02417: Time Series Analysis

Week 7 – Linear systems

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DTU Compute

Based on material previous material from the course

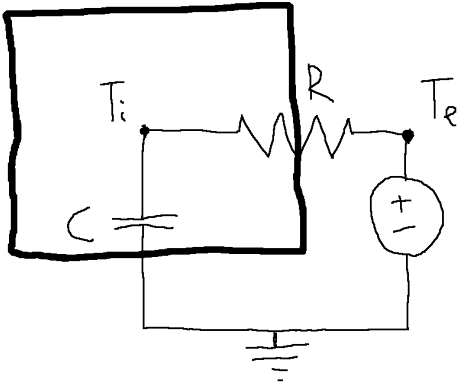
March 22, 2025

Week 7: Outline of the lecture

- Input-Output systems, Sec. 4 introduction and 4.1
- Linear system notation
- The z-transform, Sec. 4.4
- Cross Correlation Functions – from Sec. 6.2.2
- Transfer function models; identification, estimation, validation, prediction, Chap. 8

Simplest first order RC-system

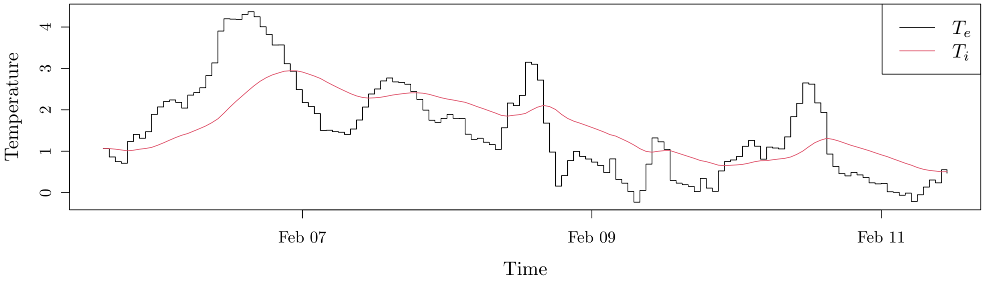
Single state model of the temperature in a box:

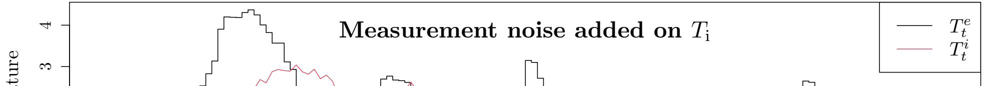


Simplest RC-system

- T_t^{e} external and T_t^{i} internal temperature at time $t = [1, 2, \ldots, n]$
- ODE model

$$\frac{dT_{\mathrm{i}}}{dt} = \frac{1}{RC}(Te - T_{\mathrm{i}})$$



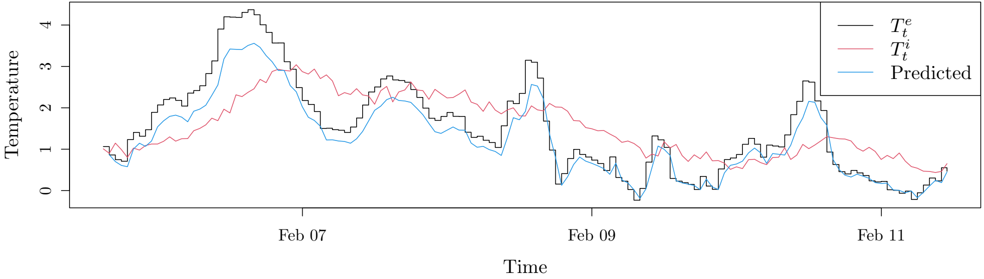


Try a static model

- A simple linear regression model (ε_t is the error)

$$T^{\mathrm{i}}_t = \omega_e T^{\mathrm{e}}_t + \varepsilon_t$$

- Are the dynamics well described by the model?
  No, the predicted temperature is just proportional to the input (T^{i}_t)

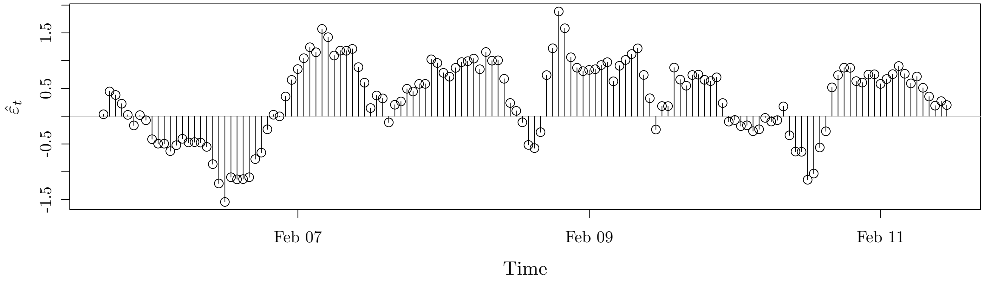


Model validation: check i.i.d. of residuals

Are residuals like white noise?

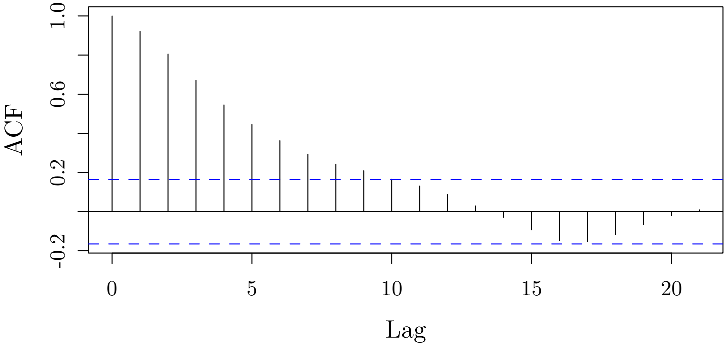
- Check if they are *independent and identically distributed*
- Is $\hat{\varepsilon}_t$ independent of $\hat{\varepsilon}_{t-k}$ for all t and k?

