# Course 02323 Introduction to Statistics

# Lecture 2: Random variables and discrete distributions

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Technical University of Denmark
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## Agenda

1. Random variables and density functions

2. Distribution functions

3. Specific (discrete) distributions I: The binomial
   - Example 1

4. Specific distributions II: The hypergeometric
   - Example 2

5. Specific distributions III: The Poisson
   - Example 3

6. Distributions in R

7. Mean and variance of discrete distributions

## Overview

1. **Random variables and density functions**

2. Distribution functions

3. Specific (discrete) distributions I: The binomial
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# Random variables

A random variable represents a value of an outcome *before* the corresponding *experiment* is carried out.

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A random variable represents a value of an outcome *before* the corresponding *experiment* is carried out.

- A throw of a dice.

- The number of six'es in ten dice throws.

- Fuel consumption of a car.

- Measurement of glucose level in blood sample.

- ...

# Discrete and continuous random variables

- We distinguish between *discrete* and *continuous* random variables.
- <u>Discrete:</u>
  - Number of people in this room who wear glasses.
  - Number of planes departing from CPH within the next hour.

- <u>Continuous:</u>
  - Wind speed measurement.
  - Transport time to DTU.

- Today: Discrete. Next week: Continuous.

## Random variable

Before the experiment is carried out, we have a random variable

$$X \text{ (or } X_1, \ldots, X_n)$$

indicated with capital letters.

## Random variable

Before the experiment is carried out, we have a random variable

$$X \text{ (or } X_1, \ldots, X_n)$$

indicated with capital letters.

After the experiment is carried out, we have a *realization* or *observation*

$$x \text{ (or } x_1, \ldots, x_n)$$

indicated with lowercase letters.

# Simulate rolling a dice in R

```r
# One random draw from (1,2,3,4,5,6)
# with equal probability for each outcome
sample(1:6, size = 1)
```

[1] 1

# Discrete distributions

- Random variables are used to describe an experiment before it is carried out.

- How to do this without yet knowing the outcome?

## Discrete distributions

- Random variables are used to describe an experiment before it is carried out.

- How to do this without yet knowing the outcome?

- Solution: Use a *density function*.

# Density function, discrete random variable: Definition 2.6

The *density function* (probability density function, pdf) of a discrete random variable:

## Definition

$$f(x) = P(X = x)$$

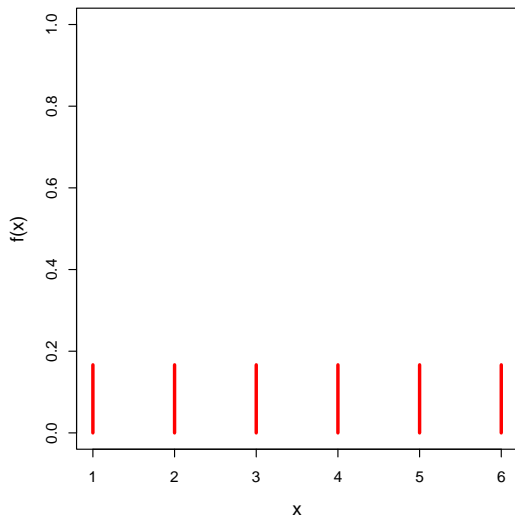Describes the probability that $X$ takes the value $x$ when the experiment is carried out.

# Density function, discrete random variable, Definition 2.6

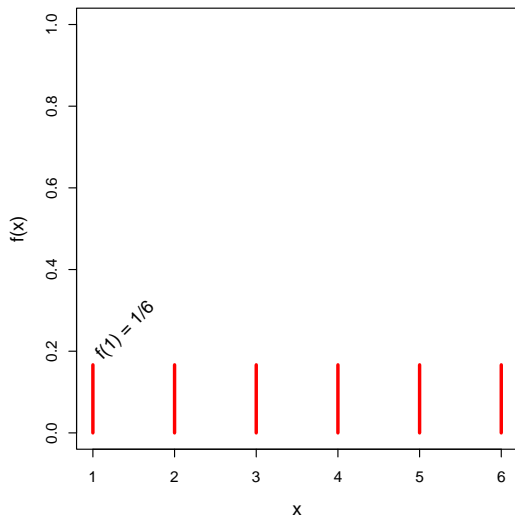The density function of a discrete random variable satisfies two properties:

## Definition

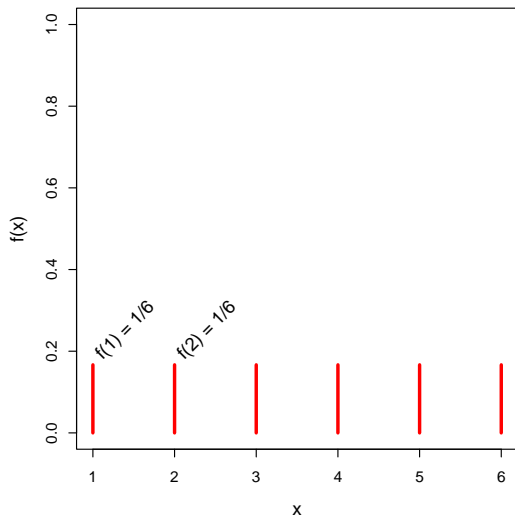$$f(x) \geq 0 \text{ for all } x \quad \text{and} \quad \sum_{\text{all } x} f(x) = 1$$

# Density function for a fair dice

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# Sample

If we only have a single observation, can we see the distribution?

## Sample

If we only have a single observation, can we see the distribution? No!

But if we have $n$ observations, then we have a *sample*

$$\{x_1, x_2, ..., x_n\}$$

and we can begin to get an idea of the distribution.
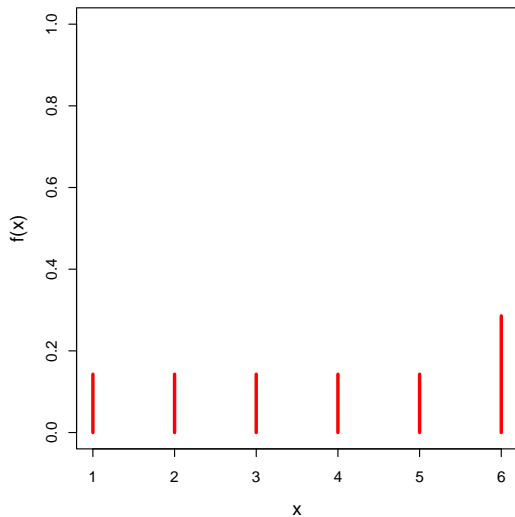
# Simulation of $n$ rolls with a fair dice

```r
# Number of simulated realizations (sample size)
n <- 30

# n independent random draws from the set (1,2,3,4,5,6)
# with equal probability of each outcome
xFair <- sample(1:6, size = n, replace = TRUE)
xFair

# Count number of each outcome using the 'table' function
table(xFair)

# Plot the empirical pdf
plot(table(xFair)/n, lwd = 10, ylim = c(0,1), xlab = "x",
     ylab = "Density f(x)")
# Add the true pdf to the plot
lines(rep(1/6,6), lwd = 4, type = "h", col = 2)
# Add a legend to the plot
legend("topright", c("Empirical pdf","True pdf"), lty = 1, col = c(1,2),
       lwd = c(5, 2), cex = 0.8)
```

