Course 02402 Introduction to Statistics

Lecture 4: Confidence intervals

DTU Compute
Technical University of Denmark
2800 Lyngby – Denmark

Overview

Overview

Example: Heights

Sample, $n = 10$:

168	161	167	179	184	166	198	187	191	179

Sample mean and standard deviation:

$$\bar{x} = 178$$
$$s = 12.21$$

Estimate population mean and standard deviation:

$$\hat{\mu} = 178$$
$$\hat{\sigma} = 12.21$$

NEW: **Confidence interval for μ**:

$$178 \pm 2.26 \cdot \frac{12.21}{\sqrt{10}} = [169.3;\ 186.7]$$

NEW: **Confidence interval for σ**:

$$[8.4;\ 22.3]$$

Overview

(Empirical) distribution of sample mean

```r
# 'True' mean and standard deviation
mu <- 178
sigma <- 12

# Sample size
n <- 10

# Simulate normal distributed X_i for n = 10
x <- rnorm(n = n, mean = mu, sd = sigma)
x

# Empirical density
hist(x, prob = TRUE, col = 'blue')
# Compute sample mean
mean(x)

# Repeat the simulated sampling many times (100 samples)
mat <- replicate(100, rnorm(n = n, mean = mu, sd = sigma))

# Compute the sample mean for each sample
xbar <- apply(mat, 2, mean)
xbar

# See the distribution of the sample means
hist(xbar, prob = TRUE, col = 'blue')
# Empirical mean and variance of sample means
mean(xbar)
var(xbar)
```


Theorem 3.3: Distribution of the sample mean of i.i.d. normal random variables

The distribution of $\bar{X}$

Assume that $X_1,\dots,X_n$ are independent and identically distributed (*i.i.d.*) normal random variables, $X_i \sim N(\mu,\sigma^2), i = 1,\dots,n$, then:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Mean, variance and 'normality' follow from 'rules':

The mean of $\bar{X}$ (Theorem 2.56):

$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}n\mu = \mu$$

The variance of $\bar{X}$ (Theorem 2.56):

$$\text{Var}(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n}$$

The 'normality' of $\bar{X}$ (Theorem 2.40):

By this theorem, the distribution of $\bar{X}$ is a normal distribution with mean μ and variance σ^2/n as specified above.

Distribution of the error $\bar{X} - \mu$

The standard deviation of $\bar{X}$:

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

The standard deviation of $\bar{X} - \mu$:

$$\sigma_{(\bar{X}-\mu)} = \frac{\sigma}{\sqrt{n}}$$

Standardized version of the above, Theorem 3.4

Distribution of the standardized sample mean (or standardized error):

Assume that $X_1, \ldots, X_n$ are i.i.d. normal random variables, $X_i \sim N\left(\mu, \sigma^2\right)$ for $i = 1, \ldots, n$, then:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N\left(0, 1^2\right)$$

That is, the standardized sample mean Z follows a standard normal distribution.

Practical problem (and solution)

How do we use the results from the previous slides to say something about μ ...

... when the 'true', unknown, population standard deviation σ enters into all the formulas?

Obvious solution:

Use the estimate s instead of σ in formulas.

BUT:

Then, we need new theory! (There is also uncertainty linked to s.)

Theorem 3.5, a more applicable extension of the above

The t-distribution takes the uncertainty of s into account:

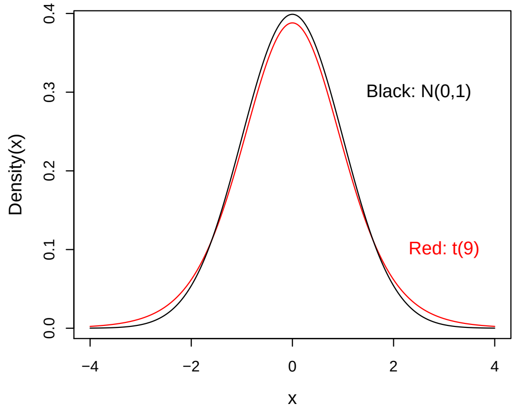
Assume that $X_1, \ldots, X_n$ are i.i.d. normal distributed random variables, where $X_i \sim N\left(\mu, \sigma^2\right)$ for $i = 1, \ldots, n$, then:

