02465: Introduction to reinforcement learning and control

Linearization and iterative LQR

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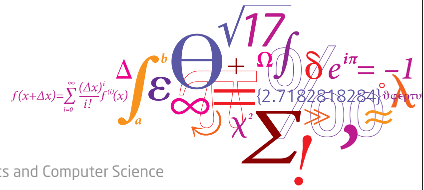


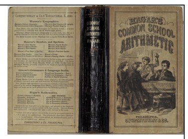
Lecture Schedule

Syllabus: `https://02465material.pages.compute.dtu.dk/02465public`
Help improve lecture by giving feedback on DTU learn

Housekeeping

- Old exam sets online
- Most of the feedback for project 1 is online on DTU Learn
  - The rest will be available in a few days

A bit of analysis



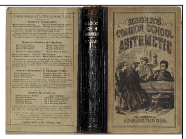
- Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a well-behaved function
- The **gradient** is defined as:

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\boldsymbol{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\boldsymbol{x}) \end{bmatrix}$$

- The **Hessian** is defined as

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

More analysis



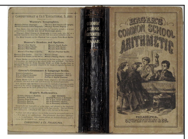
- Let $\boldsymbol{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a well-behaved multi-variate function defined as

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

- The **Jacobian matrix** is defined as:

$$\boldsymbol{J_f}(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Approximations



- Given the gradient and Hessian we can approximate f around $\boldsymbol{x}$

$$f(\mathbf{x} + \Delta) \approx f(\mathbf{x}) + \nabla f(\mathbf{x})^{\mathrm{T}} \Delta + \frac{1}{2} \Delta^{\mathrm{T}} \mathbf{H}(\mathbf{x}) \Delta$$

- A similar expression can be obtained for a multi-variate $\boldsymbol{f}$:

$$\mathbf{f}(\mathbf{x} + \boldsymbol{\Delta}) \approx \mathbf{f}(\boldsymbol{x}) + \mathbf{J}_{\mathbf{f}}(\boldsymbol{x}) \boldsymbol{\Delta}$$

Fundamental relations that are the basis for gradient descent, many higher-order optimization methods and all sorts of ML

From last time: The Linear-quadratic regulator

- For $k = 0, 1, \ldots, N-1$

$$
\begin{aligned}
x_{k+1} &= f_k(x_k, u_k, w_k) = A_k x_k + B_k u_k, \\
g_k(x_k, u_k, w_k) &= \frac{1}{2} x_k^\top Q_k x_k + \frac{1}{2} u_k^\top R_k u_k, \\
g_N(x_k) &= \frac{1}{2} x_N^\top Q_N x_N
\end{aligned}
$$

- The accumulated cost is:

$$J_{\boldsymbol{u}}(\boldsymbol{x}_0) = g_N(\boldsymbol{x}_N) + \sum_{k=0}^{N-1} g_k(\boldsymbol{x}_k, \boldsymbol{u}_k)$$

- We put this into the dynamical programming algorithm and...

Apply dynamical programming:

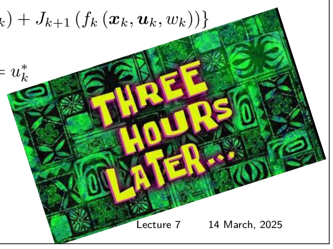
- Define $V_N \equiv Q_N$ and initialize:

$$J_N^*(\boldsymbol{x}_N) = \frac{1}{2} \boldsymbol{x}_N^T Q_N \boldsymbol{x}_N = \frac{1}{2} \boldsymbol{x}_N^T V_N \boldsymbol{x}_N$$

- DP iteration (start at $k = N-1$)

$$J_k(\boldsymbol{x}_k) = \min_{\boldsymbol{u}_k} \mathbb{E}_{w_k} \{ g_k(\boldsymbol{x}_k, \boldsymbol{u}_k, w_k) + J_{k+1}(f_k(\boldsymbol{x}_k, \boldsymbol{u}_k, w_k)) \}$$

- Remember to store optimal u_k^* as $\pi_k(x_k) = u_k^*$



LQR, simplified form

This gives the controller:

