Course 02402 Introduction to Statistics

Lecture 3: Random variables and continuous distributions

DTU Compute
Technical University of Denmark
2800 Lyngby – Denmark

Overview

1. Continuous random variables and distributions
   - Density and distribution functions
   - Mean, variance, and covariance
2. Specific continuous distributions
   - The uniform distribution
   - The normal distribution
   - The log-normal distribution
   - The exponential distribution
3. Calculation rules for random variables

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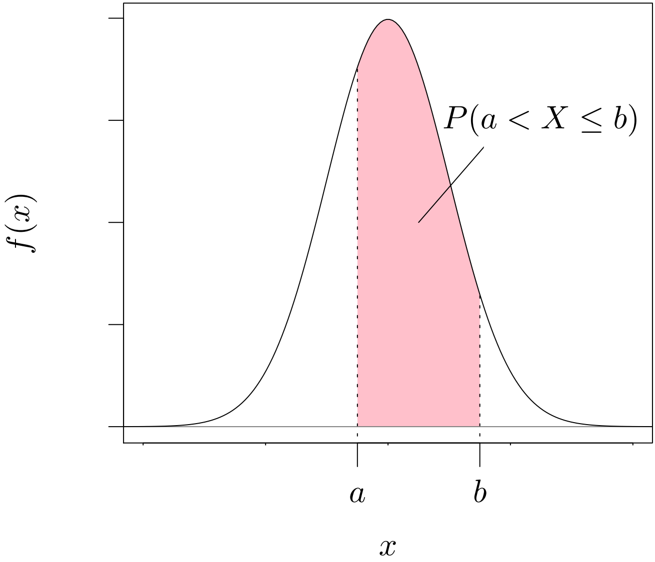
- The density function for a continuous random variable does *not* correspond directly to a probability. In fact, $f(x) \neq P(X = x)$ and $P(X = x) = 0$ for all x.
- The density function $f(x)$ for the distribution of a continuous random variable satisfies that

$$f(x) \geq 0 \text{ for all } x \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\,dx = 1\,.$$

The density function



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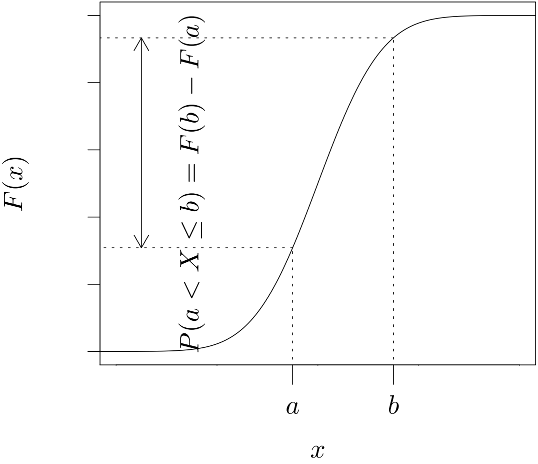
- Note that as a consequence of this definition,

$$f(x) = F^{'}(x)\,.$$

- It's particularly useful to note that

$$P(a < X \leq b) = F(b) - F(a) = \int_{a}^{b} f(x)\,dx\,.$$

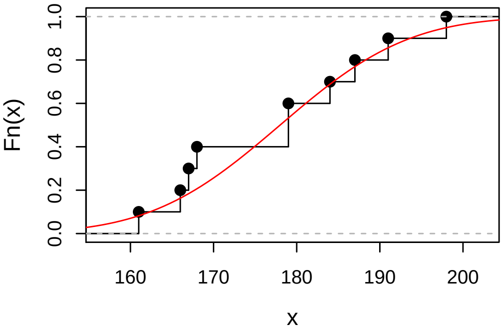
The distribution function



The empirical cumulative distribution function (ecdf)

```
# Empirical cdf for sample of height data from Chapter 1
x <- c(168, 161, 167, 179, 184, 166, 198, 187, 191, 179)
plot(ecdf(x), verticals = TRUE, main = "")

# 'True cdf' for normal distribution (with sample mean and variance)
xp <- seq(0.9*min(x), 1.1*max(x), length = 100)
lines(xp, pnorm(xp, mean(x), sd(x)), col = 2)
```