Nope! There is a pattern left...



Model validation: Test for i.i.d. with ACF

TEST if residuals are white noise?



It's not white nose!

How do we find a better model? The exponential decay in ACF points to an AR part!

Discretize the ODE

$$\frac{dT_{\mathrm{i}}}{dt} = \frac{1}{RC}(T_{\mathrm{e}} - T_{\mathrm{i}})$$

It has the solution

$$T_{\mathrm{i}}(t+\Delta t) = T_{\mathrm{e}}(t) + e^{-\frac{\Delta t}{RC}}\big(T_{\mathrm{i}}(t) - T_{\mathrm{e}}(t)\big)$$

if $\Delta t = 1$ and T_{e} is constant between the sample points then

$$T^{\mathrm{i}}_{t+1} = e^{-\frac{1}{RC}}\, T^{\mathrm{i}}_{t} + (1 - e^{-\frac{1}{RC}})\, T^{\mathrm{e}}_{t}$$

since $e^{-\frac{1}{RC}}$ is between 0 and 1, then write it as

$$T^{\mathrm{i}}_{t+1} = \phi_1\, T^{\mathrm{i}}_{t} + \omega_1\, T^{\mathrm{e}}_{t}$$

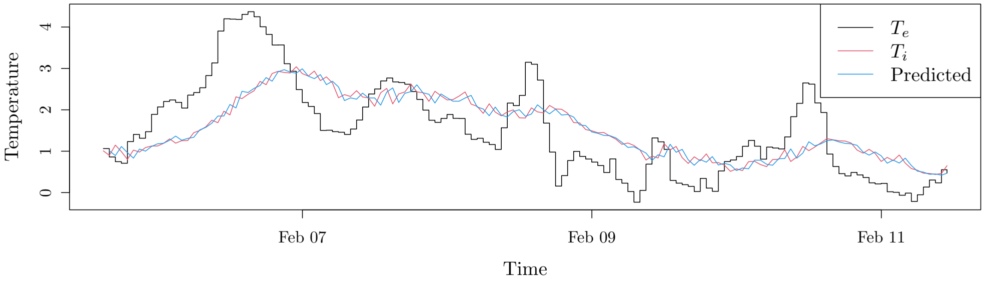
where ϕ_1 and ω_1 are between 0 and 1.

Add a noise term and we have the Auto-Regressive with eXogeneous input (ARX) model

$$T^{\mathrm{i}}_{t+1} = \phi_1\, T^{\mathrm{i}}_{t} \quad + \omega_1\, T^{\mathrm{e}}_{t} \quad + \varepsilon_{t+1}\, T^{\mathrm{i}}_{t} \qquad\qquad = \phi_1\, T^{\mathrm{i}}_{t-1} + \omega_1\, T^{\mathrm{e}}_{t-1} + \varepsilon_{t}$$

An ARX model

$$T_t^{\mathrm{i}} = \phi_1 \, T_{t-1}^{\mathrm{i}} + \omega_1 \, T_{t-1}^{\mathrm{e}} + \varepsilon_t$$

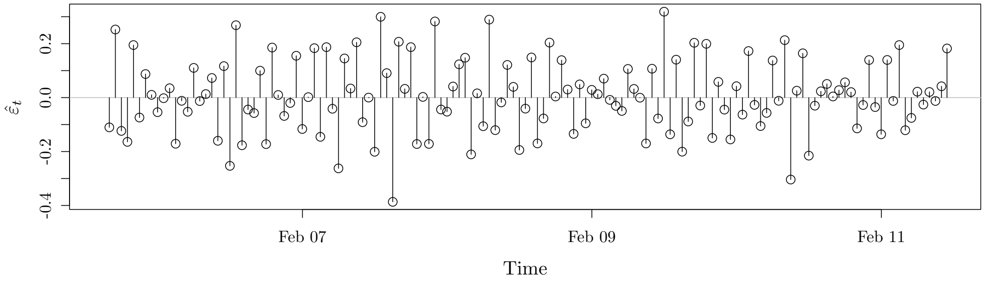


ARX model

- The residuals

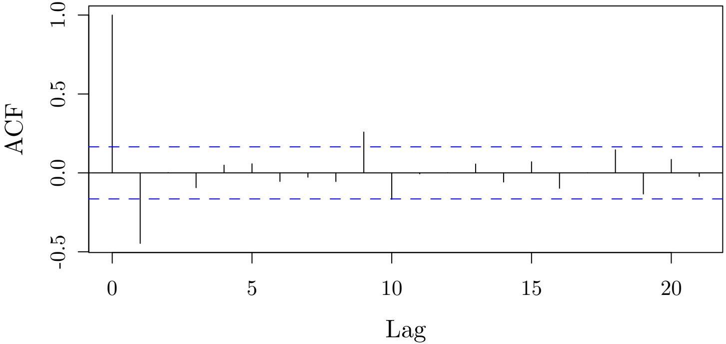
$$\hat{\varepsilon}_t = \text{obs.} - \text{pred.} = T^{\mathrm{i}}_t - \hat{T}^{\mathrm{i}}_t = T^{\mathrm{i}}_t - \frac{\hat{\omega}_1 \mathrm{B}}{1 - \hat{\phi}_1 \mathrm{B}} T^{\mathrm{e}}_t$$

- Are the residuals now white noise?