1. $V_N = Q_N$
2. $L_k = -(R_k + B_k^T V_{k+1} B_k)^{-1} (B_k^T V_{k+1} A_k)$
3. $V_k = Q_k + L_k^T R_k L_k + (A_k + B_k L_k)^T V_{k+1} (A_k + B_k L_k)$
4. $\boldsymbol{u}_k^* = L_k \boldsymbol{x}_k$
5. $J_k^*(\boldsymbol{x}_k) = \frac{1}{2} \boldsymbol{x}_k^T V_k \boldsymbol{x}_k$

Double Integrator Example

- True dynamics

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \boldsymbol{u}(t) \tag{1}$$

- **Euler discretization** using $\Delta = 1$ System evolves according to:

$$\boldsymbol{x}_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{=A} \boldsymbol{x}_k + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=B} \boldsymbol{u}_k$$

- Quadratic cost function:

$$J(\boldsymbol{x}_0) = \sum_{k=0}^{N} \boldsymbol{x}_k^\top Q \boldsymbol{x}_k + \frac{1}{2} u_k^2$$

- Where:

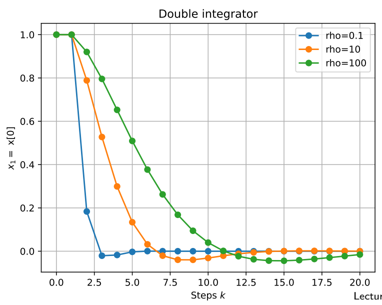
$$Q_k = Q_N = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1$$

Exponential integrator

- Apply discrete LQR
- Simulate starting in $\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ using policy

$$\pi_k(\boldsymbol{x}_k) = L_k \boldsymbol{x}_k$$

- What about the true system $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$?



The most general form of LQR

- General dynamics:

$$\boldsymbol{x}_{k+1} = A_k \boldsymbol{x}_k + B_k \boldsymbol{u}_k + \boldsymbol{d}_k$$

- General quadratic cost:

$$
\begin{aligned}
c_k(\boldsymbol{x}_k, \boldsymbol{u}_k) &= \frac{1}{2} \boldsymbol{x}_k^T Q_k \boldsymbol{x}_k + \frac{1}{2} \boldsymbol{u}_k^T R_k \boldsymbol{u}_k + \boldsymbol{u}_k^T H_k \boldsymbol{x}_k + \boldsymbol{q}_k^T \boldsymbol{x}_k + \boldsymbol{r}_k^T \boldsymbol{u}_k + q_k \\
c_N(\boldsymbol{x}_k) &= \frac{1}{2} \boldsymbol{x}_k^T Q_N \boldsymbol{x}_k + \boldsymbol{q}_N^T \boldsymbol{x}_k + q_N
\end{aligned}
$$

General discrete LQR algorithm

How to start living in luxury and never work again!

$\cdots(V_{k+1}+\mu I)\cdots$

1. $V_N = Q_N$; $\boldsymbol{v}_N = \boldsymbol{q}_N$; $v_N = q_N$
2. $$L_k = -S^{-1}_{\boldsymbol{uu},k} S_{\boldsymbol{ux},k} \qquad S_{\boldsymbol{u},k} = \boldsymbol{r}_k + B_k^T \boldsymbol{v}_{k+1} + B_k^T V_{k+1} \boldsymbol{d}_k$$
   $$\boldsymbol{l}_k = -S^{-1}_{\boldsymbol{uu},k} S_{\boldsymbol{u},k} \qquad S_{\boldsymbol{uu},k} = R_k + B_k^T V_{k+1} B_k$$
   $$S_{\boldsymbol{ux},k} = H_k + B_k^T V_{k+1} A_k.$$
3. $$V_k = Q_k + A_k^T V_{k+1} A_k - L_k^T S_{\boldsymbol{uu},k} L_k$$
   $$\boldsymbol{v}_k = \boldsymbol{q}_k + A_k^T(\boldsymbol{v}_{k+1} + V_{k+1}\boldsymbol{d}_k) + S_{\boldsymbol{ux},k}^T \boldsymbol{l}_k$$
   $$v_k = v_{k+1} + q_k + \boldsymbol{d}_k^T \boldsymbol{v}_{k+1} + \frac{1}{2}\boldsymbol{d}_k^T V_{k+1} \boldsymbol{d}_k + \frac{1}{2}\boldsymbol{l}_k^T S_{\boldsymbol{u},k}$$
4. $\boldsymbol{u}_k^* = \boldsymbol{l}_k + L_k \boldsymbol{x}_k$
5. $J_k(\boldsymbol{x}_k) = \frac{1}{2}\boldsymbol{x}_k^T V_k \boldsymbol{x}_k + \boldsymbol{v}_k^T \boldsymbol{x}_k + v_k.$

