

Course 02402 Introduction to Statistics

Lecture 3: Random variables and continuous distributions

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Overview

- 1 Continuous random variables and distributions
 - Density and distribution functions
 - Mean, variance, and covariance
- 2 Specific continuous distributions
 - The uniform distribution
 - The normal distribution
 - The log-normal distribution
 - The exponential distribution
- 3 Calculation rules for random variables

Overview

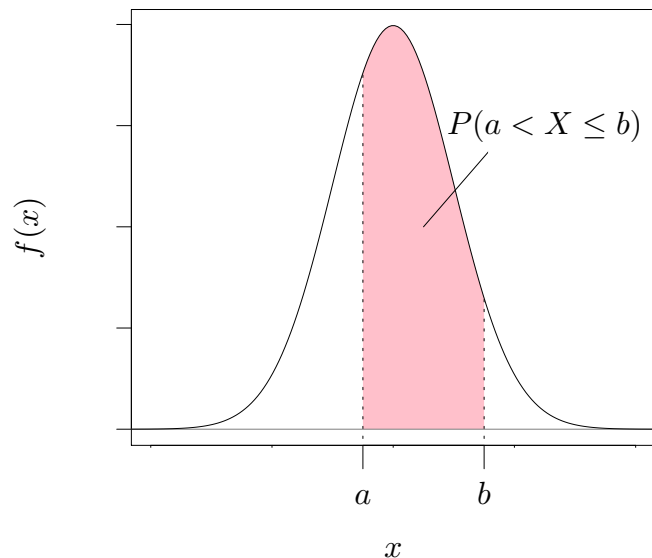
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The density function, Definition 2.32

- The density function (probability density function, pdf) for a random variable is denoted by $f(x)$.
- The density function says something about the frequency of the outcome x for the random variable X .
- The density function for a continuous random variable does *not* correspond directly to a probability. In fact, $f(x) \neq P(X = x)$ and $P(X = x) = 0$ for all x .
- The density function $f(x)$ for the distribution of a continuous random variable satisfies that

$$f(x) \geq 0 \text{ for all } x \text{ and } \int_{-\infty}^{\infty} f(x) dx = 1.$$

The density function



The distribution function, Definition 2.33

- The distribution function (cumulative density function, cdf) for a continuous random variable is denoted by $F(x)$.
- The distribution function is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt.$$

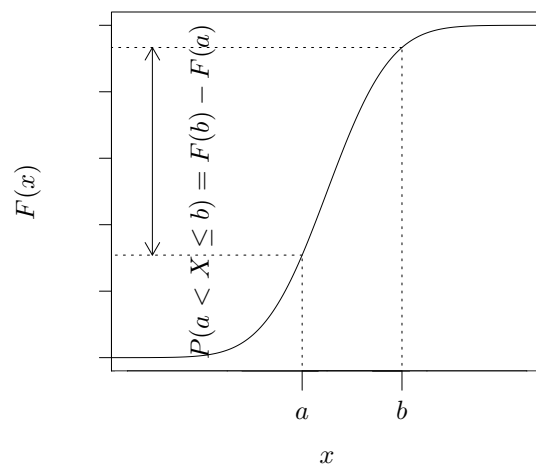
- Note that as a consequence of this definition,

$$f(x) = F'(x).$$

- It's particularly useful to note that

$$P(a < X \leq b) = F(b) - F(a) = \int_a^b f(x) dx.$$

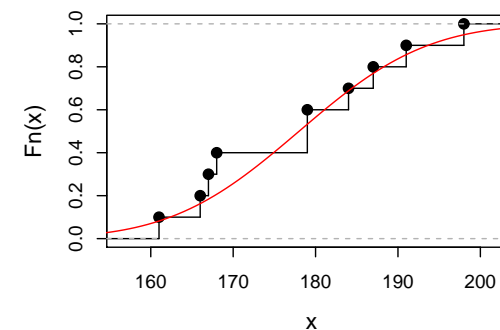
The distribution function



The empirical cumulative distribution function (ecdf)

```
# Empirical cdf for sample of height data from Chapter 1
x <- c(168, 161, 167, 179, 184, 166, 198, 187, 191, 179)
plot(ecdf(x), verticals = TRUE, main = "")

# 'True cdf' for normal distribution (with sample mean and variance)
xp <- seq(0.9*min(x), 1.1*max(x), length = 100)
lines(xp, pnorm(xp, mean(x), sd(x)), col = 2)
```