Check for i.i.d. of residuals

Is it likely that this is white noise? Almost!



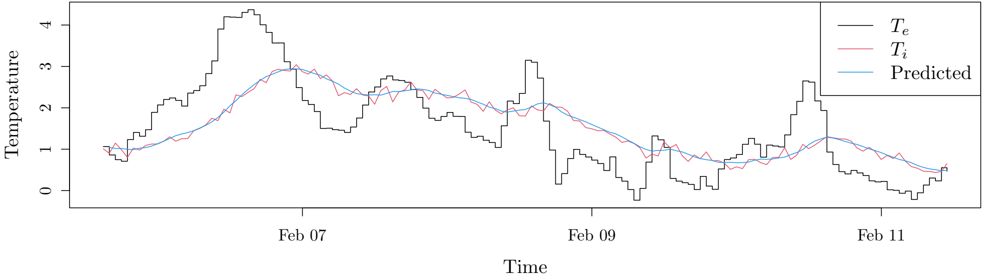
Actually we miss an MA part!

An ARMAX model

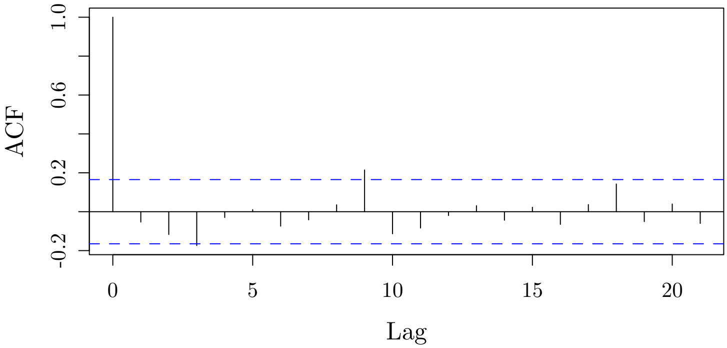
$$T_t^{\mathrm{i}} = \phi_1 T_{t-1}^{\mathrm{i}} + \omega_1 T_t^{\mathrm{e}} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

An ARMAX model

$$Y_t = \phi_1 Y_{t-1} + \omega_1 x_t + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$



Validate the model with the residuals ACF



Now we have *white noise residuals* :-)

Remember, we are validating the *one-step prediction* residuals: $\hat{\varepsilon}_{t+1} = y_{t+1} - \hat{y}_{t+1|t}$

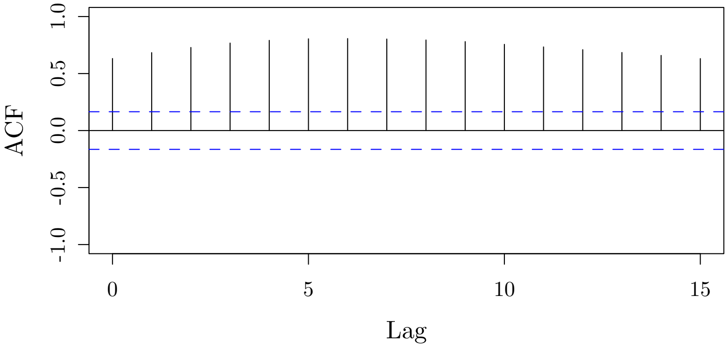
$\hat{\varepsilon}_t \; = y_t - \hat{y}_{t|t-1}$

Dependence between variables: Cross-correlation function

Calculate the Cross-Correlation Function (CCF) by simply shifting the index and lag *another* series:

t	Y_t	X_t	X_{t-1}
1	4	2	
2	5	3	2
3	2	8	3
4	3	3	8
5	4	1	3
6	5	7	1
7	5	8	7
8			8

Cross-Correlation Function (CCF) between input and output



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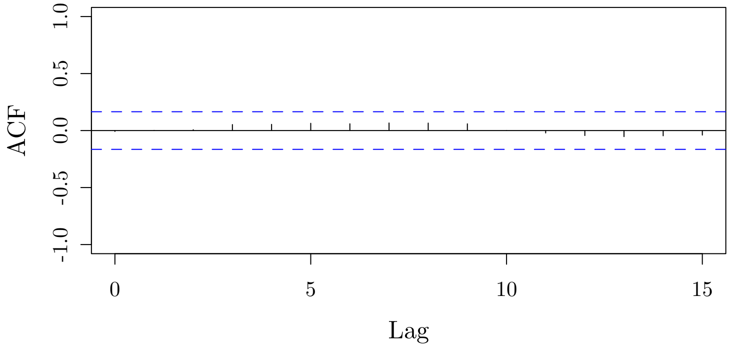
8			8

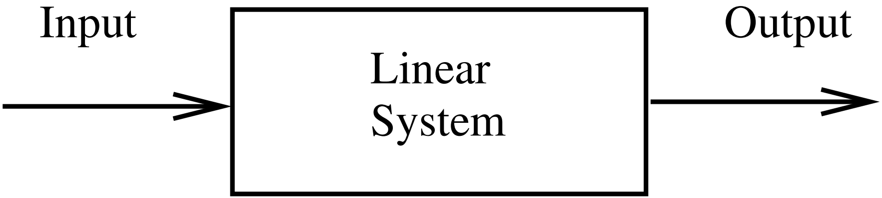
Cross-Correlation Function (CCF) between input and residuals.