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1)$$

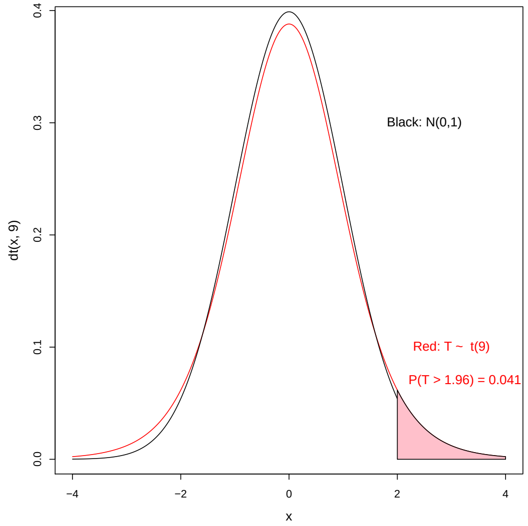
where $t(n-1)$ is the t-distribution with $n-1$ degrees of freedom.

The t-distribution with 9 degrees of freedom $(n = 10)$

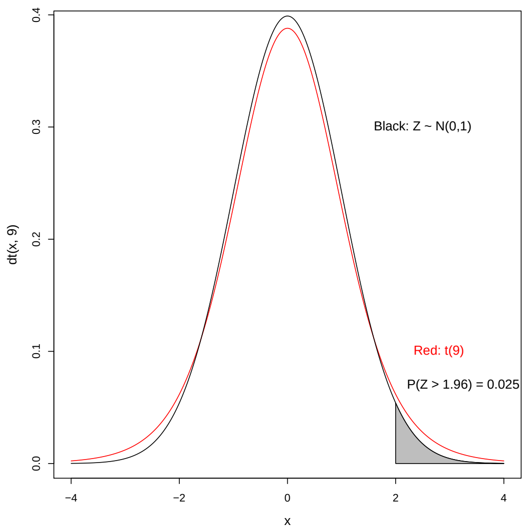
```
x <- seq(-4, 4, by = 0.01)
plot(x, dt(x, df = 9), type = "l", col = "red", ylab = "Density(x)")
lines(x, dnorm(x), type = "l")
text(2.5, 0.3,"Black: N(0,1)")
text(3, 0.1,"Red: t(9)", col = "red")
```



The t-distribution with 9 degrees of freedom and standard normal distribution



The t-distribution with 9 degrees of freedom and standard normal distribution



Overview

Method 3.9: One-sample Confidence Interval (CI) for μ

Use the correct t-distribution to construct the confidence interval:

For a sample $x_1,\ldots,x_n$ the $100(1-\alpha)\%$ confidence interval is given by:

$$\bar{x} \pm t_{1-\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

where $t_{1-\alpha/2}$ is the $100(1-\alpha/2)\%$ quantile from the t-distribution with $n-1$ degrees of freedom.

Most commonly using $\alpha = 0.05$:

The most commonly used is the 95% confidence interval:

$$\bar{x} \pm t_{0.975} \cdot \frac{s}{\sqrt{n}}$$

Example: Heights, 95% CI

```r
# 0.975 quantile for the t(9) distribution (n = 10):
qt(0.975, df = 9)
```

Gives the result $t_{0.975} = 2.26$.

Now, we can recognize the already given result

$$178 \pm 2.26 \cdot \frac{12.21}{\sqrt{10}}$$

which is

$$178 \pm 8.74 = [169.3,\ 186.7].$$

Example: Heights, 99% CI

```r
# 0.995 quantile for the t(9) distribution (n = 10):
qt(0.995, df = 9)
```

Gives the result $t_{0.995} = 3.25$.