Mean, continuous random variable, Definition 2.34

The mean/expected value of a continuous random variable:

$$\mu = \int_{-\infty}^{\infty} xf(x) dx$$

Compare with the mean of a discrete random variable:

$$\mu = \sum_{\text{all } x} xf(x)$$

Variance, continuous random variable, Definition 2.34

The variance of a continuous random variable:

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

Compare with the variance of a discrete random variable:

$$\sigma^2 = \sum_{\text{all } x} (x - \mu)^2 f(x)$$

Covariance, Definition 2.58

The covariance between two random variables:

Let X and Y be two random variables. Then, the covariance between X and Y is

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Relationship between covariance and independence:

If two random variables are *independent* their covariance is 0. *The reverse is not necessarily true!*

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Specific continuous distributions

A number of statistical distributions exist (both continuous and discrete) that can be used to describe and analyze different types of problems.

Today, we'll take a closer look at the following [continuous](#) distributions:

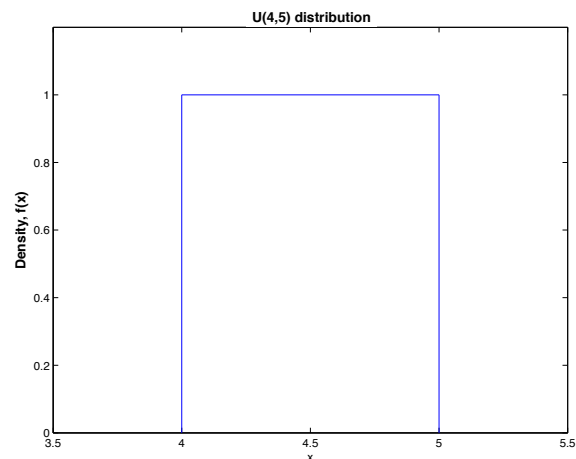
- The uniform distribution
- The normal distribution
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- The exponential distribution

Continuous distributions in R

R	Distribution
norm	The normal distribution
unif	The uniform distribution
lnorm	The log-normal distribution
exp	The exponential distribution

- d Probability density function, $f(x)$.
- p Cumulative distribution function, $F(x)$.
- q Quantile function.
- r Random numbers from the distribution.

Density of a uniform distribution (example)



The uniform distribution, Def. 2.35 & Theo. 2.36

Syntax:

$$X \sim U(\alpha, \beta)$$

Density function:

$$f(x) = \frac{1}{\beta - \alpha} \text{ for } \alpha \leq x \leq \beta$$

Mean:

$$\mu = \frac{\alpha + \beta}{2}$$

Variance:

$$\sigma^2 = \frac{1}{12}(\beta - \alpha)^2$$

Example 1

Students attending a stats course arrive at a lecture between 8.00 and 8.30. It is assumed that the arrival times can be described by a uniform distribution.

Question:

What is the probability that a randomly selected student arrives between 8.20 and 8.30?

Answer:

$$10/30 = 1/3$$

Let $X \sim U(0, 30)$ represent arrival time. Then:

$$P(20 \leq X \leq 30) = P(X \leq 30) - P(X \leq 20) = 1 - 2/3 = 1/3$$

```
punif(30, 0, 30) - punif(20, 0, 30)
```

[1] 0.33

Example 1 (continued)

Question:

What is the probability that a randomly selected student arrives after 8.30?

Answer:

0

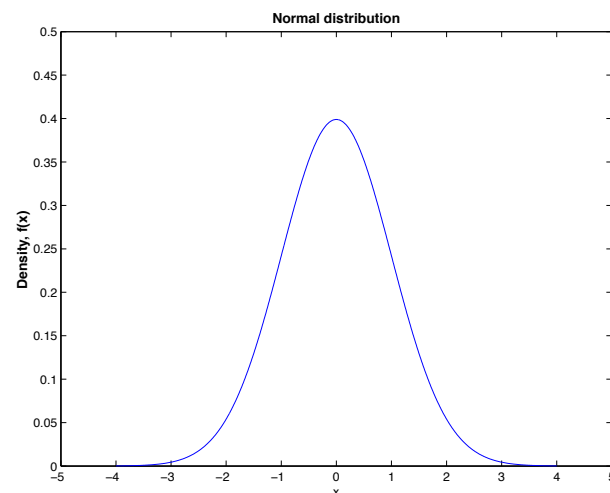
Let $X \sim U(0, 30)$ represent arrival time. Then:

$$P(X > 30) = 1 - P(X \leq 30) = 1 - 1 = 0$$

```
1 - punif(30, 0, 30)
```

[1] 0

Density of a normal distribution (example)