So we can use the CCF in model validation: If no correlation between input(s) and residuals, then the model is good!

Linear Dynamic Systems – notation

x	$\mathcal{F}[\cdot]$	y



$x(t)$	Differential eq., $h(u)$	$y(t)$
x_t	Difference eq., h_k, $h(B)$	y_t
$X(\omega)$	$\mathcal{H}(\omega)$	$Y(\omega)$
$X(z)$	$H(z)$	$Y(z)$

Dynamic Systems – Some characteristics

Def. Linear system:

$$\mathcal{F}\left[\lambda_1 x_1(t) + \lambda_2 x_2(t)\right] = \lambda_1 \mathcal{F}\left[x_1(t)\right] + \lambda_2 \mathcal{F}\left[x_2(t)\right]$$

Def. Time invariant system:

$$y(t) = \mathcal{F}\left[x(t)\right] \Rightarrow y(t-\tau) = \mathcal{F}\left[x(t-\tau)\right]$$

Def. Stable system: A system is said to be *stable* if any constrained input implies a constrained output.

Def. Causal system: A system is said to be *physically feasible* or *causal*, if the output at time t does not depend on future values of the input.

Example: "ARX(1)" system

- *System*: $y_t - ay_{t-1} = bx_t$
- Can be written: $y_t = bx_t + ay_{t-1} = bx_t + a(bx_{t-1} + ay_{t-2})$ or

$$y_t = b(x_t + ax_{t-1} + a^2 x_{t-2} + a^3 x_{t-3} + \ldots) = b \sum_{k=0}^{\infty} a^k x_{t-k}$$

- Is the system *linear* and *time invariant*?

  Yes: y_t is a linear function of the input values $x_t, x_{t-1}, \ldots$

  Yes: Time invariant since the coefficients don't change in time

- Is the system *causal*?

  Yes: y_t depend only on past input values $x_t, x_{t-1}, \ldots$

- Is the system *stable*?

  Yes, for $|a| < 1$ the coefficients sum is bounded

$$\sum_{k=0}^{\infty} |a|^k = \begin{cases} 1/(1-|a|) & ; \quad |a| < 1 \\ \infty & ; \quad |a| \geq 1 \end{cases}$$

  (stability does not depend on b)

Def. discrete impulse and step reponse

For *linear time invariant systems* the input can be convoluted to get the output:

- Discrete time:

$$y_t = \sum_{k=-\infty}^{\infty} h(k) x_{t-k} \tag{1}$$

  Causal, then:

$$y_t = \sum_{k=0}^{t} h(k) x_{t-k} \tag{2}$$

- $h(k)$ is called the *impulse response*, why? What happens if $x_0 = 1$ and $x_k = 0$ for $k \neq 0$?
- $S_k = \sum_{j=-\infty}^{k} h_j$ is called the *step response*, why? What happens if $x_k = 1$ for all k?

Example: Calculating the impulse response function

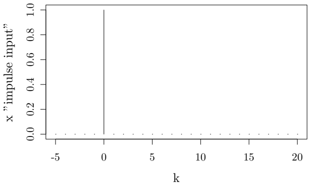
The impulse response can be determined by 'sending a 1 through the system'. Take the "ARX(1)", time-invariant system

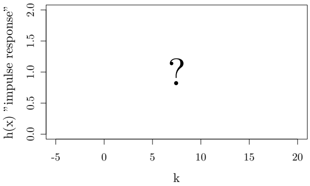
$$y_t = 0.8 y_{t-1} + 2 x_t$$

We want to rewrite this model as

$$y_t = \sum_{k=0}^{t} h_k x_{t-k}$$

An impulse: Put $x_0 = 1$ and $x_k = 0$ for $k \neq 0$, send it through the system, and observe the response:





We see that $h_k = y_k = 0$ for $k < 0$. What about y_0? and y_1?

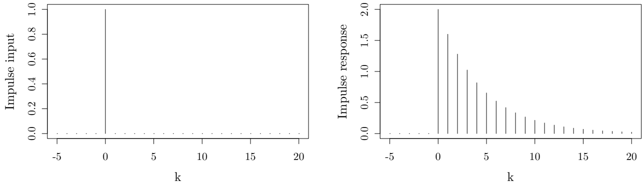
$$\begin{aligned} h_0 = y_0 &= 0.8 y_{-1} + 2 x_0 = 0.8 \cdot 0 + 2 \cdot 1 = 2 \\ h_1 = y_1 &= 0.8 y_0 + 2 x_1 = 0.8 \cdot 2 + 2 \cdot 0 = 1.6 \end{aligned}$$

Hence, the impulse response function is $h_k = 0.8^k \cdot 2$ for $k > 0$ which represents a causal system.

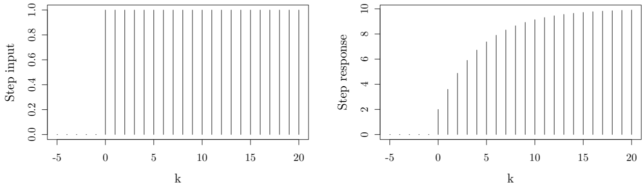
Example: Calculating the impulse response function

"ARX(1)": $y_t = 0.8y_{t-1} + 2x_t$

Impulse response: Send an impulse through the system:



Step response: Send a step through the system!