(more seriously μ is a regularization term: $\mu \to \infty \Rightarrow \boldsymbol{u} \to 0$)

Quiz: LQR

Which one of the following statements is correct?

a. Control problems where the continuous-time dynamics takes the form $\ddot{x} = a\dot{x} + bx + c + u$ falls outside the scope of the linear quadratic regulator

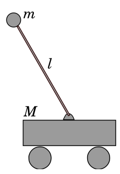
b. The linear-quadratic regulator is an example of model-free control

c. In a linear-quadratic control problem of the form $x_{k+1} = Ax_k + Bu_k$, the matrices A and B must both be square.

d. The cost-functions suitable for a linear-quadratic regulator can potentially produce negative values

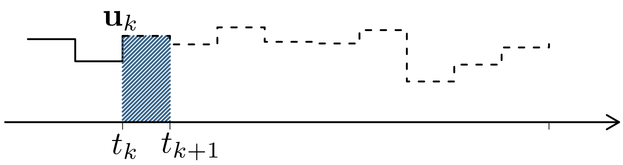
e. Don't know.

Controlling non-linear systems: Cartpole



- Continuous coordinates $\boldsymbol{x}(t) = \begin{bmatrix} x(t) & \dot{x}(t) & \theta(t) & \dot{\theta}(t) \end{bmatrix}$
- Action u is one-dimensional; the force applied to cart

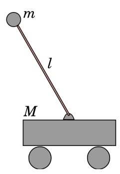
Discretization



- Choose grid size N: $t_0, t_1, \dots, t_N = t_F$, $t_{k+1} - t_k = \Delta$
- $\boldsymbol{x}_k = \boldsymbol{x}(t_k), \boldsymbol{u}_k = \boldsymbol{u}(t_k)$
- Eulers method $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta f(\boldsymbol{x}_k, \boldsymbol{u}_k)$
- Discretized dynamics will have the form:

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_k(\boldsymbol{x}_k, \boldsymbol{u}_k)$$

Cartpole cost function



- We also apply a variable transformation:

$$\phi_x : \begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix} \mapsto \begin{bmatrix} x & \dot{x} & \sin(\theta) & \cos(\theta) & \dot{\theta} \end{bmatrix}. \qquad (2)$$

- The cost function is of the form:

$$c(\boldsymbol{x}_k, \boldsymbol{u}_k) = \frac{1}{2}\left(\boldsymbol{x} - \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}\right)^\top Q \left(\boldsymbol{x} - \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}\right) + \frac{1}{2}\|\boldsymbol{u}_k\|^2$$

`lecture_06_linearize.py`

Controlling a non-linear system

- **We know** how to solve a linear/quadratic control problems of the form

$$\boldsymbol{x}_{k+1} = A_k \boldsymbol{x}_k + B_k \boldsymbol{u}_k + \boldsymbol{d}_k$$
$$c_k(\boldsymbol{x}_k, \boldsymbol{u}_k) = \frac{1}{2}\boldsymbol{x}_k^\top Q \boldsymbol{x}_k + \frac{1}{2}\boldsymbol{u}_k^\top R \boldsymbol{u}_k + \cdots$$

- **How** can we use that to solve a problem with non-linear dynamics?

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_k(\boldsymbol{x}_k, \boldsymbol{u}_k)$$
$$\boldsymbol{c}_k(\boldsymbol{x}_k, \boldsymbol{u}_k) = \cdots$$

Solution: Linearization!

