Course 02402 Introduction to Statistics Lecture 11:

Twoway Analysis of Variance, ANOVA

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Agenda

1. Intro: Small example and TV-data from B&O
2. Model
3. Computation - decomposition and the ANOVA table
4. Hypothesis test (F-test)
5. Post hoc analysis
6. Model control
7. A complete example - from the book

Oversigt

TV set development at Bang & Olufsen

Sound and image quality is measured by th human perceptual instrument:

We developed a tool that is used by B&O to ANOVA (among other things) PanelCheck (*Show Panelcheck programme with TV data*)

Bang & Olufsen data in R:

```
##  # Getting the Bang and Olufsen data from the lmerTest-package:
library(lmerTest) # (Developed by us)
data(TVbo)

# Each of 8 assessors scored each of 12 combinations 2 times
# Let's look at only a single picture and one of the two reps:
# And let us look at the sharpness
TVbosubset <- subset(TVbo,Picture==1 & Repeat==1)[,c(1, 2, 9)]

sharp <- matrix(TVbosubset$Sharpness, nrow=8, byrow=T)
colnames(sharp) <- c("TV3", "TV2", "TV1")
rownames(sharp) <- c("Person 1", "Person 2", "Person 3",
                     "Person 4", "Person 5", "Person 6",
                     "Person 7", "Person 8")

library(xtable)
xtable(sharp)
```

Bang & Olufsen data in R:

	TV3	TV2	TV1
Person 1	9.30	4.70	6.60
Person 2	10.20	7.00	8.80
Person 3	11.50	9.50	8.00
Person 4	11.90	6.60	8.20
Person 5	10.70	4.20	5.40
Person 6	10.90	9.10	7.10
Person 7	8.50	5.00	6.30
Person 8	12.60	8.90	10.70

Twoway ANOVA - example

- Same data as for oneway, but now we know that the experiment was split in blocks

	Group A	Group B	Group C
Block 1	2.8	5.5	5.8
Block 2	3.6	6.3	8.3
Block 3	3.4	6.1	6.9
Block 4	2.3	5.7	6.1

  - hence three *Groups* on four *blocks*
  - or three *treatments* on four *persons*
  - or three *varieties* on four *fields* (hence blocks)
  - or similarly

- *oneway* vs. *twoway* ANOVA
- *Completely randomized design* vs. *Randomized block design*

Twoway ANOVA - example

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Block 4	2.3	5.7	6.1

- Question: Is there a significant difference (in means) between the groups A, B and C?

- ANOVA can be used if the observations in each group are (approximately) normally distributed - OR if n_is are large enough (CLT)

The toy data in R

```
###################################
## Input data and plot

## Observations
y <- c(2.8, 3.6, 3.4, 2.3,
       5.5, 6.3, 6.1, 5.7,
       5.8, 8.3, 6.9, 6.1)

## treatments (Groups, varieties)
treatm <- factor(c(1, 1, 1, 1,
                   2, 2, 2, 2,
                   3, 3, 3, 3))

## blocks (persons, fields)
block <- factor(c(1, 2, 3, 4,
                  1, 2, 3, 4,
                  1, 2, 3, 4))

## for later formulas
(k <- length(unique(treatm)))
(l <- length(unique(block)))

## Plots
par(mfrow=c(1,2))

## Plot histogramms by treatments
plot(treatm, y, xlab="Treatments", ylab="y")
## Plot histogrammer by  blocks
plot(block, y, xlab="Blocks", ylab="y")
```

Oversigt

Twoway ANOVA, model

- Express a model

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2)$$

where the deviations

$$\varepsilon_{ij} \sim N(0, \sigma^2) \text{ and i.i.d.}$$

  - μ is the overall mean
  - α_i is the effect of treatment i
  - β_j is the level for Block i
  - there are k treatments and l blocks
  - j indicates the observations in the groups, from 1 to n_i for treatment i

Estimates of parameters in the model

- We can compute the estimates of the parameters ($\hat{\mu}$ and $\hat{\alpha}_i$, and $\hat{\beta}_j$)

$$\hat{\mu} = \bar{y} = \frac{1}{k \cdot l} \sum_{i=1}^{k} \sum_{j=1}^{l} y_{ij}$$

$$\hat{\alpha}_i = \left(\frac{1}{l} \sum_{j=1}^{l} y_{ij} \right) - \hat{\mu}$$

$$\hat{\beta}_j = \left(\frac{1}{k} \sum_{i=1}^{k} y_{ij} \right) - \hat{\mu}$$

```
####################################
## Compute estimates of parameters in the model

## Sample mean
(muHat <- mean(y))
## Sample mean for each treatment
(alphaHat <- tapply(y, treatm, mean) - muHat)
## Sample mean for each  Block
(betaHat <- tapply(y, block, mean) - muHat)
```