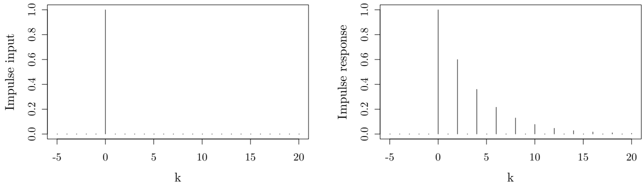


Is the system stable? YES, since $\sum_0^\infty |h_k| = 10 < \infty$

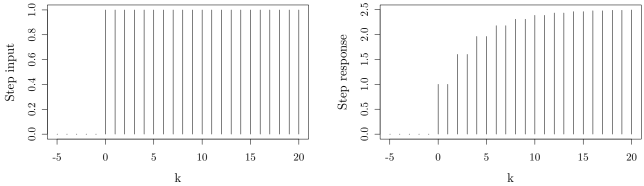
Example: Calculating the impulse response function

$$\text{"ARX(2)": } y_t = 0y_{t-1} + 0.6y_{t-2} + x_t$$

Impulse response: Send an impulse through the system:



Step response: Send a step through the system!

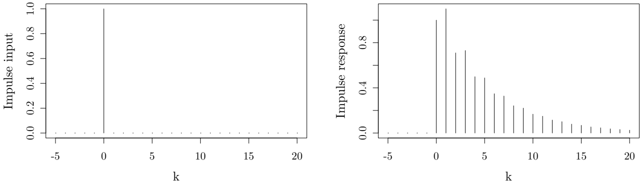


The system is stable since $\sum_0^\infty |h_k| = \frac{10}{6} < \infty$

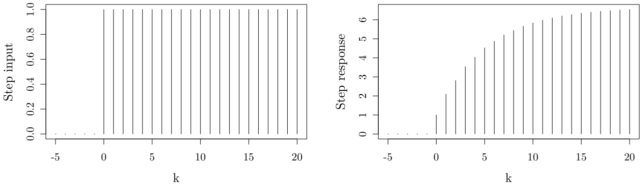
Example: Calculating the impulse response function

"ARX(2)": $y_t = 0.1y_{t-1} + 0.6y_{t-2} + x_t + x_{t-1}$

Impulse response: Send an impulse through the system:



Step response: Send a step through the system!

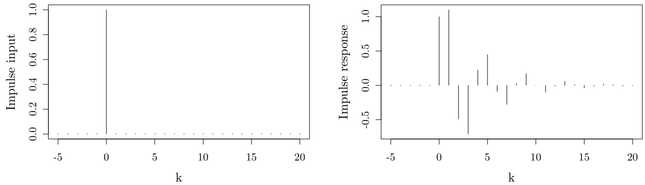


The system is stable since $\sum_0^\infty |h_k| = \frac{20}{6} < \infty$

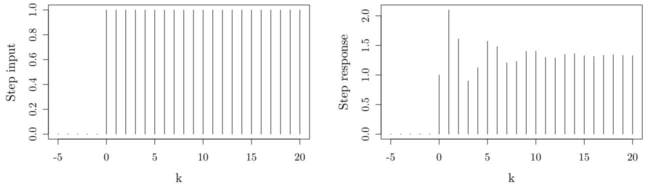
Example: Calculating the impulse response function

"ARX(2)": $y_t = 0.1y_{t-1} - 0.6y_{t-2} + x_t + x_{t-1}$

Impulse response: Send an impulse through the system:



Step response: Send a step through the system!

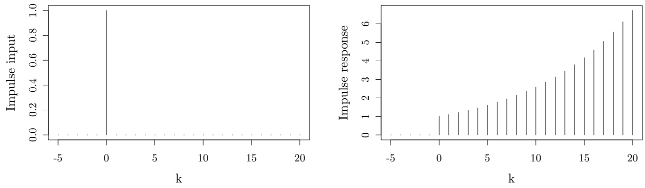


The system is stable since $\sum_0^\infty |h_k| = \frac{4}{3} < \infty$

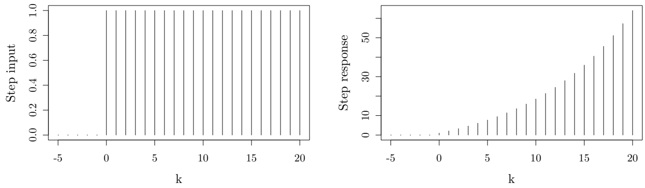
Example: Calculating the impulse response function

$$\text{"ARX(2)": } y_t = 0.1 y_{t-1} + 1.1 y_{t-2} + x_t + x_{t-1}$$

Impulse response: Send an impulse through the system:



Step response: Send a step through the system!



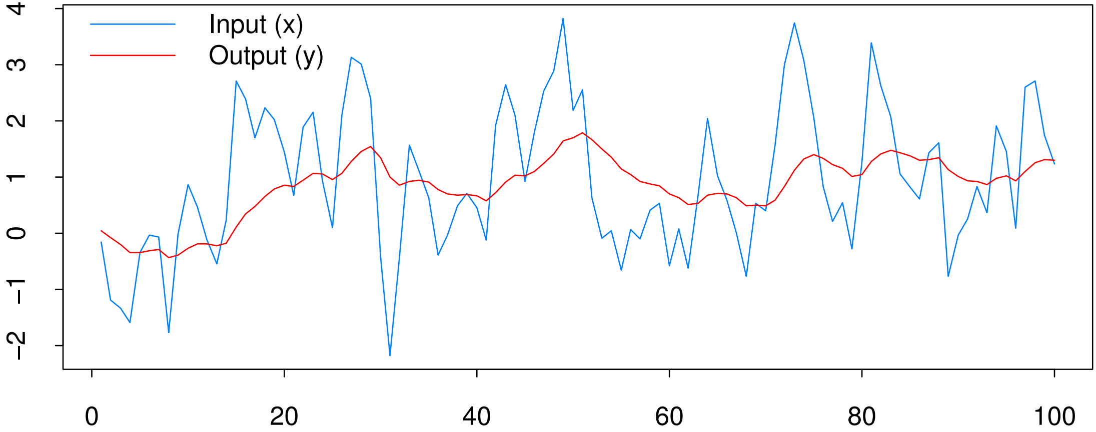
The system is NOT stable since $\sum_0^\infty |h_k| = \infty$

Where do we try to observe the impulse or step response directly. Can you name some examples where it can be possible and useful to do so?