The normal distribution, Def. 2.37 & Theo. 2.38

Syntax:

$$X \sim N(\mu, \sigma^2)$$

Density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{ for } -\infty < x < \infty$$

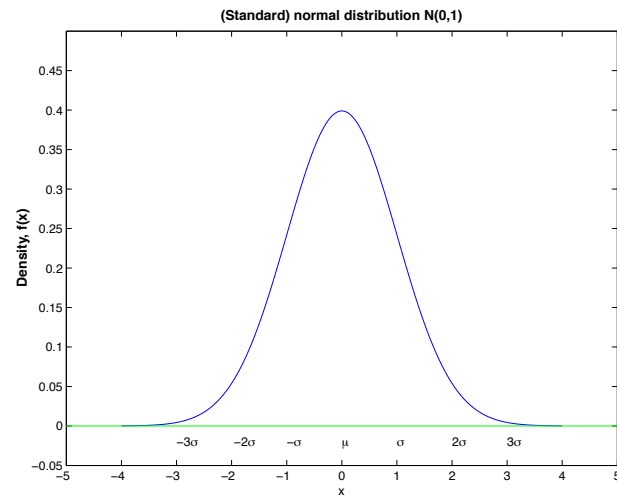
Mean:

$$\mu = \mu$$

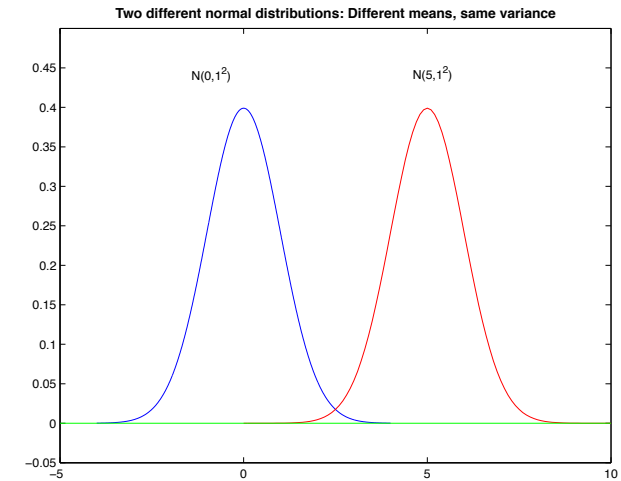
Variance:

$$\sigma^2 = \sigma^2$$

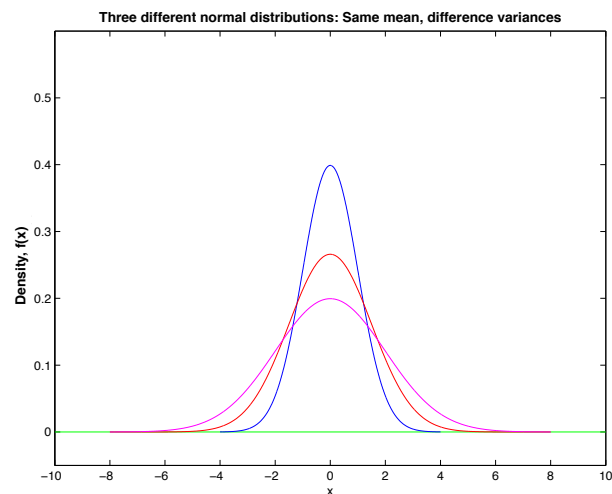
Density of a standard normal distribution



Density of two normal distributions (example)



Density of three normal distributions (example)



The standard normal distribution

The [standard normal distribution](#):

$$Z \sim N(0, 1^2)$$

The normal distribution with mean 0 and variance 1.

Standardization:

An arbitrary normal distributed variable $X \sim N(\mu, \sigma^2)$ can be *standardized* by

$$Z = \frac{X - \mu}{\sigma}$$

Example 2

Measurement error:

A scale has a measurement error, Z , that can be described by the standard normal distribution, i.e.

$$Z \sim N(0, 1^2).$$

That is, the mean measurement error is $\mu = 0$ with standard deviation $\sigma = 1$ gram. The scale is used to measure the weight of a product.

