02417: Time Series Analysis

Week 10 – State space models, 1st part

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DTU Compute

Based on previous material from the course

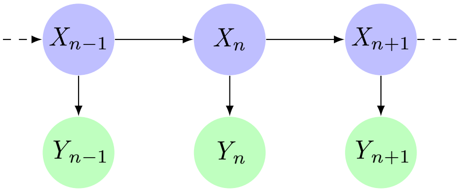
April 8, 2026

Week 10: Outline of the lecture

State space models, 1st part:

- The advantages
- The linear state space model
- Determining model structure
- Example
- An example on application of the Kalman filter.

State space models



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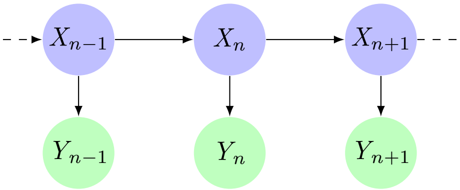

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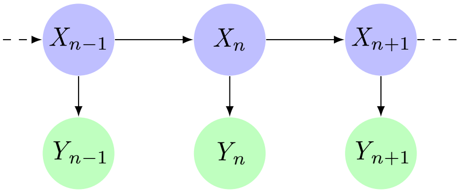

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- Goal; reconstruct and predict the state of the system

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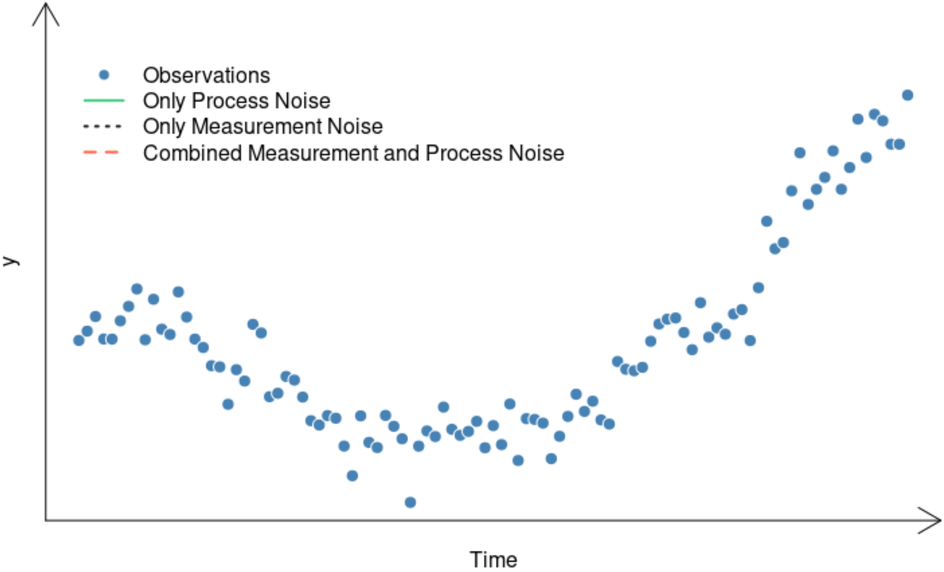

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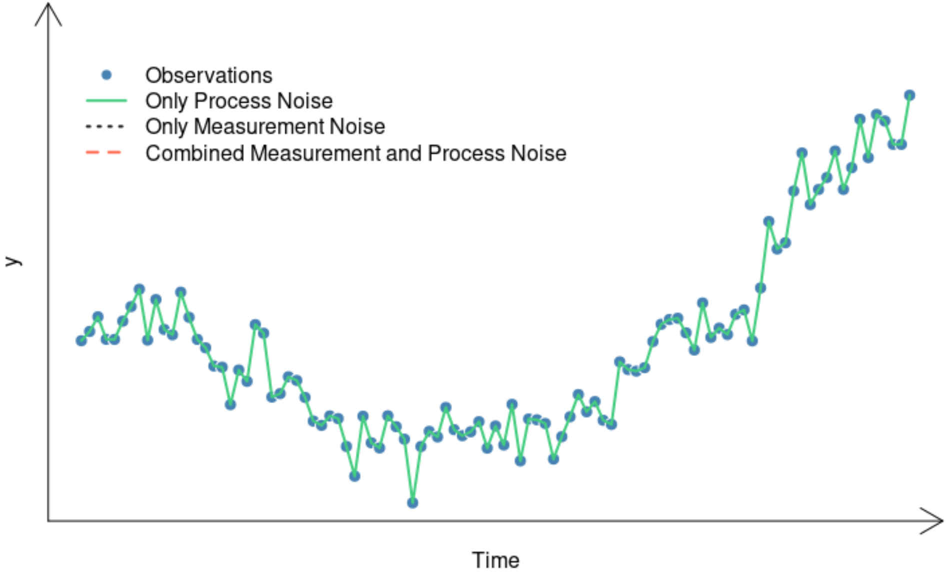