# Density function for an unfair dice

# Density function for an unfair dice
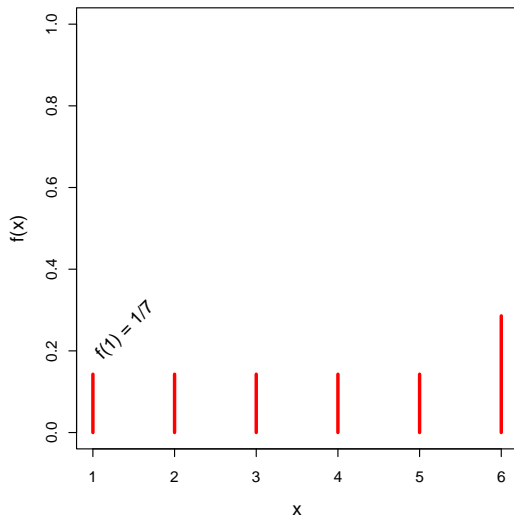
# Density function for an unfair dice

# Simulation of $n$ rolls with an unfair dice

```
# Number of simulated realizations (sample size)
n <- 30

# n independent random draws from the set (1,2,3,4,5,6)
# with higher probability of getting a six
xUnfair <- sample(1:6, size = n, replace = TRUE, prob = c(rep(1/7,5),2/7))
xUnfair

# Plot the empirical pdf
plot(table(xUnfair)/n, lwd = 10, ylim = c(0,1), xlab = "x",
     ylab = "Density f(x)")
# Add the true pdf to the plot
lines(c(rep(1/7,5),2/7), lwd = 4, type = "h", col = 2)
# Add a legend to the plot
legend("topright", c("Empirical pdf","True pdf"), lty = 1, col = c(1,2),
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```
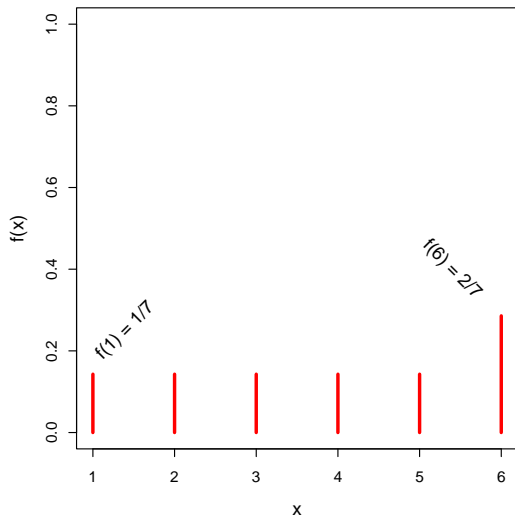
## Some questions

Let $X$ describe one throw with the *unfair* dice. What is:

- The probability of getting a 4?

- The probability of getting a 5 or a 6?

- The probability of getting less than 3?
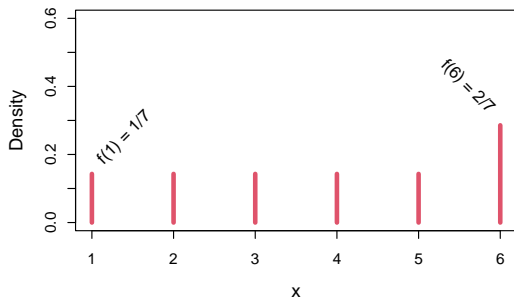
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# Distribution function, discrete random variable: Definition 2.9

The *distribution function* (cumulative distribution function, cdf) of a discrete random variable:

## Definition

$$F(x) = P(X \leq x) = \sum_{j \text{ where } x_j \leq x} f(x_j)$$

# Fair dice example

Let $X$ represent one throw with a fair dice.

Find the probability of throwing less than 3:

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$$P(X < 3) = P(X \leq 2)$$
$$= F(2) \text{ the distribution function}$$

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Let $X$ represent one throw with a fair dice.

Find the probability of throwing less than 3:

$$\begin{aligned} P(X < 3) &= P(X \leq 2) \\ &= F(2) \text{ \textit{the distribution function}} \\ &= P(X = 1) + P(X = 2) \end{aligned}$$

# Fair dice example

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Find the probability of throwing less than 3:

$$
\begin{aligned}
P(X < 3) &= P(X \le 2) \\
&= F(2) \text{ \textit{the distribution function}} \\
&= P(X = 1) + P(X = 2) \\
&= f(1) + f(2) \text{ \textit{the density function}}
\end{aligned}
$$

# Fair dice example

Let $X$ represent one throw with a fair dice.

Find the probability of throwing less than 3:

$$
\begin{aligned}
P(X < 3) &= P(X \leq 2) \\
&= F(2) \text{ \textit{the distribution function}} \\
&= P(X = 1) + P(X = 2) \\
&= f(1) + f(2) \text{ \textit{the density function}} \\
&= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
\end{aligned}
$$

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Find the probability of throwing greater than or equal to 3:

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Find the probability of throwing greater than or equal to 3:

$$P(X \geq 3) = 1 - P(X \leq 2)$$
$$= 1 - F(2) \text{ \textit{the distribution function}}$$
$$= 1 - \frac{1}{3} = \frac{2}{3}$$

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# Specific discrete distributions

- A number of different statistical distributions exist, which may be used to describe and analyse different types of problems.

- Today, we consider only <u>discrete</u> distributions:

  - The binomial distribution
  - The hypergeometric distribution
  - The Poisson distribution

## The Binomial distribution

- An experiment with two outcomes, "success" or "failure", is repeated (independent repetitions).

- $X$ is <u>the number of successes</u> after $n$ repetitions.

## The Binomial distribution

- An experiment with two outcomes, "success" or "failure", is repeated (independent repetitions).

- $X$ is the number of successes after $n$ repetitions.

- Then $X$ follows a binomial distribution:

$$X \sim B(n,p)$$

  - $n$: number of repetitions
  - $p$: probability of success in each repetition

# The density function of the binomial distribution

The probability of $x$ successes:

$$f(x;n,p) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

## Example: Binomial distribution

Suppose that $X \sim B(4, p)$, i.e. $n = 4$. Find the probability of 3 successes.

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- Probability of 3 successes: $P(X = 3)$.
- Three successes can be obtained in four "ways":
  SSSF, SSFS, SFSS, FSSS.

## Example: Binomial distribution

Suppose that $X \sim B(4, p)$, i.e. $n = 4$. Find the probability of 3 successes.

- Probability of 3 successes: $P(X = 3)$.

- Three successes can be obtained in four "ways":
  SSSF, SSFS, SFSS, FSSS.

- Thus,
$$\binom{n}{x} = \binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 1} = 4,$$

  and
$$P(X = 3) = 4p^3(1-p).$$

# Simulation from a binomial distribution