Question a):

What is the probability that the scale yields a measurement which is at least 2 grams smaller than the true weight of the product?

Answer:

$$P(Z \leq -2) = 0.02275$$

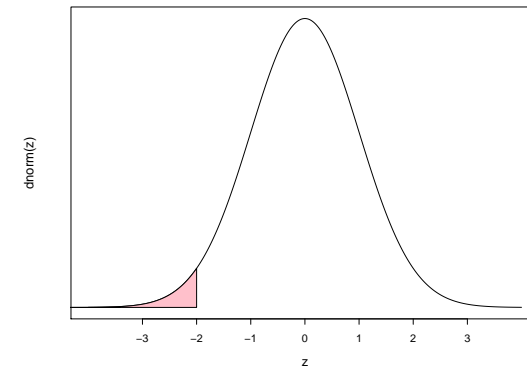
```
pnorm(-2)
```

Example 2

Answer:

```
pnorm(-2)
```

```
[1] 0.023
```



Example 2

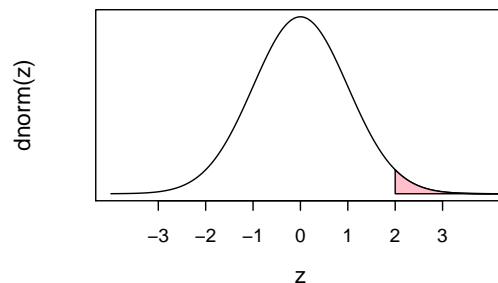
Question b):

What is the probability that the scale yields a measurement which is at least 2 grams larger than the true weight of the product?

Answer:

$$P(Z \geq 2) = 0.02275$$

```
1 - pnorm(2)
```



Example 2

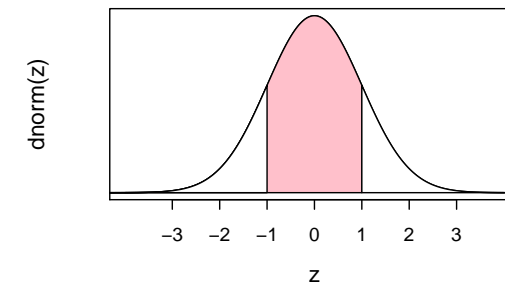
Question c):

What is the probability that the scale is off by at most ± 1 gram?

Answer:

$$P(|Z| \leq 1) = P(-1 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq -1) = 0.683$$

```
pnorm(1) - pnorm(-1)
```



Example 3

Income distribution:

It is assumed that the annual salary distribution of elementary school teachers can be described using a normal distribution with mean $\mu = 290$ (in DKK thousand) and standard deviation $\sigma = 4$ (DKK thousand).

Question a):

What is the probability that a randomly selected teacher earns more than DKK 300.000?

Example 4

(Same income distribution):

It is assumed that the annual salary distribution of elementary school teachers can be described using a normal distribution with mean $\mu = 290$ (DKK thousand) and standard deviation $\sigma = 4$ (DKK thousand).

"Opposite question"

Give a salary interval (symmetric around the mean) which covers 95% of all teachers' salary.

Answer:

```
qnorm(c(0.025, 0.975), m = 290, s = 4)
```

```
[1] 282 298
```

Example 3

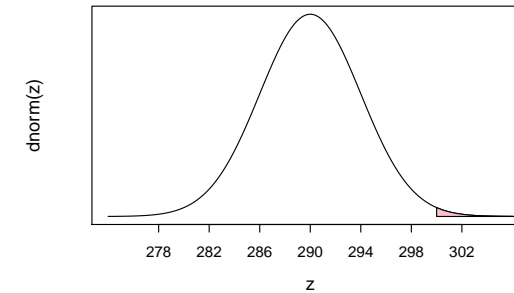
Question a):

What is the probability that a randomly selected teacher earns more than DKK 300.000?