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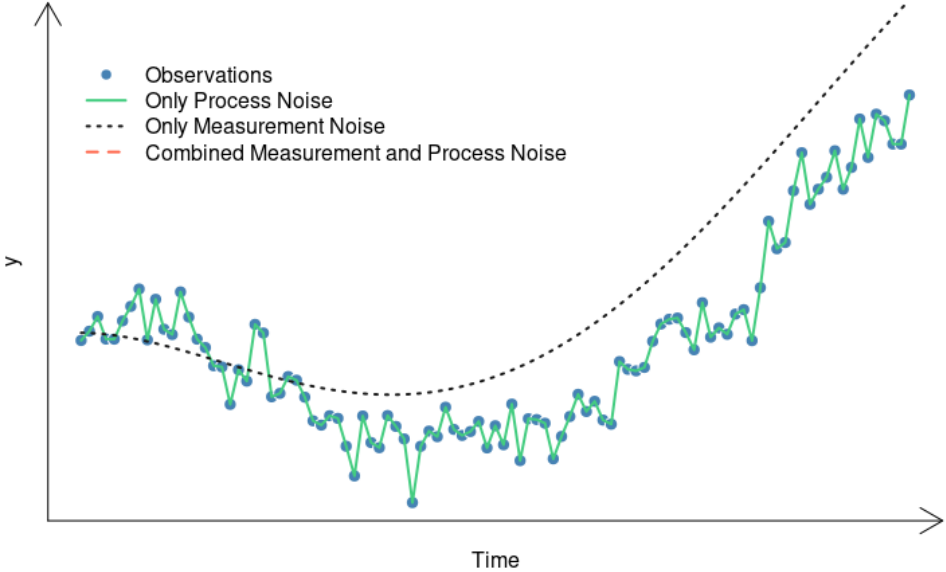

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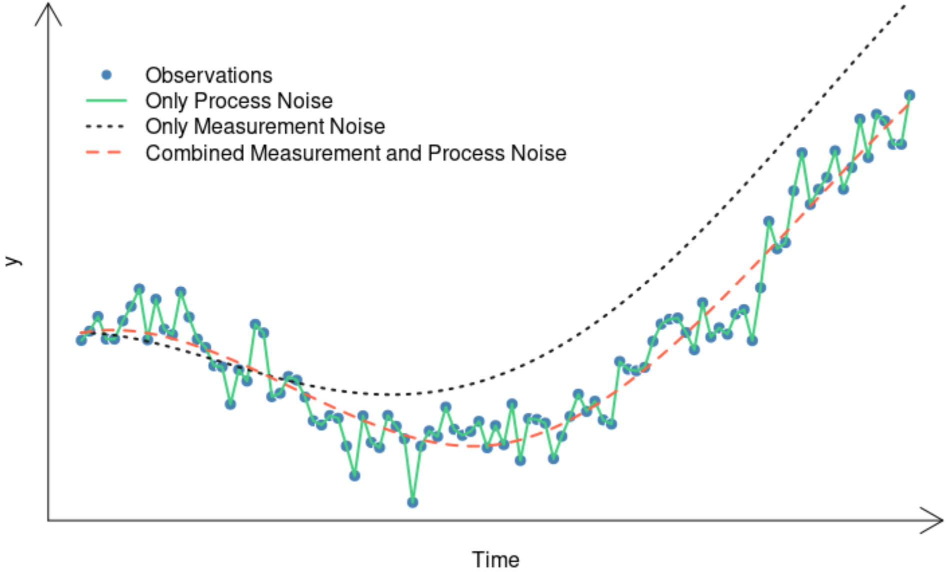

The linear stochastic state space model

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- $V[\boldsymbol{e}_1] = \boldsymbol{\Sigma}_1$, $V[\boldsymbol{e}_2] = \boldsymbol{\Sigma}_2$
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- The state vector contains all information available for future evaluation; the process is a *Markov process*.

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- Amount of persons in room: State CO2 concentration, amount of persons in room Observation: CO2 concentration (with noise)

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- Formulate observation equation.