Mean, continuous random variable, Definition 2.34

The mean/expected value of a continuous random variable:

$$\mu = \int_{-\infty}^{\infty} x f(x) \, dx$$

Mean, continuous random variable, Definition 2.34

The mean/expected value of a continuous random variable:

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

Compare with the mean of a discrete random variable:

$$\mu = \sum_{\text{all } x} x f(x)$$

Variance, continuous random variable, Definition 2.34

The variance of a continuous random variable:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

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The variance of a continuous random variable:

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Compare with the variance of a discrete random variable:

$$\sigma^2 = \sum_{\text{all } x} (x-\mu)^2 f(x)$$

Covariance, Definition 2.58

The covariance between two random variables:

Let X and Y be two random variables. Then, the covariance between X and Y is

$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$

Relationship between covariance and independence:

If two random variables are *independent* their covariance is 0. *The reverse is not necessarily true!*

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Specific continuous distributions

A number of statistical distributions exist (both continuous and discrete) that can be used to describe and analyze different types of problems.

Today, we'll take a closer look at the following continuous distributions:

- The uniform distribution
- The normal distribution
- The log-normal distribution
- The exponential distribution

Continuous distributions in R

R	Distribution
`norm`	The normal distribution
`unif`	The uniform distribution
`lnorm`	The log-normal distribution
`exp`	The exponential distribution

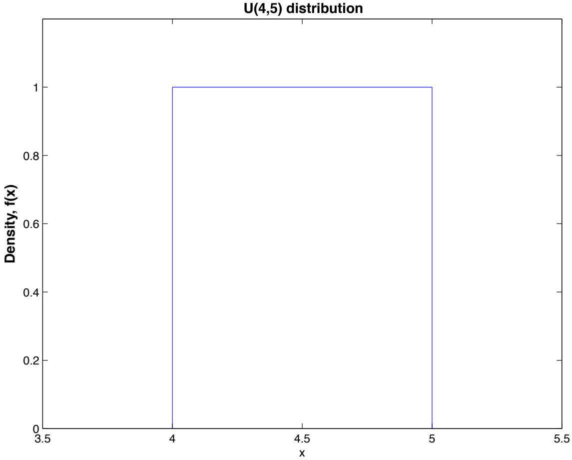
`d` Probability density function, $f(x)$.

`p` Cumulative distribution function, $F(x)$.

`q` Quantile function.

`r` Random numbers from the distribution.

Density of a uniform distribution (example)



The uniform distribution, Def. 2.35 & Theo. 2.36

Syntax:

$X \sim U(\alpha, \beta)$

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$f(x) = \frac{1}{\beta - \alpha}$ for $\alpha \leq x \leq \beta$

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$\mu = \frac{\alpha + \beta}{2}$

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Mean:

$\mu = \frac{\alpha + \beta}{2}$

Variance:

$\sigma^2 = \frac{1}{12}(\beta - \alpha)^2$

Example 1

Students attending a stats course arrive at a lecture between 8.00 and 8.30. It is assumed that the arival times can be described by a uniform distribution.

Question:

What is the probability that a randomly selected student arrives between 8.20 and 8.30?

Example 1

Students attending a stats course arrive at a lecture between 8.00 and 8.30. It is assumed that the arival times can be described by a uniform distribution.

Question:

What is the probability that a randomly selected student arrives between 8.20 and 8.30?

Answer:

$10/30 = 1/3$

Let $X \sim U(0, 30)$ represent arrival time. Then:
$P(20 \leq X \leq 30) = P(X \leq 30) - P(X \leq 20) = 1 - 2/3 = 1/3$

```
punif(30, 0, 30) - punif(20, 0, 30)
```

```
[1] 0.33
```

Example 1 (continued)

Question:

What is the probability that a randomly selected student arrives after 8.30?

Example 1 (continued)

Question:

What is the probability that a randomly selected student arrives after 8.30?