- Sound: Clap or gun shot or blow-up balloon
- Exercise: From some day start doing regular exercise
- Experiment: Make a step increase in a set-point
- Biking: Letting go you hands from the bar, observe how you instinctive make a sort of impulse with your but to learn the response!
- Almost any activity where the system has some dynamics

Dynamic response characteristics from data

- While easy, direct observations of the impulse or step responses is not always possible at all – and do not yield a lot of statistical information.
- Instead, we use parameter estimation from data with varying inputs.



Stability based on the impulse response function

If the impulse response function is absolutely convergent, the system is stable (Theorem 4.3).

- Continuous time:

$$\int_{-\infty}^{\infty} |h(u)| \mathrm{d}u < \infty$$

- Discrete time:

$$\sum_{k=-\infty}^{\infty} |h_k| < \infty$$

The z-transform

- A way to describe dynamical systems in *discrete time in the frequency domain*:

$$Z(\{x_t\}) = \sum_{t=-\infty}^{\infty} x_t z^{-t} = X(z) \qquad (z \in \mathbb{C})$$

- $z = Ae^{i(\omega t + \phi)}$: A is the amplitude, ω is the angular frequency, t is time, and ϕ is the phase. The frequency f can be derived from the angular frequency ω using the relationship:

$$f = \frac{\omega}{2\pi}$$

- The z-transform of a time delay: $Z(\{x_{t-\tau}\}) = z^{-\tau} X(z)$

- The *transfer function* of the system is called $H(z) = \sum_{t=-\infty}^{\infty} h_t z^{-t}$

$$y_t = \sum_{k=-\infty}^{\infty} h_k x_{t-k} \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

Time domain $\Leftrightarrow$ *Frequency domain*

Linear Difference Equation

$$y_t + a_1 y_{t-1} + \cdots + a_p y_{t-p} = b_0 x_{t-\tau} + b_1 x_{t-\tau-1} + \cdots + b_q x_{t-\tau-q}$$
$$(1 + a_1 z^{-1} + \cdots + a_p z^{-p}) Y(z) = z^{-\tau}(b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}) X(z)$$

Transfer function:

$$\begin{aligned} H(z) &= \frac{z^{-\tau}(b_0 + b_1 z^{-1} + \cdots + b_q z^{-q})}{(1 + a_1 z^{-1} + \cdots + a_p z^{-p})} \\ &= \frac{z^{-\tau}(1 - n_1 z^{-1})(1 - n_2 z^{-1}) \cdots (1 - n_q z^{-1}) b_0}{(1 - \lambda_1 z^{-1})(1 - \lambda_2 z^{-1}) \cdots (1 - \lambda_p z^{-1})} \end{aligned}$$

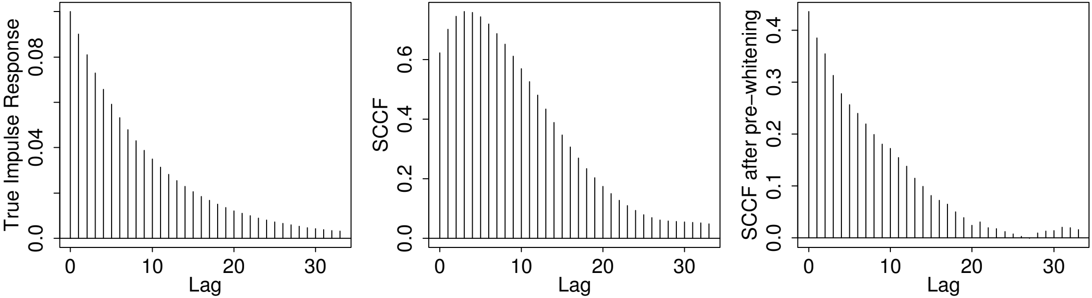
Where the roots $n_1, n_2, \ldots, n_q$ are called the *zeros of the system* and $\lambda_1, \lambda_2, \ldots, \lambda_p$ are called the *poles of the system*. What does these roots say about stability and invertibility of the system?

The system is stable if all poles lie within the unit circle
The system is invertible if all zeroes lie within the unit circle

In the course, we don't move longer into the frequency domain!

Estimating the impulse response

- The shape of the impulse response function is dictated by what kind of relationship there is between the input, X and the output, Y.
- The CCF (cross-correlation function) can be used to reveal this relationship, but requires pre-whitening



- Alternative is to simply make an LS with multiple inputs and all lags (up to some max lag)! Works for FIR and ARX models.

Estimating the impulse response

- Pre-whitening:

  Identify a suitable ARMA to the input: Filter both input and output with the ARMA, and on residuals CCF is impulse response estimate.

  Pros: Identify structure of ARMA, i.e. also MA part

  Cons: Only works on single input and requires some (manual) modelling decision while doing it

- LS-estimates:

  $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\theta} + \boldsymbol{\varepsilon}$, where columns of $\boldsymbol{X}$ are the lagged inputs (Equation (8.49) in the Book).