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- For the falling body (from the discrete-time description of the system):

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\left[C^T \vdots (CA)^T\right] = \left[\begin{matrix} 1 \\ 0 \end{matrix} \vdots \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)^T\right] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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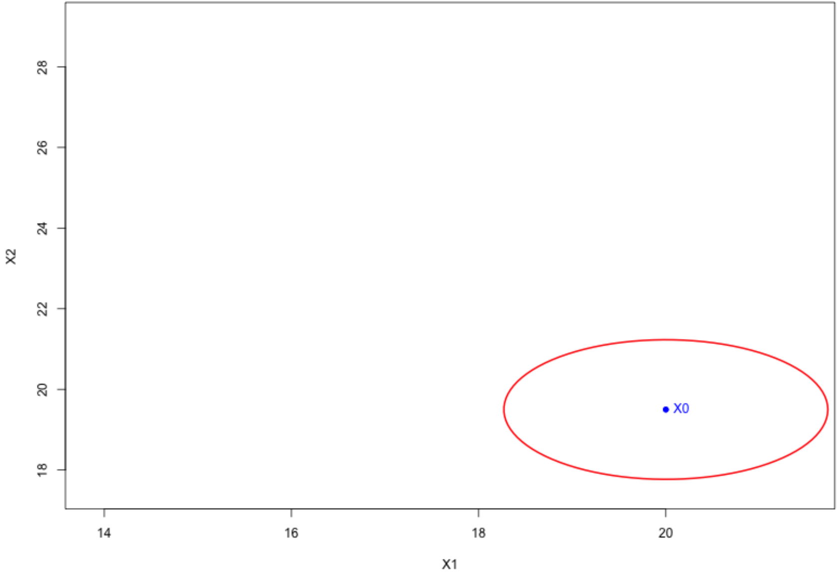

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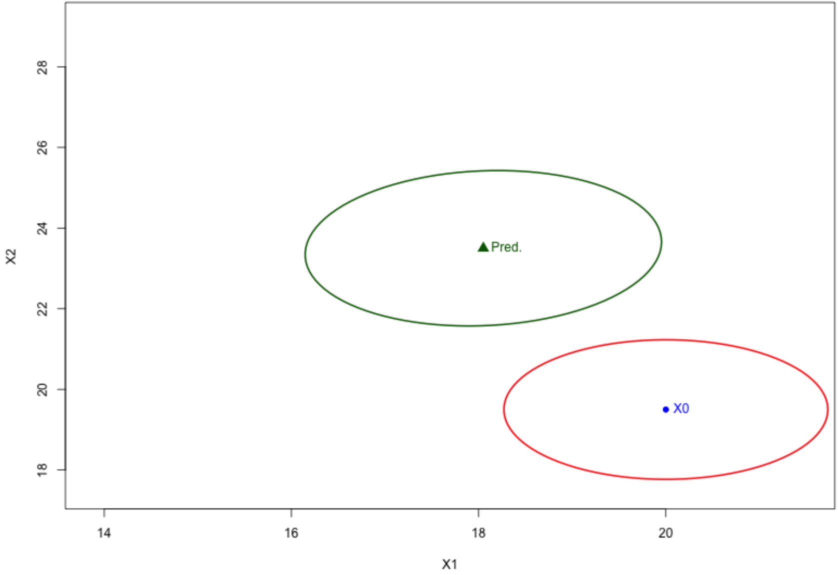

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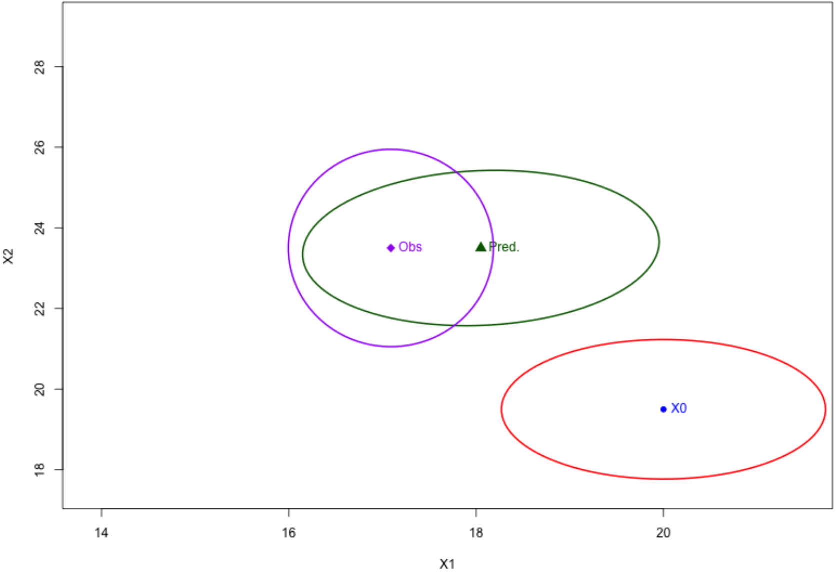

