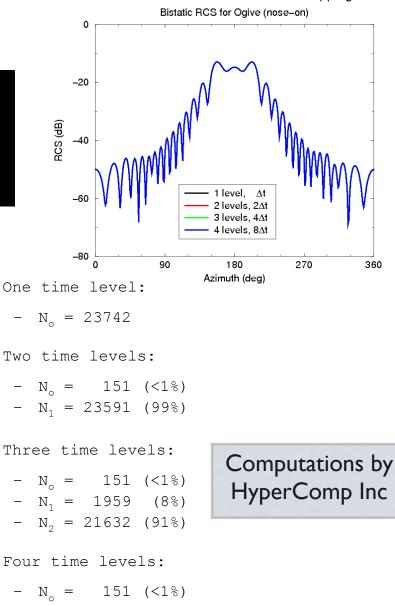
This generalizes to many levels and arbitrary time-step fractions



Four Time-Level Local Time-Stepping







 $N_1 = 1959$  (8%) —  $N_2 = 12622$  (53%) —

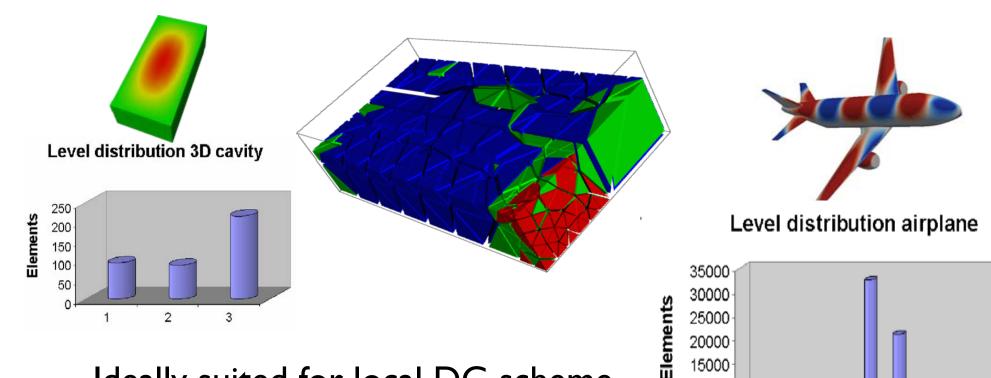
—

—

 $-N_3 = 9010 (38\%)$ 



#### Segmentation is done in preprocessing



10000 5000

2

3

5

6

7

8

9

4

Ideally suited for <u>local</u> DG scheme

<u>Known problems</u>: No known stability proof Time-step is not optimal (about 80%)



The potential speed up is considerable -- and the more complex the better !

Example	Simulation time with		
	Adams-Bashford	Adams-Bashford	LSERK
	(global time step)	(local time step)	(global time step)
Resonator	100%	59%	45%
3dB-Coupler	100%	29%	45%
Airplane	100%	15%	45%

#### Computations by Nico Godel, Hamburg

# A brief summary

We now have a good understanding all key aspects of the DG-FEM scheme for linear first order problems

- We understand both accuracy and stability and what we can expect.
- The dispersive properties are excellent.
- The discrete stability is a little less encouraging. A scaling like

$$\Delta t \le C \frac{h}{aN^2}$$

is the Achilles Heel -- but there are ways!

... but what about nonlinear problems ?