  Pros: No (manual) modelling decisions and works for multiple input models

  Cons: Can not alone be used for identifying an MA part.

Example: CCF vs. LS estimate of the impulse response

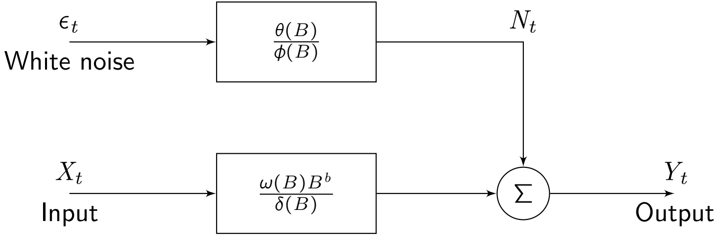
```
# Generate an AR(1) process as input
n <- 500
x <- c(arima.sim(list(ar=c(0.7,0,0)), n))
plot(x)

# Make an output vector "filtered" by the system: y_t = 0.8 * y_{t-1} + x_t
y <- filter(x,0.8,"recursive") + rnorm(n)

# Calculate the true impulse response and plot with CCF
k <- 0:20
par(mfrow=c(2,1))
plot(k, 0.8^k, type="h", main="TRUE IR")
ccf(y,x, xlim=c(0,max(k)))

# Make lags for LS estimation
library(onlineforecast)
D <- as.data.frame(y=y, lagdf(x, k))
# See the model and the estimated result
(frml <- paste0("y ~ 0+",paste0("k",k,collapse="+")))
fit <- lm(frml, D)
summary(fit)

# Finally, Plot the true and the LS estimated IR
plot(k, 0.8^k, type="h", main="TRUE IR")
plot(0:20, fit$coef, type="h", xlab="lag", ylab="Impulse response")
```

Example: Loadspeaker and microphone system

Sound through loudspeaker and into mic example:

Open `impulse_reponse_record/Record.aup3`
and then the analysis `example_IR_sound.R`.

"Complete" Transfer function models



$$Y_t = \frac{\omega(B)}{\delta(B)} B^b X_t + \frac{\theta(B)}{\varphi(B)} \varepsilon_t$$

- Also called Box-Jenkins models

Some names

The following are all sub-models of transfer function models:

- FIR: Finite Impulse Response (impulse response function(s) of finite length): $y_t = \sum_{k=-\infty}^{\infty} h(k) x_{t-k}$.
- ARX: Auto Regressive with eXogenous input: $\varphi(B) Y_t = \omega(B) u_t + \epsilon_t$.
- ARMAX: Auto Regressive Moving Average, eXogenous input: $\varphi(B) Y_t = \omega(B) X_t + \theta(B) \varepsilon_t$.
- OE: Output Error model: $Y_t = \frac{\omega(B)}{\delta(B)} B^b X_t + \varepsilon_t$.
- Regression models with ARMA noise (the `xreg` option to `arima` in R). Parameters are estimated in the same optimization:

$$(Y_t - \beta_0 + \beta_1 X_t) = (Y_t - \beta_0 + \beta_1 X_t) + \frac{\theta(B)}{\varphi(B)} \varepsilon_t$$

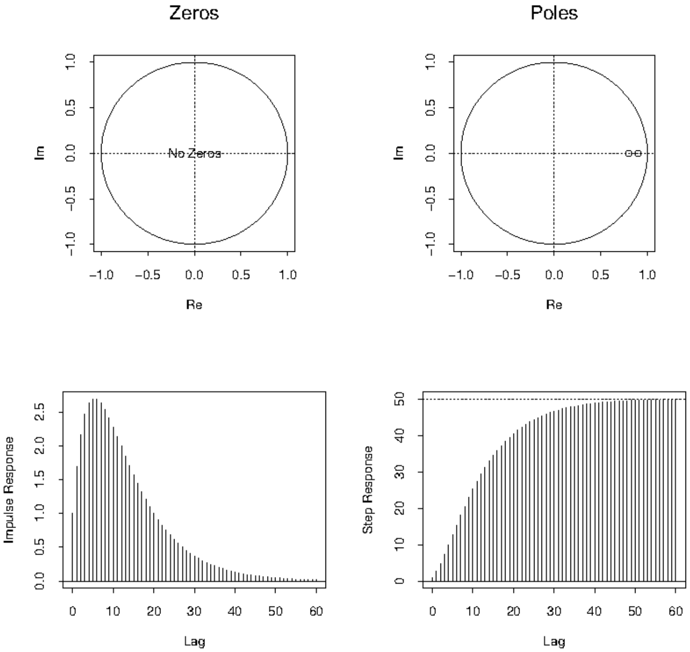
Identification of transfer function models

$$h(B) = \frac{\omega(B)B^b}{\delta(B)} = h_0 + h_1 B + h_2 B^2 + h_3 B^3 + h_4 B^4 + \dots$$

- Estimate the impulse response (pre-whitening or LS-estimate) and "guess" an appropriate structure of $h(B)$ based on this.