Answer:

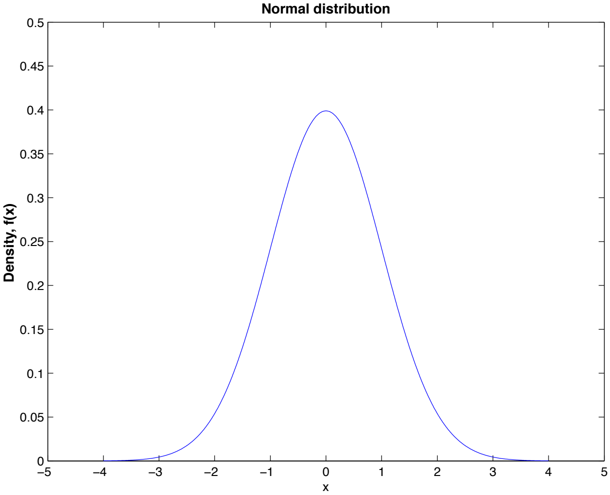
0

Let $X \sim U(0,30)$ represent arrival time. Then:
$P(X > 30) = 1 - P(X \leq 30) = 1 - 1 = 0$

```
1 - punif(30, 0, 30)
```

```
[1] 0
```

Density of a normal distribution (example)



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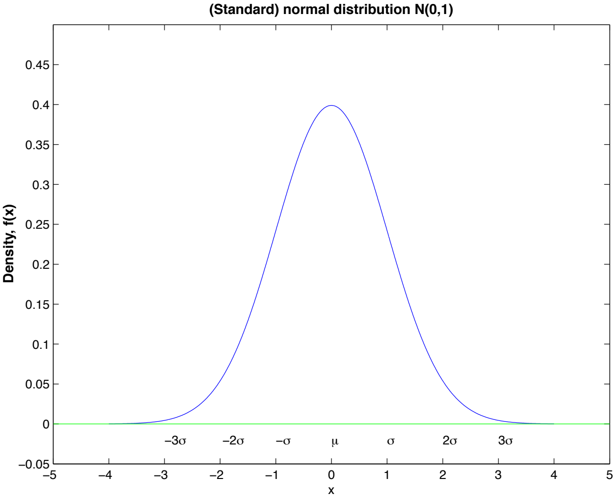
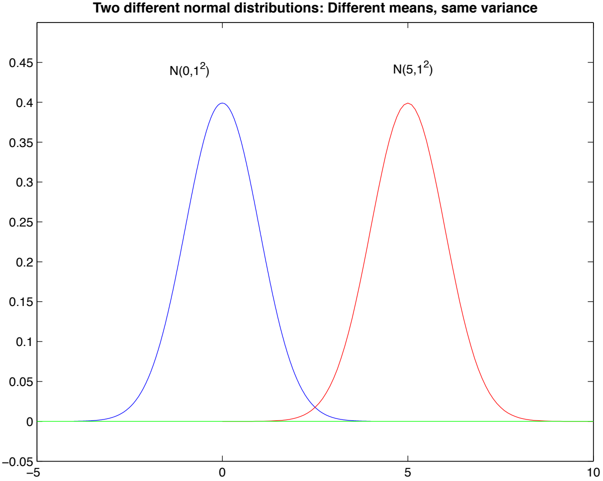
Variance:

$\sigma^2 = \sigma^2$

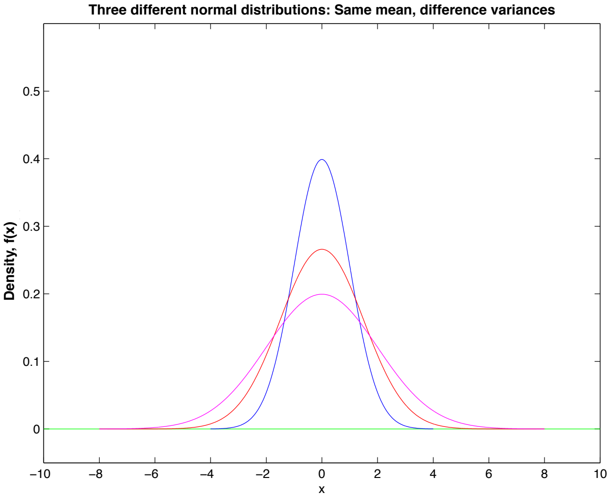
Density of a standard normal distribution



Density of two normal distributions (example)



Density of three normal distributions (example)



The standard normal distribution

The standard normal distribution:

$$Z \sim N(0, 1^2)$$

The normal distribution with mean 0 and variance 1.