Assume a general dynamics:

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_k(\boldsymbol{x}_k, \boldsymbol{u}_k), \quad c(\boldsymbol{x}_k, \boldsymbol{u}_k)$$

Assume system is near $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{u}}$. Expand using **Jacobians**

$$\boldsymbol{f}_k(\boldsymbol{x}_k, \boldsymbol{u}_k) \approx \boldsymbol{f}_k(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) + \underbrace{\frac{\partial \boldsymbol{f}_k}{\partial \boldsymbol{x}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}_{A_k}(\boldsymbol{x}_k - \bar{\boldsymbol{x}}) + \underbrace{\frac{\partial \boldsymbol{f}_k}{\partial \boldsymbol{u}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}_{B_k}(\boldsymbol{u}_k - \bar{\boldsymbol{u}})$$

Simplifies to:

$$\boldsymbol{x}_{k+1} = A_k \boldsymbol{x}_k + B_k \boldsymbol{u}_k + \boldsymbol{f}_k(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) - A_k \bar{\boldsymbol{x}} - B_k \bar{\boldsymbol{u}}$$

Linearization and iLQR

Algorithm 1 Linearized LQR

Require: Given a problem horizon N, and an expansion point $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$ corresponding to where the system should be
Compute $A_k, B_k, \boldsymbol{d}_k$ by expansion
Cost function is the same as usual because it is already quadratic
Use LQR, with dynamics A_k, B_k, d_k and cost matrices $Q_k, R_k, \boldsymbol{q}_k$ to obtain controller $L_k, \boldsymbol{l}_k$ for $k = 0, \dots, N-1$.
In a state $\boldsymbol{x}_k$, the control law is $\boldsymbol{u}_k^* = \boldsymbol{l}_k + L_k \boldsymbol{x}_k$

- Select expansion point $\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}$ as desired state
- Usually $A_k = A, B_k = B$ so just choose a large N and use $L_0, \boldsymbol{l}_0$

`lecture_06_linearize_b.py`

Quiz: Linearized LQR?

Which one of the following statements is **correct**?

a. We should apply Exponential Integration to the linearized dynamics $A_k (= J_{\boldsymbol{x}} \boldsymbol{f}_k(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}))$ and B_k before applying LQR

b. Assuming Δ is small enough, the error incurred by Euler discretization can be managed.

c. Assuming we plan on a sufficiently long horizon, the linear approximation to the dynamics does not result in major issues

d. This is a computationally inefficient method compared to e.g. Direct control

e. Don't know

(Note: Quiz changed from lecture due to double-negation of answers being beyond my abilities; In the lecture, option a was actually the right option, but I read it incorrectly and thought I had an error. In this formulation quiz, option b is correct)

Fixing linearization method

- **Problem:** The system may be far from $\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}$ giving a poor approximation
- **Idea:** Select expansion points $\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}$ near current trajectory $\boldsymbol{x}_k, \boldsymbol{u}_k$
- **How?**
  - Start with initial guess $\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k$ **(nominal trajectory)**
  - Approximate around this guess
  - Use LQR on approximation to get initial control law
  - Simulate trajectory based on this control law
  - Use the trajectory as a new guess and repeat

LQR Tracking around Nonlinear Trajectory

Given initial guess $\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k$ (**nominal trajectory**) for $k = 1, 2, \dots, N-1$

$$\boldsymbol{x}_{k+1} \approx \underbrace{\boldsymbol{f}_k(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k)}_{\bar{\boldsymbol{x}}_{k+1}} + \underbrace{\frac{\partial \boldsymbol{f}_k}{\partial \boldsymbol{x}}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k)}_{A_k} \underbrace{(\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k)}_{\delta \boldsymbol{x}} + \underbrace{\frac{\partial \boldsymbol{f}_k}{\partial \boldsymbol{u}}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k)}_{B_k} \underbrace{(\boldsymbol{u}_k - \bar{\boldsymbol{u}}_k)}_{\delta \boldsymbol{u}}$$

Introduce new variables signifying deviation around the **nominal trajectory**:

$$\delta \boldsymbol{x}_k = \boldsymbol{x}_k - \bar{\boldsymbol{x}}_k, \quad \delta \boldsymbol{u}_k = \boldsymbol{u}_k - \bar{\boldsymbol{u}}_k.$$

