02465: Introduction to reinforcement learning and control

Linear-quadratic problems in control

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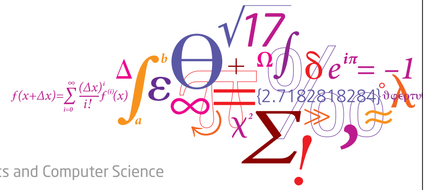


DTU Compute
Department of Applied Mathematics and Computer Science

Lecture Schedule

Syllabus: `https://02465material.pages.compute.dtu.dk/02465public`
Help improve lecture by giving feedback on DTU learn

Reading material:

- [Her24, Chapter 16]

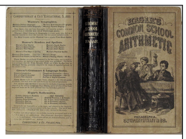
Learning Objectives

- Linear-quadratic regulator (LQR)
- Derivation of the LQR from DP
- Applications and variations

Recap
Practicals

- Project evaluations will be ready in about a week
- Programming evaluations see `https://02465material.pages.compute.dtu.dk/02465public/projects/project1.html`
- Part 2:
  - Less programming
  - A bit more emphasis on linear algebra; don't be afraid to write short answers if they are correct.
  - Be inspired by existing examples

Recap
Useful linear algebra



- A matrix A is **positive semi-definite** if it is symmetric and $\boldsymbol{x}^\top A\boldsymbol{x} \geq 0$ for all $\boldsymbol{x}$
  - This means A behaves like a positive number: $ax^2 \geq 0$.
- if A is a symmetric matrix then:

$$\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} = \frac{1}{2}\left(\mathbf{x} + \mathbf{A}^{-1}\mathbf{b}\right)^T\mathbf{A}\left(\mathbf{x} + \mathbf{A}^{-1}\mathbf{b}\right) - \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}$$

- This allows us to quickly find minimum

Recap
Recap: Dynamical programming algorithm

The Dynamical Programming algorithm

For every initial state x_0, the optimal cost $J^*(x_0)$ is equal to $J_0(x_0)$, and optimal policy π^* is $\pi^* = \{\mu_0, \dots, \mu_{N-1}\}$, computed by the following algorithm, which proceeds backward in time from $k = N$ to $k = 0$ and for each $x_k \in S_k$ computes

$$J_N(x_N) = g_N(x_N) \tag{1}$$

$$J_k(x_k) = \min_{u_k \in \mathcal{A}_k(x_k)} \mathbb{E}_{w_k}\left\{g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k))\right\} \tag{2}$$

$$\mu_k(x_k) = u_k^* \quad (u_k^* \textit{ is the } u_k \textit{ which minimizes the above expression}). \tag{3}$$

Assumptions today

- For $k = 0, 1, \ldots, N-1$

$$x_{k+1} = f_k(x_k, u_k, w_k) = A_k x_k + B_k u_k,$$
$$g_k(x_k, u_k, w_k) = \frac{1}{2} x_k^\top Q_k x_k + \frac{1}{2} u_k^\top R_k u_k,$$
$$g_N(x_k) = \frac{1}{2} x_N^\top Q_N x_N$$

- **Note: This is not the most general case, but will illustrate the main ideas**

Apply dynamical programming!

- Define $V_N \equiv Q_N$ and initialize:

$$J_N^*(\boldsymbol{x}_N) = \frac{1}{2} \boldsymbol{x}_N^T Q_N \boldsymbol{x}_N = \frac{1}{2} \boldsymbol{x}_N^T V_N \boldsymbol{x}_N$$

- DP iteration (start at $k = N-1$)

$$J_k(\boldsymbol{x}_k) = \min_{\boldsymbol{u}_k} \mathbb{E}_{w_k} \{ g_k(\boldsymbol{x}_k, \boldsymbol{u}_k, w_k) + J_{k+1}(f_k(\boldsymbol{x}_k, \boldsymbol{u}_k, w_k)) \}$$

- Remember to store optimal u_k^* as $\pi_k(x_k) = u_k^*$

LQR, simplified form

DP solution gives the controller:

1. $V_N = Q_N$
2. $L_k = -(R_k + B_k^T V_{k+1} B_k)^{-1} (B_k^T V_{k+1} A_k)$
3. $V_k = Q_k + L_k^T R_k L_k + (A_k + B_k L_k)^T V_{k+1} (A_k + B_k L_k)$
4. $\boldsymbol{u}_k^* = L_k \boldsymbol{x}_k$
5. $J_k^*(\boldsymbol{x}_k) = \frac{1}{2} \boldsymbol{x}_k^T V_k \boldsymbol{x}_k$

Double Integrator Example

- True dynamics

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \boldsymbol{u}(t) \quad (4)$$

- **Euler discretization** using $\Delta = 1$ System evolves according to:

$$\boldsymbol{x}_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{=A} \boldsymbol{x}_k + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=B} \boldsymbol{u}_k$$

- Cost function:

$$J(\boldsymbol{x}_0) = \sum_{k=0}^{N} \frac{1}{2\rho} x_{k,1}^2 + \sum_{k=0}^{N-1} \frac{1}{2} u_k^2$$

- Can be put into standard form using matrices/start position:

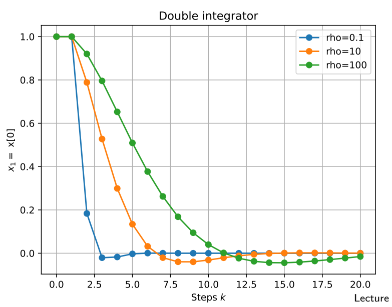
$$Q_k = Q_N = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & 0 \end{bmatrix} \quad R = 1$$

Exponential integrator

- Apply discrete LQR
- Simulate starting in $\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ using policy

$$\pi_k(\boldsymbol{x}_k) = L_k \boldsymbol{x}_k$$

- What about the true system $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$?



Quiz: LQR

Consider a (generic) LQR problem of the form:

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{u}_k \quad (5)$$

$$\text{cost} = \sum_{k=0}^{N-1} \frac{1}{2} \boldsymbol{x}_k^\top Q \boldsymbol{x}_k + \frac{1}{2} R_0 \boldsymbol{u}_k^\top \boldsymbol{u}_k \quad (6)$$

Where $R_0 > 0$ is a constant. After LQR, the controller selects actions using $\boldsymbol{u}_k = L_k \boldsymbol{x}_k$. What do you think typically happens with the matrix L_k when $R_0 \to \infty$ (**very big** R_0)

a. The entries in L_k becomes very small, negative numbers

b. The entries in L_k becomes very big, positive numbers

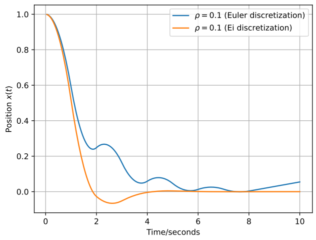
c. It is not possible to say anything about the typical case

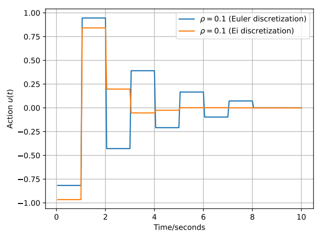
d. The entries in L_k gets closer to zero

e. Don't know.

Double integrator example

- **Blue:** LQR using Euler $x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$
- **Red:** LQR using Exponential $x_{k+1} = e^{A\Delta} x_k + A^{-1} \left(e^{A\Delta} - I \right) B u_k$





- LQR is optimal in discrete problem
- Discrete controller can be bad in real problem (always check!)
- Always use EI for linear dynamics

Example: The locomotive

Steer locomotive (starting at $x = -1$) to goal ($x^* = 0$)

$$\ddot{x}(t) = \frac{1}{m} u(t) \quad (7)$$

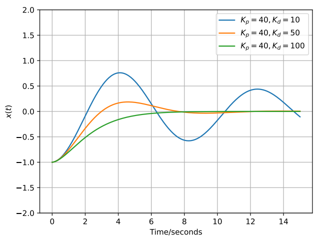
Can be re-written as:

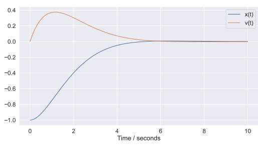
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \quad (8)$$

Discretized to $x_{k+1} = A x_k + B u_k$.