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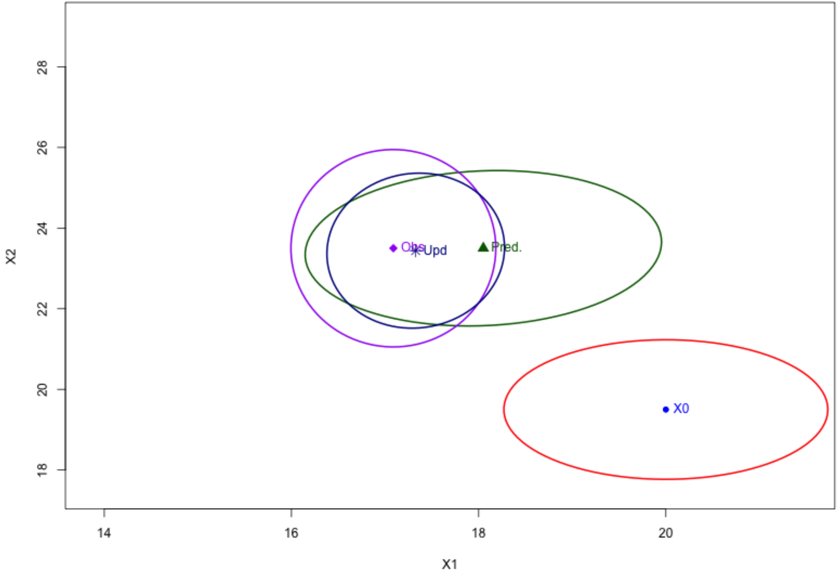

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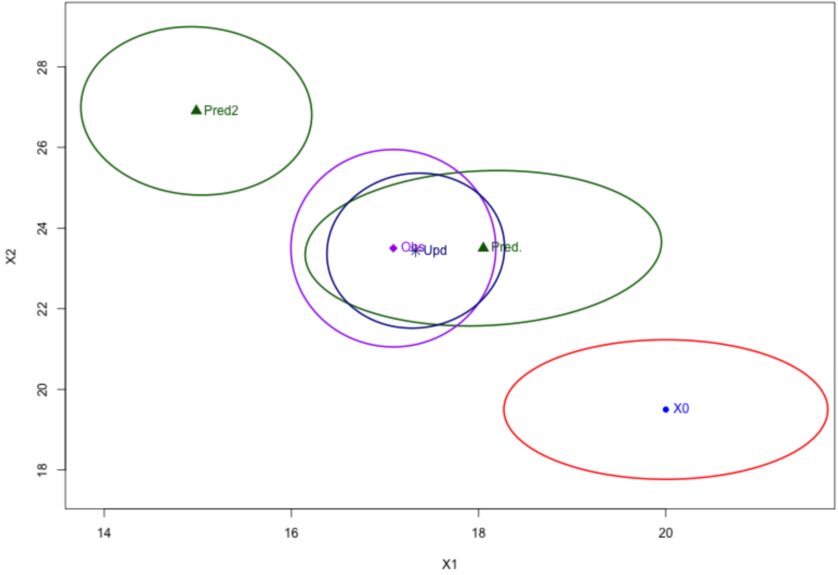

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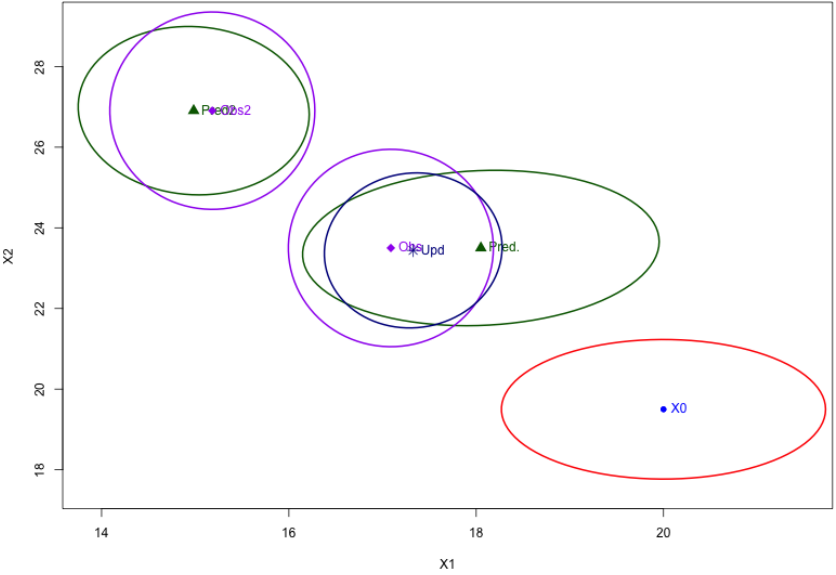