In this case,

$$178 \pm 3.25 \cdot \frac{12.21}{\sqrt{10}}$$

giving

$$178 \pm 12.55 = [165.5;\ 190.5]$$

An R function for computing these CI (and more):

```r
# Data
x <- c(168, 161, 167, 179, 184, 166, 198, 187, 191, 179)

# 99% CI for mu
t.test(x, conf.level = 0.99)

##
##  One Sample t-test
##
## data:  x
## t = 46, df = 9, p-value = 5e-12
## alternative hypothesis: true mean is not equal to 0
## 99 percent confidence interval:
##  165.5 190.5
## sample estimates:
## mean of x
##       178
```

Overview

The formal framework for *statistical inference*

From eNote, Chapter 1:

- An *observational unit* is the single entity/level at which information is sought (e.g. a person). (**Observationsenhed**)
- The *statistical population* consists of all possible "measurements" on each possible *observational unit*. (**Population**)
- The *sample* from a statistical population is the actual set of data collected. (**Stikprøve**)

Language and concepts:

- μ and σ are parameters describing the population.
- $\bar{x}$ is the *estimate* of μ (specific realization).
- $\bar{X}$ is the *estimator* of μ (now seen as a random variable).
- The word *'statistic(s)'* is used for both.

The formal framework for *statistical inference* - Example

From eNote, Chapter 1. Example: Heights

We measured the heights of 10 randomly selected students.

The sample:

The 10 specific numbers (heights): $x_1, \ldots, x_{10}$.

The population:

The heights for all people in Denmark.

Observational unit:

A person.

Statistical inference = Learning from data

Learning from data:

Learning about parameters of distributions that describe populations.

Important:

The sample must, in a meaningful way, represent some well defined population.

How to ensure this:

For example, by making sure that the sample is taken completely at random.

Random Sampling

Definition 3.12:

- A random sample from an (infinite) population: A set of observations $X_1, X_2, ..., X_n$ constitutes a random sample of size n from the infinite population $f(x)$ if:
  1. Each X_i is a random variable whose distribution is given by $f(x)$.
  2. These n random variables are independent.

What does that mean?

1. All observations must come from the same population.
2. They cannot share any information with each other (e.g., shouldn't be related).

Overview

Theorem 3.14: The Central Limit Theorem (CLT)

"No matter the distribution of X_i", the distribution of the mean of i.i.d. random variables approaches a normal distribution:

Let $\bar{X}$ be the mean of a random sample of size n taken from a population with mean μ and variance σ^2. Then

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

is a random variable whose distribution function approaches that of the standard normal distribution, $N(0, 1^2)$, as $n \to \infty$.

Hence, if n is large enough, we can assume (approximately) that:

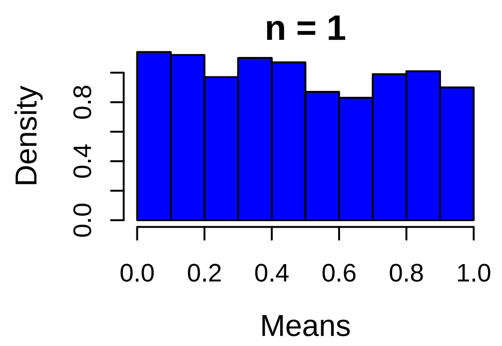
$$\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1^2)$$

CLT example: Mean of uniformly distributed observations

```r
n <- 1 # Sample size
k <- 1000 # No. of samples (i.e. no. of means to be computed)

# Simulations from U(0,1)-distribution (k = 1000 samples, each of size n = 1)
u <- matrix(runif(k*n), ncol = n)

# Empirical density of means
hist(apply(u, 1, mean), col = "blue", main = "n = 1", xlab = "Means", prob = TRUE)
```

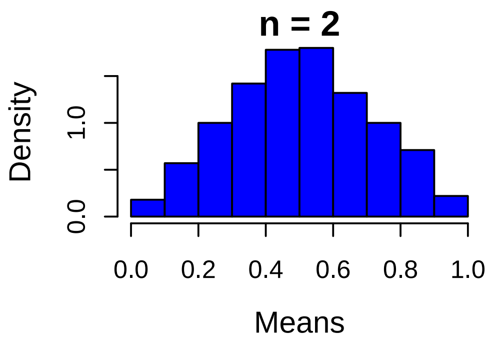


CLT example: Mean of uniformly distributed observations

```r
n <- 2 # Sample size
k <- 1000 # No. of samples (i.e. no. of means to be computed)

# Simulations from U(0,1)-distribution (k = 1000 samples, each of size n = 2)
u <- matrix(runif(k*n), ncol = n)

# Empirical density of means
hist(apply(u, 1, mean), col = "blue", main = "n = 2", xlab = "Means", xlim = c(0,1), prob = TRUE)
```