Locomotive: PID and LQR

$$e_k = x^* - x_k$$
$$u_k = e_k K_p + K_d \frac{e_k - e_{k-1}}{\Delta}$$





- Alternatively: Use a cost function $\sum_k x_k^\top Q x_k + u_k^\top u_k$ and use LQR!

`lecture_04_pid_d.py` `lecture_06_lqr_locomotive.py`

Planning on an infinite horizon

Recall LQR has the form:

1. $V_N = Q_N$
2. $L_k = -(R_k + B_k^T V_{k+1} B_k)^{-1} (B_k^T V_{k+1} A_k)$
3. $V_k = Q_k + L_k^T R_k L_k + (A_k + B_k L_k)^T V_{k+1} (A_k + B_k L_k)$
4. $u_k^* = L_k x_k$
5. $J_k^*(x_k) = \frac{1}{2} x_k^T V_k x_k$

- What happens if we repeat step 2 and 3 *many* times?
- The method will converge: $L_k \to L$
  - Select actions $u_k = L x_k$ (**"plan until convergence"**)
- If you think about it, this corresponds to planning on $N \to \infty$ horizon.
- **This is quite popular in control theory; what we will do in RL.**

Observations

- The cost term $\frac{1}{2} x^\top Q x + \frac{1}{2} u^\top R u$ is **smallest** when $x = u = 0$
- Implies that LQR will control system to state $x = u = 0$
- Suppose we want to drive system towards x_g, u_g?
  - Use $c(x, u) = \frac{1}{2}(x - x_g)^T Q (x - x_g) + \frac{1}{2}(u - u_g)^T R (u - u_g)$
- more generally assume

$$c_k(x_k, u_k) = \frac{1}{2} x_k^T Q_k x_k + \frac{1}{2} u_k^T R_k u_k + u_k^T H_k x_k + q_k^T x_k + r_k^T u_k + q_k \quad (9)$$
$$c_N(x_k) = \frac{1}{2} x_k^T Q_N x_k + q_N^T x_k + q_N \quad (10)$$

and dynamics

$$x_{k+1} = A_k x_k + B_k u_k + d_k$$

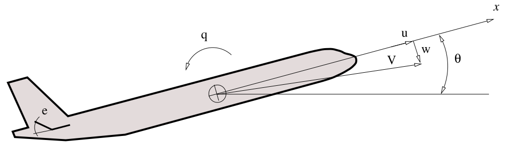
General discrete LQR algorithm

How to start living in luxury and never work again!
$\cdots (V_{k+1} + \mu I) \cdots$

1. $V_N = Q_N;\ v_N = q_N;\ v_N = q_N$
2. $L_k = -S_{uu,k}^{-1} S_{ux,k}$; $l_k = -S_{uu,k}^{-1} S_{u,k}$; $S_{u,k} = r_k + B_k^T v_{k+1} + B_k^T V_{k+1} d_k$; $S_{uu,k} = R_k + B_k^T V_{k+1} B_k$; $S_{ux,k} = H_k + B_k^T V_{k+1} A_k$.
3. $V_k = Q_k + A_k^T V_{k+1} A_k - L_k^T S_{uu,k} L_k$; $v_k = q_k + A_k^T (v_{k+1} + V_{k+1} d_k) + S_{ux,k}^T l_k$; $v_k = v_{k+1} + q_k + d_k^T v_{k+1} + \frac{1}{2} d_k^T V_{k+1} d_k + \frac{1}{2} l_k^T S_{u,k}$
4. $u_k^* = l_k + L_k x_k$
5. $J_k(x_k) = \frac{1}{2} x_k^T V_k x_k + v_k^T x_k + v_k$.