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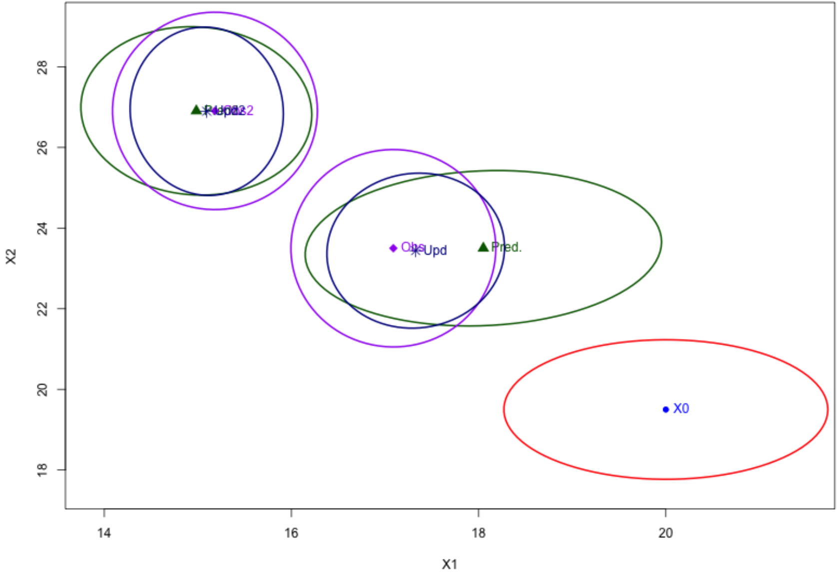

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- If noise is non-Gaussian or if the model is non-linear, the prediction/ observation densities are not Gaussian (we shall work briefly with this in the next assignment).
- If that is the case, we can use what is called *the extended* Kalman filter (enroll to Advanced Time Series Analysis (02427) to learn about this).

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- Theorem 2.6 (Linear projection)
- The theorem is concerned with the random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ for which the means, variances and covariances are used
- The state is called $\boldsymbol{X}_t$ and the observation is called $\boldsymbol{Y}_t$ and we could write down the theorem for these
- We have additional information; $\mathcal{Y}_{t-1}^T = (\boldsymbol{Y}_1^T, \ldots, \boldsymbol{Y}_{t-1}^T)$
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- We include this information by considering the random vectors $\boldsymbol{X}_t|\mathcal{Y}_{t-1}$ and $\boldsymbol{Y}_t|\mathcal{Y}_{t-1}$ instead

$$\begin{aligned} E[(\boldsymbol{X}_t|\mathcal{Y}_{t-1})\,|\,(\boldsymbol{Y}_t|\mathcal{Y}_{t-1})] = E[\boldsymbol{X}_t|\,\boldsymbol{Y}_t, \mathcal{Y}_{t-1}] = \\ E[\boldsymbol{X}_t|\mathcal{Y}_{t-1}] + \mathrm{Cov}[\boldsymbol{X}_t, \boldsymbol{Y}_t|\mathcal{Y}_{t-1}]\,V^{-1}[\boldsymbol{Y}_t|\mathcal{Y}_{t-1}](\boldsymbol{Y}_t - E[\boldsymbol{Y}_t|\mathcal{Y}_{t-1}]) \end{aligned}$$

$$\begin{aligned} V[(\boldsymbol{X}_t|\mathcal{Y}_{t-1})\,|\,(\boldsymbol{Y}_t|\mathcal{Y}_{t-1})] = V[\boldsymbol{X}_t|\,\boldsymbol{Y}_t, \mathcal{Y}_{t-1}] = \\ V[\boldsymbol{X}_t|\mathcal{Y}_{t-1}] - \mathrm{Cov}[\boldsymbol{X}_t, \boldsymbol{Y}_t|\mathcal{Y}_{t-1}]\,V^{-1}[\boldsymbol{Y}_t|\mathcal{Y}_{t-1}]\mathrm{Cov}^T[\boldsymbol{X}_t, \boldsymbol{Y}_t|\mathcal{Y}_{t-1}] \end{aligned}$$