Oversigt

Twoway ANOVA, decomposition and the ANOVA table, Theorem 8.20

- With the model

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2)$$

- the total variation in the data can be decomposed:

$$SST = SS(Tr) + SS(Bl) + SSE$$

- 'twoway' refers to the fact that there are two factors in the experiment(Two "ways" of the data table)

- The method is called analysis of variance, because the testing is carried out by comparing certain variances.

Formulas for sums of squares

- Total sum of squares ("the total variance") (same as for oneway)

$$SST = \sum_{i=1}^{k} \sum_{j=1}^{l} (y_{ij} - \hat{\mu})^2$$

- treatment sum of squares ("Variance explained by the treatment part of the model")

$$SS(Tr) = l \cdot \sum_{i=1}^{k} \hat{\alpha}_i^2$$

Formulas for sums of squares

- Sum of squares for blocks (persons) ("Variance explained by the block part of the model")

$$SS(Bl) = k \cdot \sum_{j=1}^{l} \hat{\beta}_j^2$$

- The sum of squares for the residuals ("residual variance after model fit")

$$SSE = \sum_{i=1}^{k} \sum_{j=1}^{l} (y_{ij} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\mu})^2$$

Oversigt

Twoway ANOVA: hypothesis of no effect of treatment, Theorem 8.22

- We want to compare (more than 2) means $\mu + \alpha_i$ in the model

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2)$$

- So we can express the hypothesis:

$$\begin{aligned} H_{0,Tr}: &\quad \alpha_i = 0 \quad \text{for all } i \\ H_{1,Tr}: &\quad \alpha_i \neq 0 \quad \text{for at least one } i \end{aligned}$$

- Under $H_{0,Tr}$ the following is true:

$$F_{Tr} = \frac{SS(Tr)/(k-1)}{SSE/((k-1)(l-1))}$$

is F-distributed with $k-1$ and $(k-1)(l-1)$ degrees of freedom

Twoway ANOVA: hypothesis of no effect of persons (blocks), Theorem 8.22

- We want to compare (more than 2) means $\mu + \beta_i$ in the model

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2)$$

- So we can express the hypothesis

$$H_{0,Bl}: \quad \beta_i = 0 \quad \text{for all } i$$
$$H_{1,Bl}: \quad \beta_i \neq 0 \quad \text{for at least one } i$$

- Under $H_{0,Bl}$ the following is true:

$$F_{Bl} = \frac{SS(Bl)/(l-1)}{SSE/((k-1)(l-1))}$$

follows an F-distribution with $l-1$ and $(k-1)(l-1)$ degrees of freedom

F-distribution and treatments hypothesis

```
################################
## Plot the F distribution and see the critical value for treatments

## Remember, this is "under H0" (that is we compute as if H0 is true):
## Sequence for plot
xseq <- seq(0, 10, by=0.1)
## Plot the density of the  F distribution
plot(xseq, df(xseq, df1=k-1, df2=(k-1)*(l-1)), type="l")
##The critical value for significance level 5 %
cr <- qf(0.95, df1=k-1, df2=(k-1)*(l-1))
## Mark it in the plot
abline(v=cr, col="red")
## The value of the  test statistic
  (Ftr <- (SSTr/(k-1)) / (SSE/((k-1)*(l-1))))
## The p-value hence is:
(1 - pf(Ftr, df1=k-1, df2=(k-1)*(l-1)))
```

F-distribution and blocks hypothesis

```
################################
## Plot the F distribution  and see the critical value

## Remember, this is "under H0" (that is we compute as if H0 is true):
## Sequence for plot
xseq <- seq(0, 10, by=0.1)
## Plot the density of the  F distribution
plot(xseq, df(xseq, df1=l-1, df2=(k-1)*(l-1)), type="l")
##The critical value for significance level 5 %
cr <- qf(0.95, df1=l-1, df2=(k-1)*(l-1))
## Mark it in the plot
abline(v=cr, col="red")
## The value of the  test statistic
(Fbl <- (SSBl/(l-1)) / (SSE/((k-1)*(l-1))))
## The p-value hence is:
(1 - pf(Fbl, df1=l-1, df2=(k-1)*(l-1)))
```