Back-substituting gives:

$$\delta \boldsymbol{x}_{k+1} = A_k \delta \boldsymbol{x}_k + B_k \delta \boldsymbol{u}_k$$

Expansion of the cost function

We then expand the cost-function around: $\boldsymbol{z}_k = \begin{bmatrix} \boldsymbol{x}_k \\ \boldsymbol{u}_k \end{bmatrix}$ and $\bar{\boldsymbol{z}} = \begin{bmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{u}} \end{bmatrix}$:

$$c_k(\boldsymbol{x}_k, \boldsymbol{u}_k) \approx c_k(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) + (\nabla_{\boldsymbol{z}} c_k(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}))^\top (\boldsymbol{z}_k - \bar{\boldsymbol{z}}) + \frac{1}{2}(\boldsymbol{z}_k - \bar{\boldsymbol{z}})^\top H_{\bar{\boldsymbol{z}}}(\boldsymbol{z}_k - \bar{\boldsymbol{z}})$$

Multiplying out all the terms gives a quadratic approximation in the δ-coordinates

$$\begin{aligned} c_k &= c_k(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) \\ c_{\boldsymbol{x},k} &= \nabla_{\boldsymbol{x}} c_k(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}), \quad c_{\boldsymbol{u},k} = \nabla_{\boldsymbol{u}} c_k(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) \\ c_{\boldsymbol{xx},k} &= H_{\boldsymbol{x}} c_k(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}), \quad c_{\boldsymbol{uu},k} = H_{\boldsymbol{u}} c_k(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) \\ c_{\boldsymbol{ux},k} &= J_{\boldsymbol{x}} \nabla_{\boldsymbol{u}} c_k(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) \end{aligned}$$

Expansion of the cost function

all in all we get a quadratic cost function:

$$
\begin{aligned}
c_k(\delta \boldsymbol{x}_k, \delta \boldsymbol{u}_k) &= \frac{1}{2} \delta \boldsymbol{x}_k^\top c_{\boldsymbol{xx},k} \delta \boldsymbol{x}_k + c_{\boldsymbol{x},k}^\top \delta \boldsymbol{x}_k \\
&\quad + \frac{1}{2} \delta \boldsymbol{u}_k^\top c_{\boldsymbol{uu},k} \delta \boldsymbol{u}_k + c_{\boldsymbol{u},k}^\top \delta \boldsymbol{u}_k + \delta \boldsymbol{u}_k^\top c_{\boldsymbol{ux},k} \delta \boldsymbol{x}_k + c_k \\
c_N(\delta \boldsymbol{x}_N) &= \frac{1}{2} \delta \boldsymbol{x}_N^\top c_{\boldsymbol{xx},N} \delta \boldsymbol{x}_N + c_{\boldsymbol{x},N}^\top \delta \boldsymbol{x}_N + c_N
\end{aligned}
$$

Linearized solution to actual controls

- Put linearized problem into LQR
- Once problem is solved, new control inputs obey

$$\delta \boldsymbol{u}_k^* = \boldsymbol{l}_k + L_k \delta \boldsymbol{x}_k$$

- Rearranging

$$(\boldsymbol{u}_k^* - \bar{\boldsymbol{u}}_k) = \boldsymbol{l}_k + L_k(\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k)$$

- Or

$$\boldsymbol{u}_k^* = \bar{\boldsymbol{u}}_k + \boldsymbol{l}_k + L_k(\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k)$$

Basic iLQR Algorithm

Algorithm 2 Basic iLQR

Require: Given initial state $\boldsymbol{x}_0$

Set $\bar{\boldsymbol{x}}_k = \boldsymbol{x}_0$, $\bar{\boldsymbol{u}}_k = \boldsymbol{0}$ (or a random vector), $L_k = \boldsymbol{0}$ and $\boldsymbol{l}_k = \boldsymbol{0}$

$\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k \leftarrow \textsc{Forward-Pass}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k, L_k, \boldsymbol{l}_k)$ ▷ Compute initial nominal trajectory using eq. (17.10) .