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Standardization:

An arbitrary normal distributed variable $X \sim N(\mu, \sigma^2)$ can be *standardized* by

$$Z = \frac{X - \mu}{\sigma}$$

Example 2

Measurement error:

A scale has a measurement error, Z, that can be described by the standard normal distribution, i.e.

$$Z \sim N(0, 1^2).$$

That is, the mean measurement error is $\mu = 0$ with standard deviation $\sigma = 1$ gram. The scale is used to measure the weight of a product.

Example 2

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Question a):

What is the probability that the scale yields a measurement which is at least 2 grams smaller than the true weight of the product?

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Measurement error:

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Question a):

What is the probability that the scale yields a measurement which is at least 2 grams smaller than the true weight of the product?

Answer:

$P(Z \leq -2) = 0.02275$

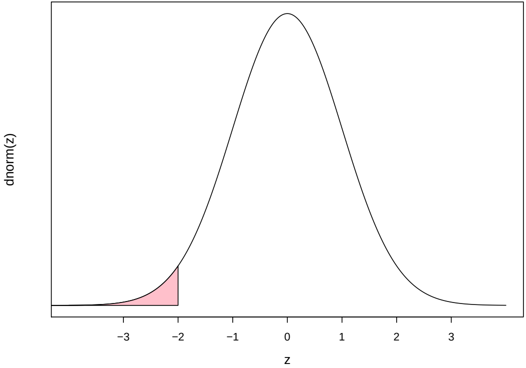
```
pnorm(-2)
```

Example 2

Answer:

```
pnorm(-2)
```

[1] 0.023



Example 2

Question b):

What is the probability that the scale yields a measurement which is at least 2 grams larger than the true weight of the product?

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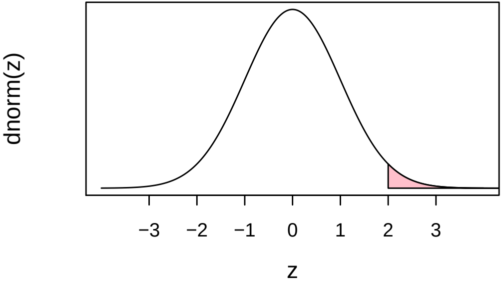
What is the probability that the scale yields a measurement which is at least 2 grams larger than the true weight of the product?

Answer:

$P(Z \geq 2) = 0.02275$

```
1 - pnorm(2)
```



Example 2

Question c):

What is the probability that the scale is off by at most ± 1 *gram?*

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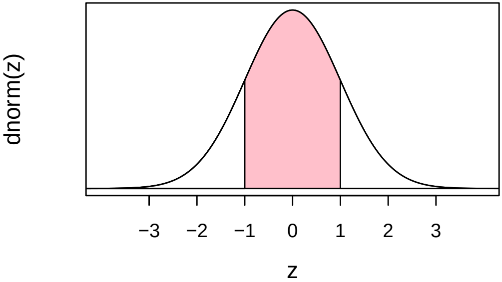
Answer:

$$P(|Z| \leq 1) = P(-1 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq -1) = 0.683$$

```
pnorm(1) - pnorm(-1)
```



Example 3

Income distribution:

It is assumed that the annual salary distribution of elementary school teachers can be described using a normal distribution with mean $\mu = 290$ (in DKK thousand) and standard deviation $\sigma = 4$ (DKK thousand).

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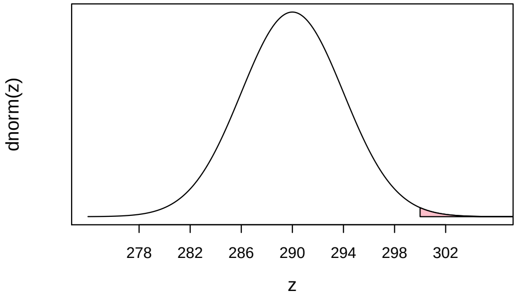
Answer:

```
1 - pnorm(300, m = 290, s = 4)
```

[1] 0.0062



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"Opposite question"

Give a salary interval (symmetric around the mean) which covers 95% of all teachers' salary.