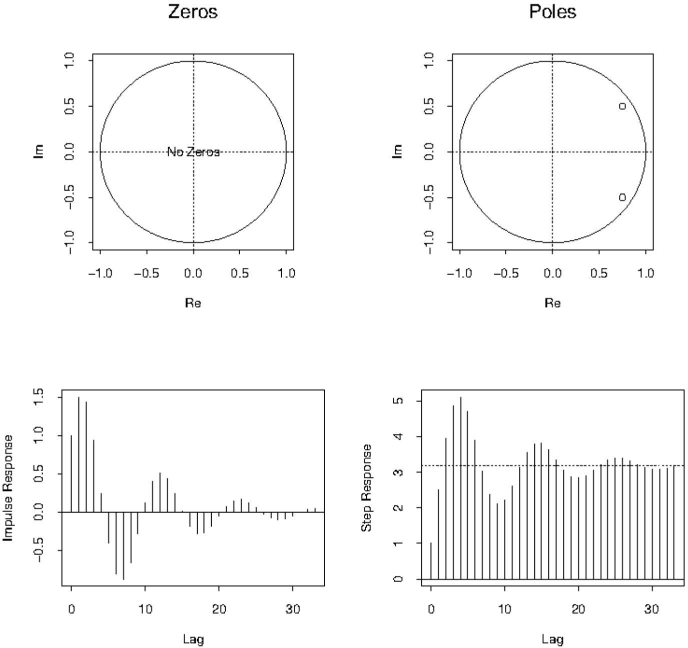
2 real poles

$$h(B) = \frac{1}{1 - 1.7B + 0.72B^2}$$



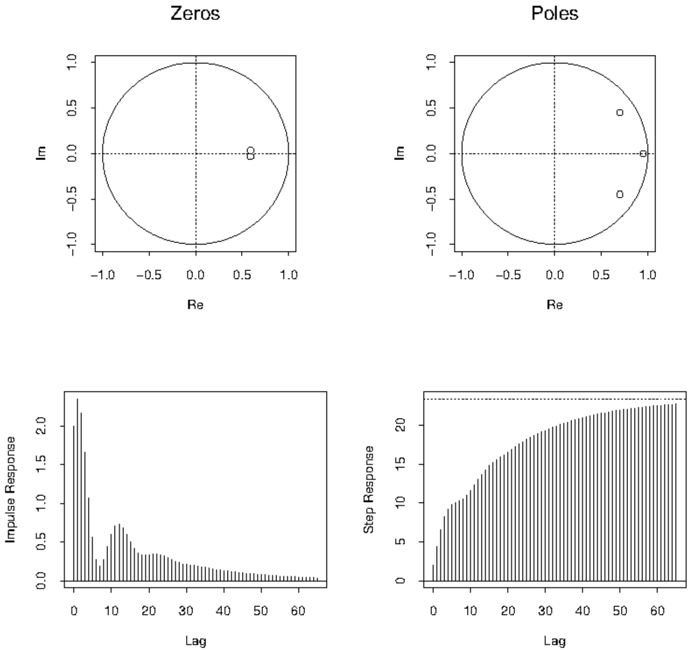
2 complex

$$h(B) = \frac{1}{1 - 1.5B + 0.81B^2}$$



1 real, 2 comp

$$h(B) = \frac{2 - 2.35B + 0.69B^2}{1 - 2.35B + 2.02B^2 - 0.66B^3}$$



Identification of the transfer function for the noise

EITHER:

- After selection of the structure of the transfer function of the input we estimate the parameters of the model (assuming N_t to be white)

$$Y_t = \frac{\omega(B)}{\delta(B)} B^b X_t + N_t$$

- Then, we extract the residuals $\{N_t\}$ and identify a structure for an ARMA model of this series

$$N_t = \frac{\theta(B)}{\varphi(B)} \varepsilon_t \quad \Leftrightarrow \quad \varphi(B) N_t = \theta(B) \varepsilon_t$$

- Finally, we have the full structure of the model and we estimate all parameters simultaneously

OR

- Simply use a forward or backward selection procedure! It's often easier!

Estimation

- Form 1-step predictions, treating the input $\{X_t\}$ as known (corresponds to conditioning on observed $\{X_t\}$ if it is actually stochastic)
- Select the parameters so that the sum of squares of these errors is as small as possible (implicit assumption of $\{\epsilon_t\}$ being gaussian – that's very close to maximum likelihood)
- If model has MA-part (i.e. lagged residuals) some recursive method is needed (Kalman filter or Spliid method)
- For FIR and ARX models we can write the model as $\boldsymbol{Y}_t = \boldsymbol{X}_t^T\boldsymbol{\theta} + \varepsilon_t$ and use LS-estimates

Model validation

As for ARMA models with some additions:

- Usual ACF of residuals and plots!
- Test for cross correlation between the residuals and the input. If $\{\varepsilon_t\}$ is white noise and when there is no correlation between the input and the residuals then (approximately)

$$\hat{\rho}_{\varepsilon X}(k) \sim \mathcal{N}(0, 1/N)$$

- A *Portmanteau test* (*Ljung-Box*) can also be performed to test for significent ccf's.

Cross covariance and cross correlation functions

Estimate of the cross covariance function:

$$
\begin{aligned}
C_{XY}(k) &= \frac{1}{N}\sum_{t=1}^{N-k}(X_t-\overline{X})(Y_{t+k}-\overline{Y}) \\
C_{XY}(-k) &= \frac{1}{N}\sum_{t=1}^{N-k}(X_{t+k}-\overline{X})(Y_t-\overline{Y})
\end{aligned}
$$

Estimate of the cross correlation function:

$$
\widehat{\rho}_{XY}(k) = C_{XY}(k)/\sqrt{C_{XX}(0)C_{YY}(0)}
$$

What is a defining property of the CCF for causal systems with no feedback? If at least one of the processes is white noise and if the processes are uncorrelated then $\widehat{\rho}_{XY}(k)$ is approximately normally distributed with mean 0 and variance $1/N$