for $i = 0$ to a pre-specified number of iterations **do**

  $A_k, B_k, c_k, c_{\boldsymbol{x},k}, c_{\boldsymbol{u},k}, c_{\boldsymbol{xx},k}, c_{\boldsymbol{ux},k}, c_{\boldsymbol{uu},k} \leftarrow \textsc{Get-derivatives}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k)$

  $L_k, \boldsymbol{l}_k \leftarrow \textsc{Backward-Pass}(A_k, B_k, c_k, c_{\boldsymbol{x},k}, c_{\boldsymbol{u},k}, c_{\boldsymbol{xx},k}, c_{\boldsymbol{ux},k}, c_{\boldsymbol{uu},k}, \mu)$

  $J^{(i)} \leftarrow \textsc{Cost-of-trajectory}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k)$

  $\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k \leftarrow \textsc{Forward-Pass}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k, L_k, \boldsymbol{l}_k)$

end for

Compute control law $\pi_k(\boldsymbol{x}_k) = \bar{\boldsymbol{u}}_k + \bar{\boldsymbol{l}}_k + L_k(\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k)$

return $\{\pi_k\}_{k=0}^{N-1}$

function $\textsc{Forward-pass}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k, L_k, \boldsymbol{l}_k)$ ▷ Forward-simulation of dynamics

  Set $\boldsymbol{x}_0 = \bar{\boldsymbol{x}}_0$

  for all $k = 0, \dots, N-1$ **do**

    $\boldsymbol{u}_k^* \leftarrow \bar{\boldsymbol{u}}_k + L_k(\boldsymbol{x}\_k - \bar{\boldsymbol{x}}_k) + \boldsymbol{l}_k$ ▷ see eq. (17.16)

    $\boldsymbol{x}_{k+1} \leftarrow f_k(\boldsymbol{x}_k, \boldsymbol{u}_k^*)$

  end for

  return $\boldsymbol{x}_k, \boldsymbol{u}_k^*$

end function

function $\textsc{Backward-pass}(A_k, B_k, c_k, c_{\boldsymbol{x},k}, c_{\boldsymbol{u},k}, c_{\boldsymbol{xx},k}, c_{\boldsymbol{ux},k}, c_{\boldsymbol{uu},k}, \mu)$ eq. (17.14)

  Compute $L_k, \boldsymbol{l}_k$ using dLQR with μ, algorithm 22 ▷ Obtain control law

end function

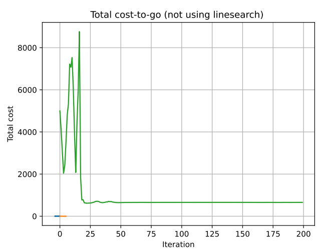
function $\textsc{Cost-of-trajectory}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k)$

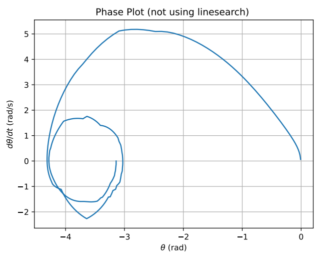
  return $c_N(\bar{\boldsymbol{x}}_N) + \sum_{k=0}^{N-1} c_k(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k)$

end function

Basic iLQR: Pendulum swingup task

Pendulum starts at $\theta = \pi$ and $\dot{\theta} = 0$ and controller tries to swing it up $\theta = 0$





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`lecture_06_pendulum_bilqr_ubar`

Iterative LQR

Basic iLQR is not very numerically stable. iLQR adds two ideas:

- Use regularization to stabilize the discrete LQR algorithm (μ)
- Search for policies that are **close** to the old ones. Recall:

$$\boldsymbol{u}_k^* = \overline{\boldsymbol{u}}_k + l_k + L_k(\boldsymbol{x}_k - \overline{\boldsymbol{x}}_k)$$

- Since $(\boldsymbol{x}_k - \overline{\boldsymbol{x}}_k)$ assumed small (and L_k stabilized by μ), decreasing l_k means new control closer to old.
- Specifically, introduce $0 \leq \alpha \leq 1$

$$\boldsymbol{u}_k^* = \overline{\boldsymbol{u}}_k + \alpha l_k + L_k(\boldsymbol{x}_k - \overline{\boldsymbol{x}}_k)$$