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(Same income distribution):

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"Opposite question"

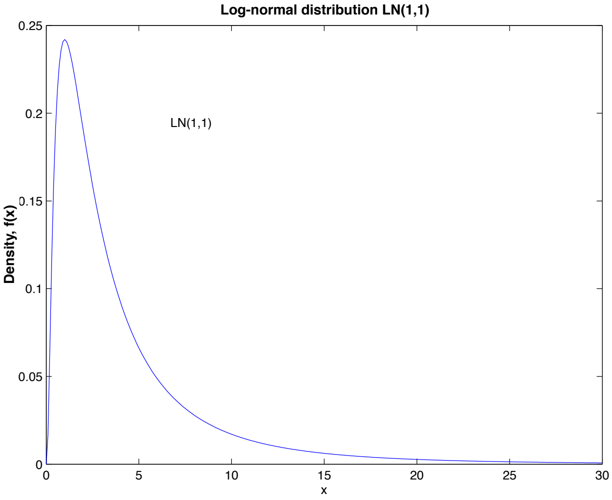
Give a salary interval (symmetric around the mean) which covers 95% of all teachers' salary.

Answer:

```
qnorm(c(0.025, 0.975), m = 290, s = 4)
```

```
[1] 282 298
```

The log-normal distribution



The log-normal distribution, Def. 2.46 & Theo. 2.47

Syntax:

$X \sim LN(\alpha, \beta^2)$ (with $\beta > 0$)

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Density function:

$$f(x) = \begin{cases} \frac{1}{\beta\sqrt{2\pi}} x^{-1} e^{-(\ln(x)-\alpha)^2/2\beta^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

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Mean:

$\mu = e^{\alpha+\beta^2/2}$

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Mean:

$\mu = e^{\alpha+\beta^2/2}$

Variance:

$\sigma^2 = e^{2\alpha+\beta^2}(e^{\beta^2} - 1)$

The log-normal distribution

Log-normal and normal distributions:

A log-normal distributed variable $Y \sim LN(\alpha, \beta^2)$ can be transformed into a normal distributed variable:

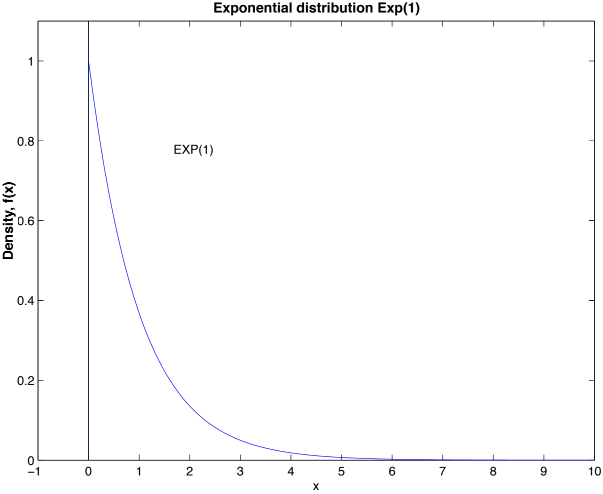
$$X = \ln(Y)$$

is normal distributed with mean α and variance β^2, i.e. $X \sim N(\alpha, \beta^2)$.

$$Z = \frac{\ln(Y) - \alpha}{\beta}$$

is standard normal distributed, i.e. $Z \sim N(0, 1)$.

The exponential distribution



The exponential distribution, Def. 2.48 & Theo. 2.49

Syntax:

$$X \sim \text{Exp}(\lambda)$$

with $\lambda > 0$.

Density function:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean:

$$\mu = \frac{1}{\lambda}$$

Variance:

$$\sigma^2 = \frac{1}{\lambda^2}$$

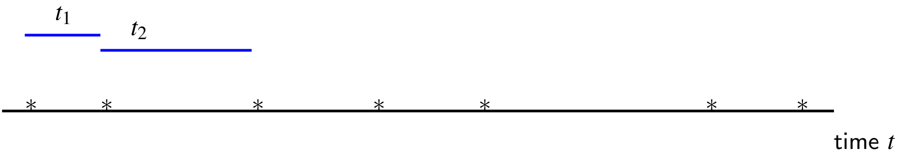
The exponential distribution

- The exponential distribution is a special case of the gamma distribution.
- The exponential distribution is used to describe lifespan and waiting times.
- The exponential distribution can be used to describe (waiting) time between Poisson events.