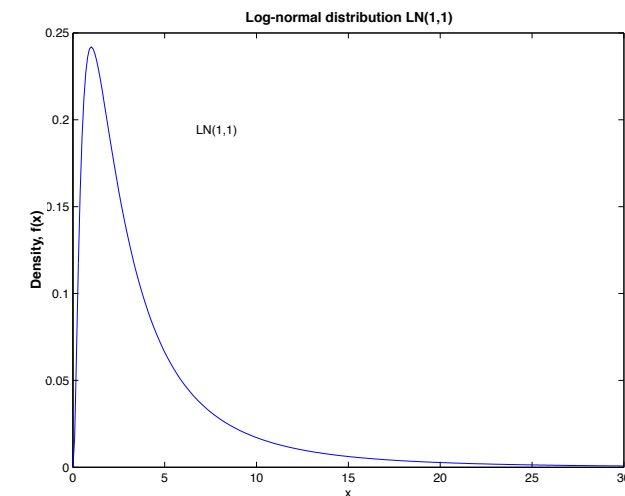
Answer:

```
1 - pnorm(300, m = 290, s = 4)
```

```
[1] 0.0062
```



The log-normal distribution



The log-normal distribution, Def. 2.46 & Theo. 2.47

Syntax:

$$X \sim LN(\alpha, \beta^2) \text{ (with } \beta > 0\text{)}$$

Density function:

$$f(x) = \begin{cases} \frac{1}{\beta\sqrt{2\pi}} x^{-1} e^{-(\ln(x)-\alpha)^2/2\beta^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean:

$$\mu = e^{\alpha + \beta^2/2}$$

Variance:

$$\sigma^2 = e^{2\alpha + \beta^2} (e^{\beta^2} - 1)$$

The log-normal distribution

Log-normal and normal distributions:

A log-normal distributed variable $Y \sim LN(\alpha, \beta^2)$ can be transformed into a normal distributed variable:

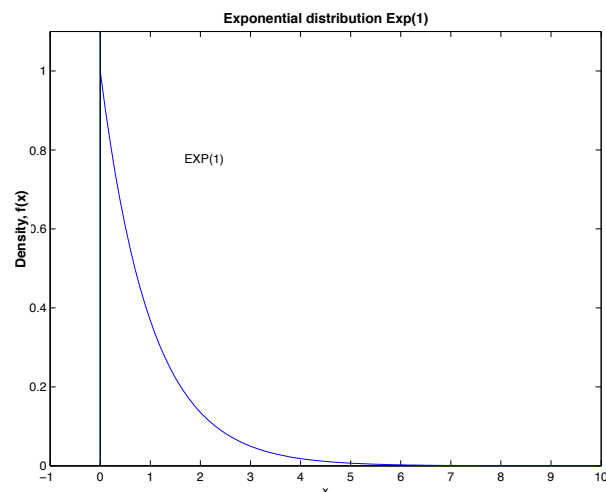
$$X = \ln(Y)$$

is normal distributed with mean α and variance β^2 , i.e. $X \sim N(\alpha, \beta^2)$.

$$Z = \frac{\ln(Y) - \alpha}{\beta}$$

is standard normal distributed, i.e. $Z \sim N(0, 1)$.

The exponential distribution



The exponential distribution, Def. 2.48 & Theo. 2.49

Syntax:

$$X \sim \text{Exp}(\lambda)$$

with $\lambda > 0$.

Density function:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean:

$$\mu = \frac{1}{\lambda}$$

Variance:

$$\sigma^2 = \frac{1}{\lambda^2}$$

The exponential distribution

- The exponential distribution is a special case of the gamma distribution.
- The exponential distribution is used to describe lifespan and waiting times.
- The exponential distribution can be used to describe (waiting) time between Poisson events.

Example 5

Queuing model – Poisson process

The time between customer arrivals at a post office is exponentially distributed with mean $\mu = 2$ minutes.

Question:

One customer has just arrived. What is the probability that no other customers will arrive during the next 2 minutes?

Answer:

$X \sim \text{Exp}(1/2)$ represents waiting time until next customer.
 $P(X > 2) = 1 - P(X \leq 2)$

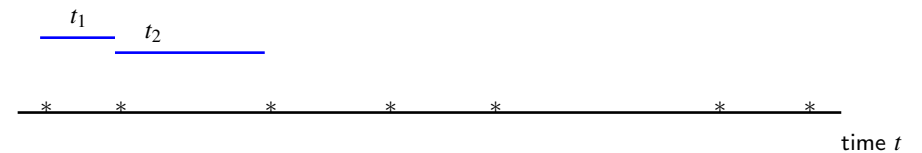
```
1 - pexp(2, rate = 1/2)
```

```
[1] 0.37
```