The twoway ANOVA table

Source of variation	Deg. of freedom	Sums of squares	Mean sum of squares	Test-statistic F	p-value
Treatment	$k-1$	$SS(Tr)$	$MS(Tr) = \frac{SS(Tr)}{k-1}$	$F_{\text{Tr}} = \frac{MS(Tr)}{MSE}$	$P(F > F_{\text{Tr}})$
Block	$l-1$	$SS(Bl)$	$MS(Bl) = \frac{SS(Bl)}{l-1}$	$F_{\text{Bl}} = \frac{MS(Bl)}{MSE}$	$P(F > F_{\text{Bl}})$
Residual	$(k-1)(l-1)$	SSE	$MSE = \frac{SSE}{(k-1)(l-1)}$		
Total	$n-1$	SST			

```
anova(lm(y ~ treatm + block))

## Analysis of Variance Table
##
## Response: y
##           Df Sum Sq Mean Sq F value  Pr(>F)
## treatm     2  30.79   15.40   74.40 5.8e-05 ***
## block      3   3.95    1.32    6.37   0.027 *
## Residuals  6   1.24    0.21
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Oversigt

Post hoc confidence interval

- As for oneway (Use methods 8.9 and 8.10) substitute $(n-k)$ degrees of freedom with $(k-1)(l-1)$ (and use MSE from twoway).
- Can be done with either treatments or blocks
- A single pre-planned confidence interval for the difference between treatment i and j is found as:

$$\bar{y}_i - \bar{y}_j \pm t_{1-\alpha/2}\sqrt{\frac{SSE}{n-k}\left(\frac{1}{n_i}+\frac{1}{n_j}\right)} \tag{1}$$

  where $t_{1-\alpha/2}$ is based on the t-distribution with $(k-1)(l-1)$ degrees of freedom.

- If all $M = k(k-1)/2$ combinations of pairwise confidence intervals are found use the formula M times but each time with $\alpha_{\text{Bonferroni}} = \alpha/M$.

Post hoc pairwise hypothesis test

- A single pre-planned level α hypothesis tests:

$$H_0: \ \mu_i = \mu_j, \ \ H_1: \ \mu_i \neq \mu_j$$

is carried out as:

$$t_{\text{obs}} = \frac{\bar{y}_i - \bar{y}_j}{\sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}} \tag{2}$$

and:

$$p - \text{value} = 2P(t > |t_{\text{obs}}|)$$

where the t-distribution with $(k-1)(l-1)$ degrees of freedom is used.

- If all $M = k(k-1)/2$ combinations of pairwise confidence intervals are found use the formula M times but each time with $\alpha_{\text{Bonferroni}} = \alpha/M$.

Oversigt

Variance homogeneity

Look at box-plot to check whether the variability seems different for the groups

```
################################
## Check assumption of  homogeneous variance



## Save the  fit
fit <- lm(y ~ treatm + block)
## Box plot
par(mfrow=c(1,2))
plot(treatm, fit$residuals, y, xlab="Treatment")
## Box plot
plot(block, fit$residuals, xlab="Block")
```

Normal assumption

Look at qq-normal plot

```
###################################
## Check the assumption  of  normality of residuals

## qq-normal plot of residuals
qqnorm(fit$residuals)
qqline(fit$residuals)

## Or with a  Wally plot
require(MESS)
qqwrap <- function(x, y, ...) {qqnorm(y, main="",...);
  qqline(y)}
## Can we see a deviating qq-norm plot?
wallyplot(fit$residuals, FUN = qqwrap)
```

Oversigt

A complete example - from the book

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Example 8.26 Car tires

In a study of 3 different types of tires (“treatment”) effect on the fuel economy, drives of 1000 km in 4 different cars ("blocks") were carried out. The results are listed in the following table in km/l.

	Car 1	Car 2	Car 3	Car 4	*Mean*
Tire 1	22.5	24.3	24.9	22.4	22.525
Tire 2	21.5	21.3	23.9	18.4	21.275
Tire 3	22.2	21.9	21.7	17.9	20.925
Mean	21.400	22.167	23.167	19.567	21.575

Let us analyse these data with a two-way ANOVA model, but first some explorative plotting:

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