Connection between the exponential and Poisson distributions

Poisson: Discrete events per unit

Exponential: Continuous distance between events



Example 5

Queuing model – Poisson process

The time between customer arrivals at a post office is exponentially distributed with mean $\mu = 2$ minutes.

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One customer has just arrived. What is the probability that no other customers will arrive during the next 2 minutes?

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Queuing model – Poisson process

The time between customer arrivals at a post office is exponentially distributed with mean $\mu = 2$ minutes.

Question:

One customer has just arrived. What is the probability that no other customers will arrive during the next 2 minutes?

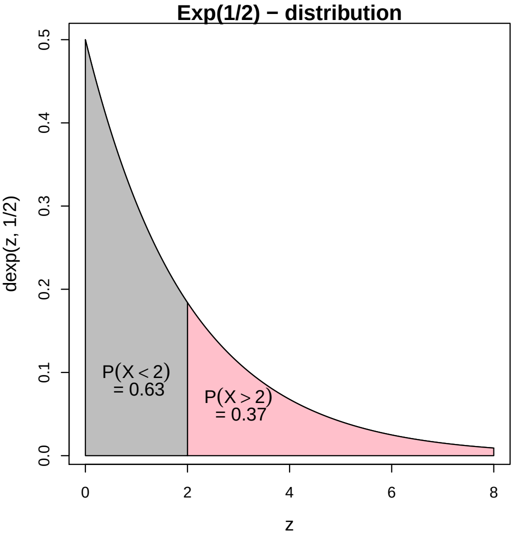
Answer:

$X \sim \text{Exp}(1/2)$ represents waiting time until next customer.
$P(X > 2) = 1 - P(X \leq 2)$

```
1 - pexp(2, rate = 1/2)
```

```
[1] 0.37
```

Example 5


Exp(1/2) − distribution
dexp(z, 1/2)
z
0.0
0.1
0.2
0.3
0.4
0.5
0
2
4
6
8
P(X < 2)
= 0.63
P(X > 2)
= 0.37

Example 6

Question:

One customer has just arrived. Use the Poisson distribution to calculate the probability that no other costumers will arrive during the next two minutes.

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Question:

One customer has just arrived. Use the Poisson distribution to calculate the probability that no other costumers will arrive during the next two minutes.

Answer:

$\lambda_{2min} = 1$, $P(X = 0) = \frac{e^{-1}}{1!} 1^0 = e^{-1}$

```r
dpois(0,1)
```

[1] 0.37

```r
exp(-1)
```

[1] 0.37

Overview

Calculation rules for random variables

These rules work for both continuous and discrete random variables!

X is a random variable, a and b are constants.

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Mean rule:

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Mean rule:

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Variance rule:

$$\mathsf{Var}(aX+b) = a^2\mathsf{Var}(X)$$

Example 7

X is a random variable with mean 4 and variance 6.

Question:

Calculate the mean and variance of $Y = -3X + 2$

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Answer:

$$\mathsf{E}(Y) = -3\mathsf{E}(X) + 2 = -3 \cdot 4 + 2 = -10$$
$$\mathsf{Var}(Y) = (-3)^2\mathsf{Var}(X) = 9 \cdot 6 = 54$$

Calculation rules for random variables

$X_1, \ldots, X_n$ are *independent* random variables.

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Mean rule:

$$\begin{aligned} &\mathsf{E}(a_1 X_1 + a_2 X_2 + \cdots + a_n X_n) \\ &= a_1 \mathsf{E}(X_1) + a_2 \mathsf{E}(X_2) + \cdots + a_n \mathsf{E}(X_n) \end{aligned}$$

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$X_1, \ldots, X_n$ are *independent* random variables.