The Foundation of the Kalman filter II

$$
\begin{aligned}
&E[\boldsymbol{X}_t | \boldsymbol{Y}_t, \mathcal{Y}_{t-1}] = \\
&\qquad E[\boldsymbol{X}_t|\mathcal{Y}_{t-1}] + \mathrm{Cov}[\boldsymbol{X}_t, \boldsymbol{Y}_t|\mathcal{Y}_{t-1}] V^{-1}[\boldsymbol{Y}_t|\mathcal{Y}_{t-1}](\boldsymbol{Y}_t - E[\boldsymbol{Y}_t|\mathcal{Y}_{t-1}]) \\
&V[\boldsymbol{X}_t | \boldsymbol{Y}_t, \mathcal{Y}_{t-1}] = \\
&\qquad V[\boldsymbol{X}_t|\mathcal{Y}_{t-1}] - \mathrm{Cov}[\boldsymbol{X}_t, \boldsymbol{Y}_t|\mathcal{Y}_{t-1}] V^{-1}[\boldsymbol{Y}_t|\mathcal{Y}_{t-1}]\mathrm{Cov}^T[\boldsymbol{X}_t, \boldsymbol{Y}_t|\mathcal{Y}_{t-1}]
\end{aligned}
$$

The Foundation of the Kalman filter II

$$
\begin{aligned}
E[\boldsymbol{X}_t | \boldsymbol{Y}_t, \mathcal{Y}_{t-1}] &= \\
&E[\boldsymbol{X}_t | \mathcal{Y}_{t-1}] + \mathrm{Cov}[\boldsymbol{X}_t, \boldsymbol{Y}_t | \mathcal{Y}_{t-1}] V^{-1}[\boldsymbol{Y}_t | \mathcal{Y}_{t-1}](\boldsymbol{Y}_t - E[\boldsymbol{Y}_t | \mathcal{Y}_{t-1}]) \\
V[\boldsymbol{X}_t | \boldsymbol{Y}_t, \mathcal{Y}_{t-1}] &= \\
&V[\boldsymbol{X}_t | \mathcal{Y}_{t-1}] - \mathrm{Cov}[\boldsymbol{X}_t, \boldsymbol{Y}_t | \mathcal{Y}_{t-1}] V^{-1}[\boldsymbol{Y}_t | \mathcal{Y}_{t-1}] \mathrm{Cov}^T[\boldsymbol{X}_t, \boldsymbol{Y}_t | \mathcal{Y}_{t-1}]
\end{aligned}
$$

Using definitions from previous slide the update equations are:

$$
\begin{aligned}
\widehat{\boldsymbol{X}}_{t|t} &= \widehat{\boldsymbol{X}}_{t|t-1} + \boldsymbol{\Sigma}^{xy}_{t|t-1} \left(\boldsymbol{\Sigma}^{yy}_{t|t-1} \right)^{-1} \left(\boldsymbol{Y}_t - \widehat{\boldsymbol{Y}}_{t|t-1} \right) \\
\boldsymbol{\Sigma}^{xx}_{t|t} &= \boldsymbol{\Sigma}^{xx}_{t|t-1} - \boldsymbol{\Sigma}^{xy}_{t|t-1} \left(\boldsymbol{\Sigma}^{yy}_{t|t-1} \right)^{-1} \left(\boldsymbol{\Sigma}^{xy}_{t|t-1} \right)^T
\end{aligned}
$$

$$
\boldsymbol{K}_t = \boldsymbol{\Sigma}^{xy}_{t|t-1} \left(\boldsymbol{\Sigma}^{yy}_{t|t-1} \right)^{-1}
$$

$\boldsymbol{K}_t$ is called the *Kalman gain*, because it determines how much the 1-step prediction error influence the update of the state estimate

The Kalman filter

Initialization:

$$\widehat{\boldsymbol{X}}_{1|0} = E\left[\boldsymbol{X}_1\right] = \boldsymbol{\mu}_0$$

$$\boldsymbol{\Sigma}^{xx}_{1|0} = V\left[\boldsymbol{X}_1\right] = \boldsymbol{V}_0$$

$$\boldsymbol{\Sigma}^{yy}_{1|0} = \boldsymbol{C}\boldsymbol{\Sigma}^{xx}_{1|0}\boldsymbol{C}^T + \boldsymbol{\Sigma}_2$$