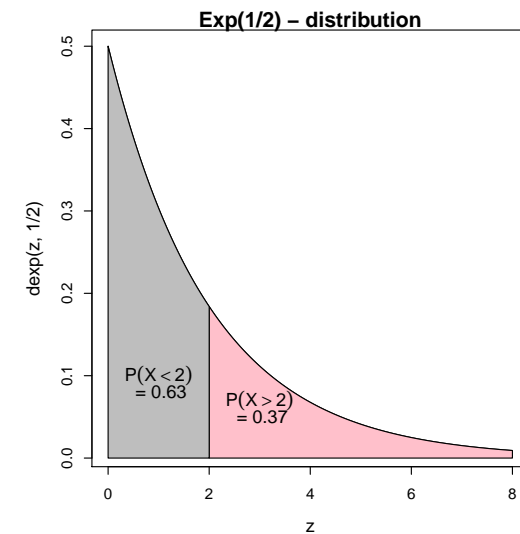
Connection between the exponential and Poisson distributions

Poisson: Discrete events per unit

Exponential: Continuous distance between events



Example 5



Example 6

Question:

One customer has just arrived. Use the Poisson distribution to calculate the probability that no other costumers will arrive during the next two minutes.

Answer:

$$\lambda_{2min} = 1, P(X = 0) = \frac{e^{-1}}{1!} 1^0 = e^{-1}$$

```
dpois(0,1)
```

```
[1] 0.37
```

```
exp(-1)
```

```
[1] 0.37
```

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Calculation rules for random variables

These rules work for both continuous and discrete random variables!

X is a random variable, a and b are constants.

Mean rule:

$$E(aX + b) = aE(X) + b$$

Variance rule:

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

Example 7

X is a random variable with mean 4 and variance 6.

Question:

Calculate the mean and variance of $Y = -3X + 2$

Answer:

$$E(Y) = -3E(X) + 2 = -3 \cdot 4 + 2 = -10$$

$$\text{Var}(Y) = (-3)^2 \text{Var}(X) = 9 \cdot 6 = 54$$

Calculation rules for random variables

X_1, \dots, X_n are *independent* random variables.

Mean rule:

$$\begin{aligned} E(a_1X_1 + a_2X_2 + \dots + a_nX_n) \\ = a_1E(X_1) + a_2E(X_2) + \dots + a_nE(X_n) \end{aligned}$$

Variance rule:

$$\begin{aligned} \text{Var}(a_1X_1 + a_2X_2 + \dots + a_nX_n) \\ = a_1^2\text{Var}(X_1) + \dots + a_n^2\text{Var}(X_n) \end{aligned}$$

Example 8

Airline Planning

The weight of each passenger on a flight is assumed to be normal distributed $X \sim N(70, 10^2)$.

A plane, which can take 55 passengers, may not have a load exceeding 4000 kg (only the weight of the passengers is considered load).

Question:

Calculate the probability that the plain is overloaded

What is $Y = \text{Total passenger weight}$?

What is Y ?

Definitely NOT: $Y = 55 \cdot X$

Example 8

What is $Y = \text{Total passenger weight}$?

$Y = \sum_{i=1}^{55} X_i$, where $X_i \sim N(70, 10^2)$ (and assumed to be independent)

Mean and variance of Y :

$$\begin{aligned} E(Y) &= \sum_{i=1}^{55} E(X_i) = \sum_{i=1}^{55} 70 = 55 \cdot 70 = 3850 \\ \text{Var}(Y) &= \sum_{i=1}^{55} \text{Var}(X_i) = \sum_{i=1}^{55} 100 = 55 \cdot 100 = 5500 \end{aligned}$$

Y is normal distributed, so we may find $P(Y > 4000)$ using:

```
1-pnorm(4000, mean = 3850, sd = sqrt(5500))
```

```
[1] 0.022
```

Example 8 - WRONG ANALYSIS

What is Y ?

Definitely NOT: $Y = 55 \cdot X$

Mean and variance of WRONG Y :

$$\begin{aligned} E(Y) &= 55 \cdot 70 = 3850 \\ \text{Var}(Y) &= 55^2 \text{Var}(X) = 55^2 \cdot 100 = 550^2 \end{aligned}$$

Wrong Y is also normal distributed. Finding $P(Y > 4000)$ using WRONG Y :

```
1 - pnorm(4000, mean = 3850, sd = 550)
```

```
[1] 0.39
```

Consequence of wrong calculation:

A LOT of wasted money for the airline company!!!

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