Iterative LQR Procedure

- Initialize regularization parameter to a fairly low value μ
- In the forward pass try smaller and smaller changes to trajectory (α-values)
- For each α-value check if the cost $J^{(i)}$ decreases relative to $J^{(i-1)}$. If so, *accept* this α and decrease the regularization parameter μ by a small amount
- If no α-value works, increase the regularization parameter μ by a small amount

iLQR Algorithm

Algorithm 3 iLQR

Require: Given initial state $\boldsymbol{x}_0$

$\mu_{\min} \leftarrow 10^{-6}$, $\mu_{\max} \leftarrow 10^{10}$, $\mu \leftarrow 1$, $\Delta_0 \leftarrow 2$ and $\Delta \leftarrow \Delta_0$
Initialize $\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k$ as before
for $i = 0$ to a pre-specified number of iterations **do**
  $A_k, B_k, c_k, c_{\boldsymbol{x},k}, c_{\boldsymbol{u},k}, c_{\boldsymbol{xx},k}, c_{\boldsymbol{ux},k}, c_{\boldsymbol{uu},k} \leftarrow \text{GET-DERIVATIVES}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k)$
  $L_k, \boldsymbol{l}_k \leftarrow \text{BACKWARD-PASS}(A_k, B_k, c_k, c_{\boldsymbol{x},k}, c_{\boldsymbol{u},k}, c_{\boldsymbol{xx},k}, c_{\boldsymbol{ux},k}, c_{\boldsymbol{uu},k}, \mu)$
  $J' \leftarrow \text{COST-OF-TRAJECTORY}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k)$
  for $\alpha = 1$ to a very low value **do**
    $\hat{\boldsymbol{x}}_k, \hat{\boldsymbol{u}}_k \leftarrow \text{FORWARD-PASS}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k, L_k, \boldsymbol{l}_k, \alpha)$
    $J^{\text{new}} \leftarrow \text{COST-OF-TRAJECTORY}(\hat{\boldsymbol{x}}_k, \hat{\boldsymbol{u}}_k)$
    if $J^{\text{new}} < J'$ **then**
      if $\frac{1}{J'}|J^{\text{new}} - J'| <$ a small number **then**
        Method has converged, terminate outer loop and return
      end if
      $J' \leftarrow J^{\text{new}}$
      $\bar{\boldsymbol{x}}_k \leftarrow \hat{\boldsymbol{x}}_k$ and $\bar{\boldsymbol{u}}_k \leftarrow \hat{\boldsymbol{u}}_k$
      α **accepted**: Update Δ and μ using eq. (17.19) ▷ Reduce regularization
      Break loop over α
    end if
  end for
  if No α-value was accepted then
    Update Δ and μ using eq. (17.18) ▷ Increase regularization
  end if
end for
Compute controller $\{\pi_k\}_{k=0}^{N-1}$ as before from $L_k, \boldsymbol{l}_k$

`lecture_06_pendulum_ilqr_L`

`lecture_06_pendulum_ilqr_ubar`

`lecture_06_cartpole`

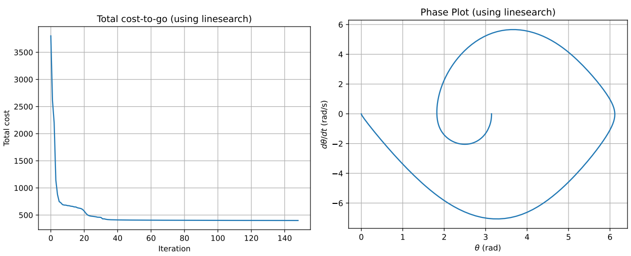
Iterative LQR

Given $\boldsymbol{x}_0$ and f_k, c_k, c_N; initialize $\overline{\boldsymbol{u}}_k$