The Kalman filter

Initialization:

$$\widehat{\boldsymbol{X}}_{1|0} = E\left[\boldsymbol{X}_1\right] = \boldsymbol{\mu}_0$$
$$\boldsymbol{\Sigma}^{xx}_{1|0} = V\left[\boldsymbol{X}_1\right] = \boldsymbol{V}_0$$
$$\boldsymbol{\Sigma}^{yy}_{1|0} = \boldsymbol{C}\boldsymbol{\Sigma}^{xx}_{1|0}\boldsymbol{C}^T + \boldsymbol{\Sigma}_2$$

For: $t = 1, 2, 3, \ldots$

Reconstruction:

$$\boxed{\begin{aligned} \boldsymbol{K}_t &= \boldsymbol{\Sigma}^{xx}_{t|t-1}\boldsymbol{C}^T\left(\boldsymbol{\Sigma}^{yy}_{t|t-1}\right)^{-1} \\ \widehat{\boldsymbol{X}}_{t|t} &= \widehat{\boldsymbol{X}}_{t|t-1} + \boldsymbol{K}_t\left(\boldsymbol{Y}_t - \boldsymbol{C}\widehat{\boldsymbol{X}}_{t|t-1}\right) \\ \boldsymbol{\Sigma}^{xx}_{t|t} &= \boldsymbol{\Sigma}^{xx}_{t|t-1} - \boldsymbol{K}_t\boldsymbol{\Sigma}^{yy}_{t|t-1}\boldsymbol{K}_t^T \end{aligned}}$$

The Kalman filter

Initialization:

$$\widehat{\boldsymbol{X}}_{1|0} = E\left[\boldsymbol{X}_1\right] = \boldsymbol{\mu}_0$$
$$\boldsymbol{\Sigma}^{xx}_{1|0} = V\left[\boldsymbol{X}_1\right] = \boldsymbol{V}_0$$
$$\boldsymbol{\Sigma}^{yy}_{1|0} = \boldsymbol{C}\boldsymbol{\Sigma}^{xx}_{1|0}\boldsymbol{C}^T + \boldsymbol{\Sigma}_2$$

For: $t = 1, 2, 3, \ldots$

Reconstruction:

$$\boldsymbol{K}_t = \boldsymbol{\Sigma}^{xx}_{t|t-1}\boldsymbol{C}^T\left(\boldsymbol{\Sigma}^{yy}_{t|t-1}\right)^{-1}$$
$$\widehat{\boldsymbol{X}}_{t|t} = \widehat{\boldsymbol{X}}_{t|t-1} + \boldsymbol{K}_t\left(\boldsymbol{Y}_t - \boldsymbol{C}\widehat{\boldsymbol{X}}_{t|t-1}\right)$$
$$\boldsymbol{\Sigma}^{xx}_{t|t} = \boldsymbol{\Sigma}^{xx}_{t|t-1} - \boldsymbol{K}_t\boldsymbol{\Sigma}^{yy}_{t|t-1}\boldsymbol{K}_t^T$$

Prediction:

$$\widehat{\boldsymbol{X}}_{t+1|t} = \boldsymbol{A}\widehat{\boldsymbol{X}}_{t|t} + \boldsymbol{B}\boldsymbol{u}_t$$
$$\boldsymbol{\Sigma}^{xx}_{t+1|t} = \boldsymbol{A}\boldsymbol{\Sigma}^{xx}_{t|t}\boldsymbol{A}^T + \boldsymbol{\Sigma}_1$$
$$\boldsymbol{\Sigma}^{yy}_{t+1|t} = \boldsymbol{C}\boldsymbol{\Sigma}^{xx}_{t+1|t}\boldsymbol{C}^T + \boldsymbol{\Sigma}_2$$

Example: The falling body revised

Description of the system:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} -1/2 \\ -1 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\boldsymbol{\Sigma}_1 = \begin{bmatrix} 2.0 & 0.8 \\ 0.8 & 1.0 \end{bmatrix} \qquad \boldsymbol{\Sigma}_2 = \begin{bmatrix} 10000 \end{bmatrix}$$

Initialization: Released 10000 m above ground at 0 m/s

$$\widehat{\boldsymbol{X}}_{1|0} = \begin{bmatrix} 10000 \\ 0 \end{bmatrix} \qquad \boldsymbol{\Sigma}^{xx}_{1|0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{\Sigma}^{yy}_{1|0} = \begin{bmatrix} 10000 \end{bmatrix}$$

Simulation of a falling body – initialization

```r
z0 <- 10000
A <- matrix(c(1,0,1,1),nrow=2)
B <- matrix(c(-.5,-1),nrow=2)
C <- matrix(c(1,0),nrow=1)
Sigma1 <- matrix(c(2,.8,.8,1),nrow=2)
Sigma2 <- matrix(10000)
g <- 9.82; N <- 300
X <- matrix(nrow=2,ncol=N) ## Allocating space
X[,1] <- c(z0,0)
Y <- numeric(N)
Y[1] <- C%*%X[,1]+sqrt(Sigma2) %*% rnorm(1)
```