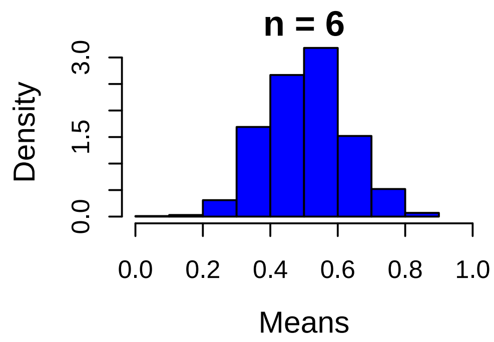


CLT example: Mean of uniformly distributed observations

```r
n <- 6 # Sample size
k <- 1000 # No. of samples (i.e. no. of means to be computed)

# Simulations from U(0,1)-distribution (k = 1000 samples, each of size n = 6)
u <- matrix(runif(k*n), ncol = n)

# Empirical density of means
hist(apply(u, 1, mean), col = "blue", main = "n = 6", xlab = "Means", xlim = c(0,1), prob = TRUE)
```

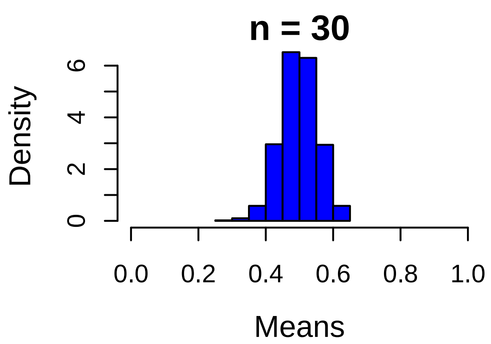


CLT example: Mean of uniformly distributed observations

```r
n <- 30 # Sample size
k <- 1000 # No. of samples (i.e. no. of means to be computed)

# Simulations from U(0,1)-distribution (k = 1000 samples, each of size n = 30)
u <- matrix(runif(k*n), ncol = n)

# Empirical density of means
hist(apply(u, 1, mean), col = "blue", main = "n = 30", xlab = "Means", xlim = c(0,1), prob = TRUE)
```



Consequence of the CLT:

Our CI-method also works for non-normal data:

We can use the confidence-interval based on the t-distribution in basically any situation, as long as n is large enough.

When is n "large enough"?

Actually difficult to say exactly, BUT:

- Rule of thumb: $n \geq 30$
- Even for smaller n the approach can be (almost) valid for non-normal data.

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1. Intro and example
2. Distribution of the sample mean
   - The t-distribution
3. Confidence interval (CI) for μ
   - Example: Heights
4. The language of statistics and the formal framework
5. Non-normal data, the Central Limit Theorem (CLT)
6. **Formal interpretation of the CI**
7. CI for variance σ^2 and standard deviation σ

'Repeated sampling' interpretation

In the long run, we catch the true value in 95% of cases (95% CI):

The confidence interval will vary in both width (s) and position ($\bar{x}$) if the study is repeated.

More formally expressed (Theorem 3.5):

$$P\left(\frac{|\bar{X}-\mu|}{S/\sqrt{n}} < t_{0.975}\right) = 0.95$$

Which is equivalent to:

$$P\left(\bar{X} - t_{0.975}\frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{0.975}\frac{S}{\sqrt{n}}\right) = 0.95$$

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Motivating Example

Production of tablets

In the production of tablets, an active matter is mixed with a powder and then the mixture is formed to tablets. It is important that the mixture is homogenous, so that each tablet has the same strength.

We consider a mixture (of the active matter and powder) from where a large amount of tablets is to be produced.

We seek to produce the mixtures (and the final tablets) so that the mean content of the active matter is 1 mg/g with the smallest variance as possible. A random sample is collected where the amount of active matter is measured. It is assumed that all the measurements follow a normal distribution with the unit mg/g.