```
## Probability of success
p <- 0.1

## Number of repetitions
nRepeat <- 30

## Simulate Bernoulli experiment 'nRepeat' times
tmp <- sample(c(0,1), size = nRepeat, prob = c(1-p,p), replace = TRUE)

# Compute 'x'
sum(tmp)

## Or: Use the binomial distribution simulation function
rbinom(1, size = 30, prob = p)
```

# Example: Fair dice

```r
# Number of simulated realizations (sample size)
n <- 30

# n independent random draws from the set (1,2,3,4,5,6)
# with equal probability for each outcome
xFair <- sample(1:6, size = n, replace = TRUE)

# Count the number of six'es
sum(xFair == 6)

## Do the same using 'rbinom()' instead
rbinom(n = 1, size = 30, prob = 1/6)
```

# Example 1

In the call center of a phone company, customer satisfaction is an issue. It is especially important that when errors/faults occur, they are corrected within the same day.

Assume that six errors occur, and that the probability of any error being corrected within the same day is 70%. **What is the probability that all six errors are corrected within the same day that they occurred?**

## Example 1

In the call center of a phone company, customer satisfaction is an issue. It is especially important that when errors/faults occur, they are corrected within the same day.

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  A binomial distribution with $n = 6$ and $p = 0.7$.

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  $\underline{P(X = 6) = f(6; 6, 0.7)}$

# Example 1: Binomial Probability

```
##########################################################
### What is the probability that all six errors are   ###
### corrected within the same day that they occurred? ###
##########################################################

# A binomial distribution with £n = 6£ and £p= 0.7£

# dbinom gives the density, pbinom gives the distribution function,
# qbinom gives the quantile function and rbinom generates random deviates.

##One way to obtain the result for P(X=6)
dbinom(6,6,0.7)

##Another way is to compute 1- P(X<=5)
1-pbinom(5,6,0.7)
```

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# The hypergeometric distribution

- Again, $X$ is the the number of successes, but now *without* replacement when repeating.

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- Again, $X$ is the the number of successes, but now *without* replacement when repeating.

- $X$ follows the hypergeometric distribution

$$X \sim H(n, a, N)$$

  - $n$ is the number of draws (repetitions)
  - $a$ is the number of successes in the population
  - $N$ is the number of elements in the (entire) population

# The hypergeometric distribution

- The probability of getting $x$ successes is

$$f(x;n,a,N) \,=\, P(X=x) \,=\, \frac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}}$$

  - $n$ is the number of draws (repetitions)
  - $a$ is the number of successes in the population
  - $N$ is the number of elements in the (entire) population

## Example 2

In a shipment of 10 harddisks, 2 of them have small scratches.

A random sample of 3 harddisks is taken. **What is the probability that at least 1 of them has scratches?**

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  Number of harddisks with scratches in the random sample.

- Step 2) What is the distribution of $X$?
  
  A hypergeometric distribution with $n = 3$, $a = 2$, $N = 10$.

- Step 3) Which probability should be computed?

## Example 2

In a shipment of 10 harddisks, 2 of them have small scratches.

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- Step 2) What is the distribution of $X$?

  A hypergeometric distribution with $n = 3$, $a = 2$, $N = 10$.

- Step 3) Which probability should be computed?

  $P(X \geq 1) = 1 - P(X = 0) = 1 - f(0; 3, 2, 10)$

# Binomial vs. hypergeometric

- The binomial distribution is used to analyse samples with replacement.

- The hypergeometric distribution is used to analyse samples without replacement.

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# The Poisson distribution

- The Poisson distribution is often used as a distribution (model) for counts, which do not have a natural upper bound.

- The Poisson distribution is often characerized by its *intensity*, which is on the form "number/unit", and often denoted $\lambda$.

# The Poisson distribution

$$X \sim Po(\lambda)$$

The density function:

$$f(x) = P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

The distribution function:

$$F(x) = P(X \leq x)$$

# Example 3

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**What is the probability that at most two patients are hospitalized in Copenhagen due to air pollution on any given day?**

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  The number of patients on a given day.

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  A Poisson distribution with $\lambda = 0.3$.

## Example 3

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  A Poisson distribution with $\lambda = 0.3$.

- Step 3) Which probability should be computed?

  $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

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| R | Name |
|---|---|
| binom | Binomial |
| hyper | Hypergeometric |
| pois | Poisson |

d $f(x)$, probability density function

p $F(x)$, cumulative distribution function

r random numbers from the distribution

q quantiles of the distribution ("inverse" of $F(x)$)

**Example:** The binomial distribution, $P(X \leq 5) = F(5; 10, 0.6)$