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$$\mathsf{E}(a_1X_1 + a_2X_2 + \cdots + a_nX_n)$$
$$= a_1\mathsf{E}(X_1) + a_2\mathsf{E}(X_2) + \cdots + a_n\mathsf{E}(X_n)$$

Variance rule:

$$\mathsf{Var}(a_1X_1 + a_2X_2 + \cdots + a_nX_n)$$
$$= a_1^2\mathsf{Var}(X_1) + \cdots + a_n^2\mathsf{Var}(X_n)$$

Example 8

Airline Planning

The weight of each passenger on a flight is assumed to be normal distributed $X \sim N(70, 10^2)$.

A plane, which can take 55 passengers, may not have a load exceeding 4000 kg (only the weight of the passengers is considered load).

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Question:

Calculate the probability that the plain is overloaded

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What is Y = Total passenger weight?

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What is Y?

Definitely NOT: $Y = 55 \cdot X$

Example 8

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$Y = \sum_{i=1}^{55} X_i$, where $X_i \sim N(70, 10^2)$ (and assumed to be independent)

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$Y = \sum_{i=1}^{55} X_i$, where $X_i \sim N(70, 10^2)$ (and assumed to be independent)

Mean and variance of Y:

$$\mathsf{E}(Y) = \sum_{i=1}^{55} \mathsf{E}(X_i) = \sum_{i=1}^{55} 70 = 55 \cdot 70 = 3850$$

$$\mathsf{Var}(Y) = \sum_{i=1}^{55} \mathsf{Var}(X_i) = \sum_{i=1}^{55} 100 = 55 \cdot 100 = 5500$$

Example 8

What is Y = Total passenger weight?

$Y = \sum_{i=1}^{55} X_i$, where $X_i \sim N(70, 10^2)$ (and assumed to be independent)

Mean and variance of Y:

$$\mathsf{E}(Y) = \sum_{i=1}^{55} \mathsf{E}(X_i) = \sum_{i=1}^{55} 70 = 55 \cdot 70 = 3850$$

$$\mathsf{Var}(Y) = \sum_{i=1}^{55} \mathsf{Var}(X_i) = \sum_{i=1}^{55} 100 = 55 \cdot 100 = 5500$$

Y is normal distributed, so we may find $P(Y > 4000)$ using:

```
1-pnorm(4000, mean = 3850, sd = sqrt(5500))
```

```
[1] 0.022
```

Example 8 - WRONG ANALYSIS

What is Y?

Definitely NOT: $Y = 55 \cdot X$

Example 8 - WRONG ANALYSIS

What is Y?

Definitely NOT: $Y = 55 \cdot X$

Mean and variance of WRONG Y:

$$\mathsf{E}(Y) = 55 \cdot 70 = 3850$$
$$\mathsf{Var}(Y) = 55^2 \mathsf{Var}(X) = 55^2 \cdot 100 = 550^2$$

Example 8 - WRONG ANALYSIS

What is Y?

Definitely NOT: $Y = 55 \cdot X$

Mean and variance of WRONG Y:

$$\mathsf{E}(Y) = 55 \cdot 70 = 3850$$

$$\mathsf{Var}(Y) = 55^2 \mathsf{Var}(X) = 55^2 \cdot 100 = 550^2$$

Wrong Y is also normal distributed. Finding $P(Y > 4000)$ using WRONG Y:

```
1 - pnorm(4000, mean = 3850, sd = 550)
```

[1] 0.39

Example 8 - WRONG ANALYSIS

What is Y?

Definitely NOT: $Y = 55 \cdot X$

Mean and variance of WRONG Y:

$$\mathsf{E}(Y) = 55 \cdot 70 = 3850$$

$$\mathsf{Var}(Y) = 55^2 \mathsf{Var}(X) = 55^2 \cdot 100 = 550^2$$

Wrong Y is also normal distributed. Finding $P(Y > 4000)$ using WRONG Y:

```
1 - pnorm(4000, mean = 3850, sd = 550)
```

[1] 0.39

Consequence of wrong calculation:

A LOT of wasted money for the airline company!!!

Overview

1. Continuous random variables and distributions
   - Density and distribution functions
   - Mean, variance, and covariance
2. Specific continuous distributions
   - The uniform distribution
   - The normal distribution
   - The log-normal distribution
   - The exponential distribution
3. Calculation rules for random variables