The sampling distribution of the variance estimator, Theorem 2.81

Assume i.i.d. normal distributed variables, $X_i \sim N(\mu, \sigma^2)$ for $i = 1, \ldots, n$.

Variance estimators behaves like a χ^2-distribution:

Let

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

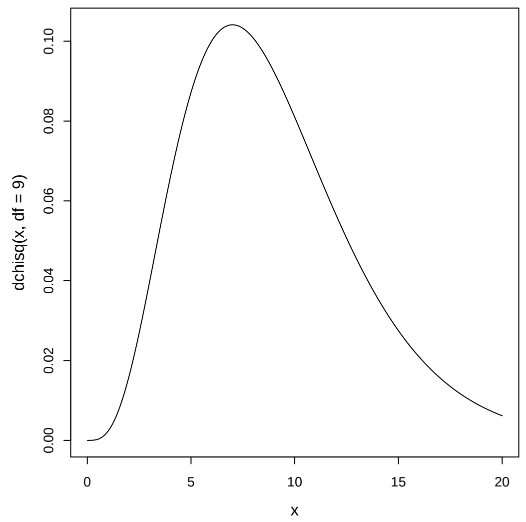
then:

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

is a stochastic variable following the χ^2-distribution with $v = n-1$ degrees of freedom.

χ^2-distribution with $\nu = 9$ degrees of freedom

```r
x <- seq(0, 20, by = 0.1)
plot(x, dchisq(x, df = 9), type = "l")
```



Method 3.19: Confidence interval for the variance and standard deviation

Assume i.i.d. normal distributed variables, $X_i \sim N(\mu, \sigma^2)$ for $i = 1, \ldots, n$.

The variance:

A $100(1-\alpha)\%$ confidence interval for the variance σ^2 is:

$$\left[\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}; \ \frac{(n-1)s^2}{\chi^2_{\alpha/2}} \right]$$

where the quantiles come from a χ^2-distribution with $\nu = n-1$ degrees of freedom.

The standard deviation:

A $100(1-\alpha)\%$ confidence interval for the standard deviation σ is:

$$\left[\sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}; \ \sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}} \right]$$

Example

Data:

A random sample with $n = 20$ tablets is taken and from this we get:

$$\hat{\mu} = \bar{x} = 1.01, \ \hat{\sigma}^2 = s^2 = 0.07^2$$

95% confidence interval for the variance - we need the χ^2-quantiles (19 degrees of freedom):

$$\chi^2_{0.025} = 8.9065, \ \chi^2_{0.975} = 32.8523$$

```r
qchisq(c(0.025, 0.975), df = 19)
```

```
[1] 8.907 32.852
```

Example

So the confidence interval for the variance σ^2 becomes:

$$\left[\frac{19 \cdot 0.07^2}{32.85};\ \frac{19 \cdot 0.07^2}{8.907}\right] = [0.002834;\ 0.01045]$$

and the confidence interval for the standard deviation σ becomes:

$$\left[\sqrt{0.002834};\ \sqrt{0.01045}\right] = [0.053;\ 0.102]$$

Example: Heights

We need the χ^2-quantiles with $\nu = 9$ degrees of freedom:

$$\chi^2_{0.025} = 2.700389,\ \chi^2_{0.975} = 19.022768$$

```
qchisq(c(0.025, 0.975), df = 9)
```

```
[1] 2.70 19.02
```

So the confidence interval for the height standard deviation σ becomes:

$$\left[\sqrt{\frac{9 \cdot 12.21^2}{19.022768}};\ \sqrt{\frac{9 \cdot 12.21^2}{2.700389}}\right] = [8.4;\ 22.3]$$

Example: Heights

Sample, $n = 10$:

168 161 167 179 184 166 198 187 191 179

Sample mean and standard deviation:

$$\bar{x} = 178$$
$$s = 12.21$$

Estimate population mean and standard deviation:

$$\hat{\mu} = 178$$
$$\hat{\sigma} = 12.21$$

NEW: **Confidence interval,** μ:

$$178 \pm 2.26 \cdot \frac{12.21}{\sqrt{10}} = [169.3;\ 186.7]$$

NEW: **Confidence interval,** σ:

$$[8.4;\ 22.3]$$

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