Doctors hate this one weird trick!
$V_k \leftarrow \frac{1}{2}(V_k^T + V_k)$

Boing 747 Example



$$\begin{bmatrix}\dot u\\ \dot w\\ \dot q\\ \dot\theta\end{bmatrix} = \underbrace{\begin{bmatrix}-0.003 & 0.039 & 0. & -0.322\\ -0.065 & -0.319 & 7.74 & 0.\\ 0.02 & -0.101 & -0.429 & 0.\\ 0. & 0. & 1. & 0.\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}u-u_w\\ w-w_w\\ q\\ \theta\end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix}0.01 & 1.\\ -0.18 & -0.04\\ -1.16 & 0.598\\ 0. & 0.\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}e\\ t\end{bmatrix}}_{\boldsymbol{u}}$$

$$\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \underbrace{\begin{bmatrix}1. & 0. & 0. & 0.\\ 0. & -1. & 0. & 7.74\end{bmatrix}}_{=P}\begin{bmatrix}u(t)-u_w(t)\\ w(t)-w_w(t)\\ q(t)\\ \theta(t)\end{bmatrix}$$

- y_1 and y_2 corresponds to the airspeed and climb rate.
- **Start**: $\boldsymbol{x} = 0$ (steady flight)
- **Want**: Air-speed of 10: $\boldsymbol{y}^* = \begin{bmatrix}10\\ 0\end{bmatrix}$

Approach

- Write dynamics as $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$
- Introduce cost function:

$$\int_0^{t_F} \left(\frac{1}{2}(\boldsymbol{y}-\boldsymbol{y}^*)^\top(\boldsymbol{y}-\boldsymbol{y}^*) + \frac{1}{2}\boldsymbol{u}^\top\boldsymbol{u}\right) dt$$

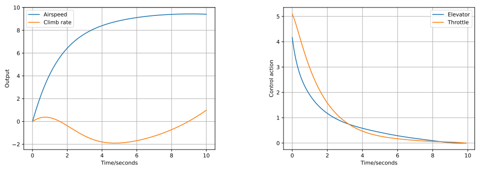
- Discretize dynamics using Exponential Integration to get $\boldsymbol{x}_{k+1} = \bar{A}\boldsymbol{x}_k + \bar{B}\boldsymbol{u}_k$
- Discretize cost to get one of the form

$$\sum_{k=0}^{\infty} \frac{1}{2}\boldsymbol{x}_k^\top Q\boldsymbol{x}_k + \boldsymbol{q}\boldsymbol{x}_k + q_0 + \frac{1}{2}\boldsymbol{u}_k^\top R\boldsymbol{u}_k$$

- Apply LQR!

Outcome and a Quiz

- Control law $\boldsymbol{u}_k = Lx_k$



Left: airspeed and climb rate. **Right:** Elevator and throttle

Why does the output adjust quickly but fail to get entirely to the goal y^*?

a. Something bad happened to the exponential integration

b. The explanation has to do with planning on a finite horizon

c. The explanation is that R in $\boldsymbol{u}_k^\top R\boldsymbol{u}_k$ should be bigger

d. Don't know.

LQR with Additive Noise

- Consider the case where there is additive Gaussian noise:

$$\boldsymbol{x}_{k+1} = A_k\boldsymbol{x}_k + B_k\boldsymbol{u}_k + \boldsymbol{\omega}_k$$

- We can still solve the problem, and (amazingly!) the noise has **no influence** on the control law

$$\boldsymbol{u}_k = L_k\boldsymbol{x}_k$$

- LQR is robust to noise

Much more to LQR

- Stability/controllability of LQR?
  - **Important subject which we ignore**
- What if matrices A_k, B_k are random?
  - **This too can be solved[Ber05, Chapter 4]**
- What about partial observation?
  - **I.e. assume we observe $o_k = D_k x_k$[Ber05, Chapter 4]**
- What about constraints? What if we know $u_L \leq \boldsymbol{u}_k \leq u_B$?
- Euler integration is often not ideal.
  - **Alternatives including error analysis**

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