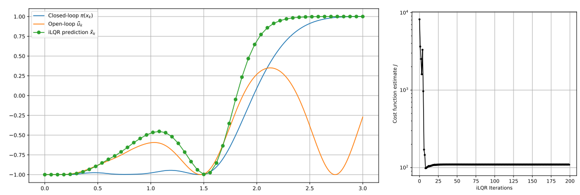
- Simulate $\overline{\boldsymbol{x}}_k$ and compute matrices for linearized problem as well as cost $J_{\overline{\boldsymbol{u}}}(\overline{x}_0)$
- Solve for $\delta \boldsymbol{u}_k^*$ using regularization μ
- Loop over α starting at $\alpha = 1$
  - Obtain controls $\boldsymbol{u}_k^*$ with α (see [TET12, Eq.(12)])
$$\boldsymbol{u}_k^* = \overline{\boldsymbol{u}}_k + \alpha l_k + L_k(\boldsymbol{x}_k - \overline{\boldsymbol{x}}_k) \qquad (7)$$
  - If cost $J_{\boldsymbol{u}^*}(\boldsymbol{x}_0) < J_{\overline{\boldsymbol{u}}}(\overline{x}_0)$ accept α/decrease μ
- (On failure to find α increase regularization μ)

Full iLQR: Pendulum swingup task

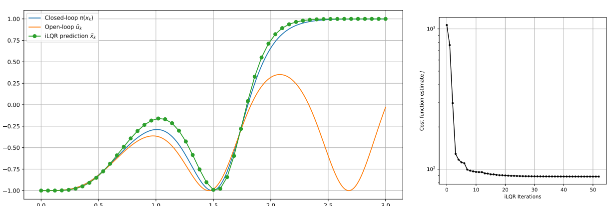
Pendulum starts at $\theta = \pi$ and $\dot{\theta} = 0$ and controller tries to swing it up $\theta = 0$



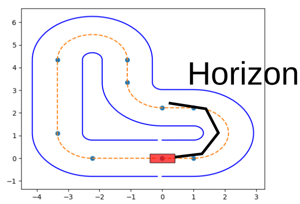
Basic iLQR Algorithm Example



iLQR Algorithm Example



Model Predictive Control



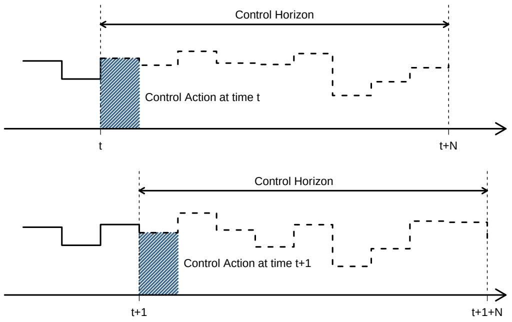
Model-predictive control/receding horizon control

Iteratively solve optimization problem on short time scale

- Long horizon equals great computation, uncertainty
- Solving problem on short horizon often sufficient

Model Predictive Control

- Solve control problem $\boldsymbol{u}_0, \dots, \boldsymbol{u}_{N-1}$ for a small number of steps N
- Apply control $\boldsymbol{u}_0$ from first step
- Repeat



Appendix: MPC can be understood as dynamical programming

DP applied in the starting state (**optimal**):

$$J^*(x_0) = \min_{u_0} \mathbb{E}\left[J_1^*(x_1) + g_0(x_0, u_0, w_0)\right]$$

d-step rollout of DP (**optimal**):

$$J^*(x_0) = \min_{\mu_0,\dots,\mu_{d-1}} \mathbb{E}\left[J_d^*(x_{k+d}) + \sum_{k=0}^{d-1} g_k(x_k, \mu_k(x_k), w_k)\right]$$

Deterministic simplification for control (**optimal**):

$$J^*(\boldsymbol{x}_0) = \min_{\boldsymbol{u}_0,\dots,\boldsymbol{u}_{d-1}} \left[J_d^*(\boldsymbol{x}_{k+d}) + \sum_{k=0}^{d-1} c_k(\boldsymbol{x}_k, \boldsymbol{u}_k)\right]$$

- **MPC: Approximate** $J_d^*(\boldsymbol{x}_{k+d})$ and just plan on d-horizon
- Re-plan at each step