```
pbinom(q = 5, size = 10, prob = 0.6)
```

[1] 0.37

```
# Get help with:
?pbinom
```

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# Mean (expectation, expected value)

Mean of a discrete random variable, Definition 2.13:

## Definition

$$\mu = E(X) = \sum_{\text{all } x} x f(x)$$

- The *"true mean"* of X (as opposed to the sample mean).
- Expresses the "center" of the distribution of $X$.

# Example: Mean of a throw with a fair dice

$$\mu = E(X)$$
$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$
$$= 3.5$$

# Link to sample mean - learning from simulations

```r
# Number of simulated realizations (sample size)
n <- 30

# Sample independently from the set (1,2,3,4,5,6)
# with equal probability of outcomes
xFair <- sample(1:6, size = n, replace = TRUE)

# Compute the sample mean
mean(xFair)
```

[1] 3.3

# Asymptotics, increasing the sample size

The more observations (the larger the sample size), the closer you get to the true mean:

$$\lim_{n \to \infty} \hat{\mu} = \mu$$

- Try increasing $n$ in the simulations in R.

## Variance

Variance of a discrete random variable, Definition 2.16:

### Definition

$$\sigma^2 = \text{Var}(X) = \sum_{\text{all } x} (x - \mu)^2 f(x)$$

- Measures average dispersion/spread.
- The "true variance" of $X$ (as opposed to the sample variance).

# Example: Variance of a throw with a fair dice

$$\sigma^2 = E[(X - \mu)^2]$$
$$= (1 - 3.5)^2 \cdot \frac{1}{6} + (2 - 3.5)^2 \cdot \frac{1}{6} + (3 - 3.5)^2 \cdot \frac{1}{6}$$
$$+ (4 - 3.5)^2 \cdot \frac{1}{6} + (5 - 3.5)^2 \cdot \frac{1}{6} + (6 - 3.5)^2 \cdot \frac{1}{6}$$
$$\approx 2.92$$

# Link to sample variance - learning from simulations

```
# Number of simulated realizations (sample size)
n <- 30

# Sample independently from the set (1,2,3,4,5,6)
# with equal probability of outcomes
xFair <- sample(1:6, size = n, replace = TRUE)

# Compute the sample variance
var(xFair)
```

[1] 2.4

# Mean and variance of specific discrete distributions

## The binomial distribution

- Mean:
  $\mu = n \cdot p$

- Variance:
  $\sigma^2 = n \cdot p \cdot (1 - p)$

# Mean and variance of specific discrete distributions

## The hypergeometric distribution

- Mean:
  $$\mu = n \cdot \frac{a}{N}$$

- Variance:
  $$\sigma^2 = \frac{n \cdot a \cdot (N-a) \cdot (N-n)}{N^2 \cdot (N-1)}$$

# Mean and variance of specific discrete distributions

The poisson distribution

- Mean:
  $\mu = \lambda$

- Variance:
  $\sigma^2 = \lambda$

## Agenda

1. Random variables and density functions
2. Distribution functions
3. Specific (discrete) distributions I: The binomial
   - Example 1
4. Specific distributions II: The hypergeometric
   - Example 2
5. Specific distributions III: The Poisson
   - Example 3
6. Distributions in R
7. Mean and variance of discrete distributions