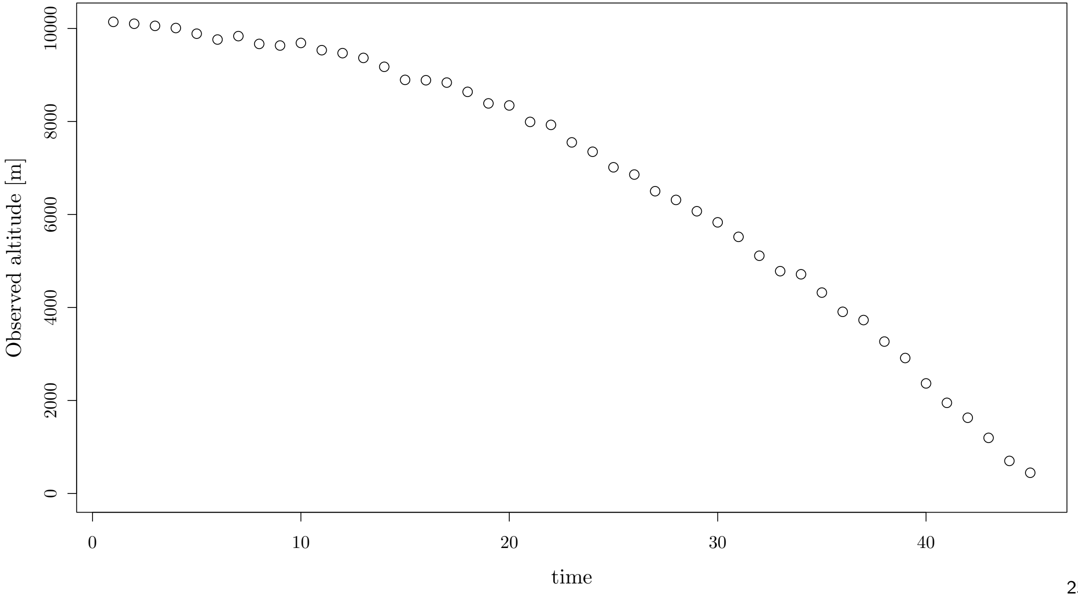
Simulation of a falling body - simulation

```
for (I in 2:N){
    X[,I] <- A %*% X[,I-1,drop=FALSE] + B%*%g +
        chol(Sigma1) %*% matrix(rnorm(2),ncol=1)
    Y[I] <- C %*% X[,I] + sqrt(Sigma2) %*% rnorm(1)
}
Nhit <- min(which(X[1,]<0))-1
X <- X[,1:Nhit]
Y <- Y[1:Nhit]
```

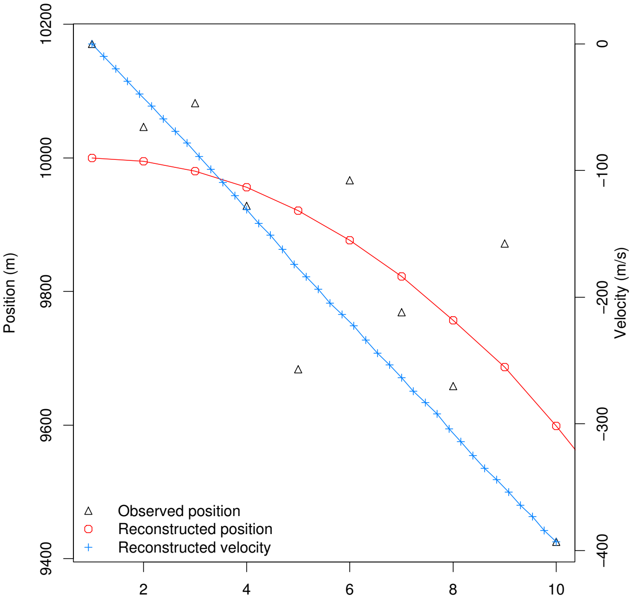
- Remember that if $Z \sim N(0, I)$, then $Y = QZ \sim N(0, QQ^T)$.
- The Cholesky factorization is one way to solve $QQ^T = \Sigma$ for Q.

The falling body – observations

```
plot(Y,xlab="time",ylab="Observed altitude [m]", ylim=c(0,max(Y)))
```



Falling body – the 10 first time points



Falling body – wrong initial state