02417: Time Series Analysis

Week 6 - ACF and PACF with a focus on model order selection

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DTU Compute

Based on material previous material from the course

March 12, 2025

Week 6: Outline of the lecture

- Estimation of auto-covariance and -correlation, Sec. 6.2.1 (and the intro. to 6.2)
- Using the SACF and SPACF for model order selection Sec. 6.5
- Model validation, Sec. 6.6

Autocorrelation and Partial Autocorrelation

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Autocorrelation and Partial Autocorrelation

Autocorrelation

$$\rho(k) = \text{Cor}[Y_t, Y_{t+k}]$$

- Sample autocorrelation function (SACF): $\widehat{\rho}(k) = r_k = C(k)/C(0)$
- For white noise and $k \neq 0$ it holds that $E[\widehat{\rho}(k)] \simeq 0$ and $V[\widehat{\rho}(k)] \simeq 1/N$, this gives the bounds $\pm 2/\sqrt{N}$ for deciding when it is not possible to distinguish a value from zero.
- R: `acf(x)`

Partial autocorrelation

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Partial autocorrelation

$$\phi_{kk} = \text{Cor}[Y_t, Y_{t+k} | Y_{t+1}, \ldots, Y_{t+k-1}]$$

- Sample partial autocorrelation function (SPACF): Use the Yule-Walker equations on $\widehat{\rho}(k)$ (exactly as for the theoretical relations Eq.(5.81)) or as in next slide
- It turns out that $\pm 2/\sqrt{N}$ is also appropriate for deciding when the SPACF is zero
- R: `acf(x, type="partial")` or `pacf(x)`

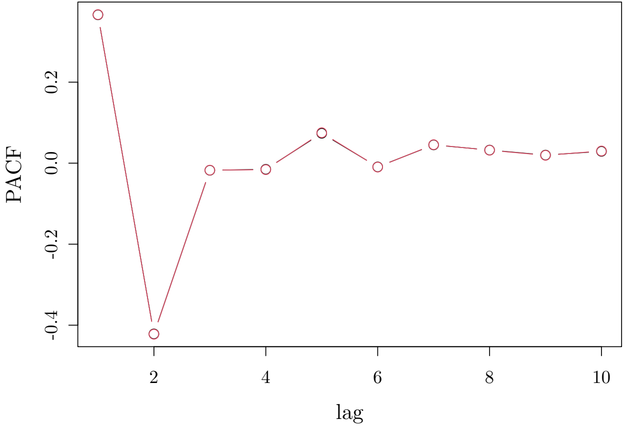
Autocorrelation and Partial Autocorrelation

```
# Example to show how the PACF is calculated
set.seed(972)
n <- 1000
x <- arima.sim(list(ar=c(0.5,-0.4)), n=n)
#acf(x)
#pacf(x)
D <- lagdf(c(x), 0:50)

# A way to calculate the PACF
lag.max <- 10
pacf1 <- numeric(lag.max)
# First, calculate it with the function
val <- pacf(x, lag.max, plot=FALSE)
# Then calc on our own
for(k in 1:lag.max){
  # Fit a regression model with 1 to k lags (and intercept)
  (frml <- pst("k0 ~ 1 + ",pst("k",1:k, collapse=" + ")))
  fit <- lm(frml, D)
  # Take the coefficient for the k lag
  pacf1[k] <- fit$coef[pst("k",k)]
}
```

Autocorrelation and Partial Autocorrelation

```r
# It's very close!
pacf1 - val$acf
plot(val$acf, type="b", xlab="lag", ylab="PACF")
lines(pacf1, type="b", col=2)
```



ARIMA models

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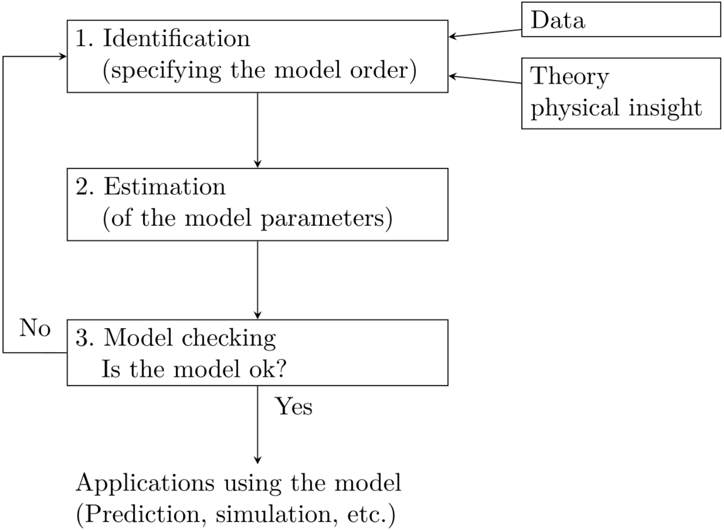
- Basically ARIMA(p, d, q) has

  Auto-**R**egression order p

  Moving-**A**verage order q

  Integration order d

Model building in general

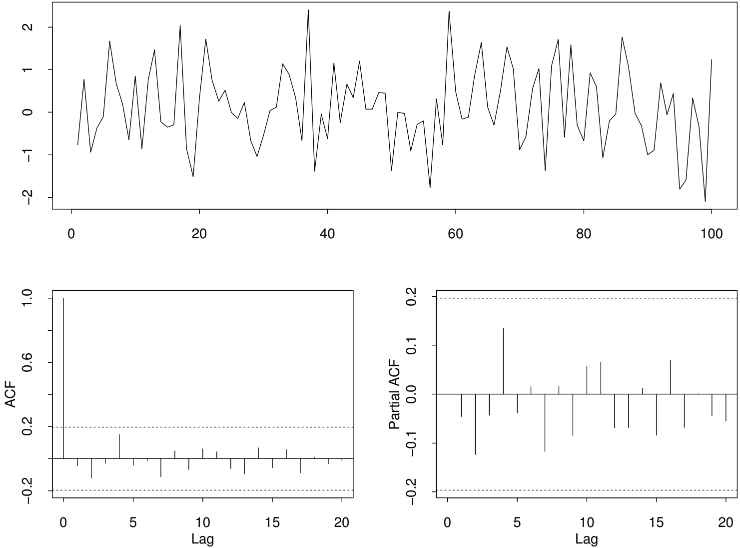


The golden table for ARMA identification

(Table 6.1)

	ACF $\rho(k)$	PACF ϕ_{kk}
AR(p)	Damped exponential and/or sine functions	$\phi_{kk} = 0$ for $k > p$
MA(q)	$\rho(k) = 0$ for $k > q$	Dominated by damped exponential and or/sine functions
ARMA(p, q)	Damped exponential and/or sine functions after lag max$(0, q - p)$	Dominated by damped exponential and/or sine functions after lag max$(0, p - q)$

1. What would be an appropriate structure?



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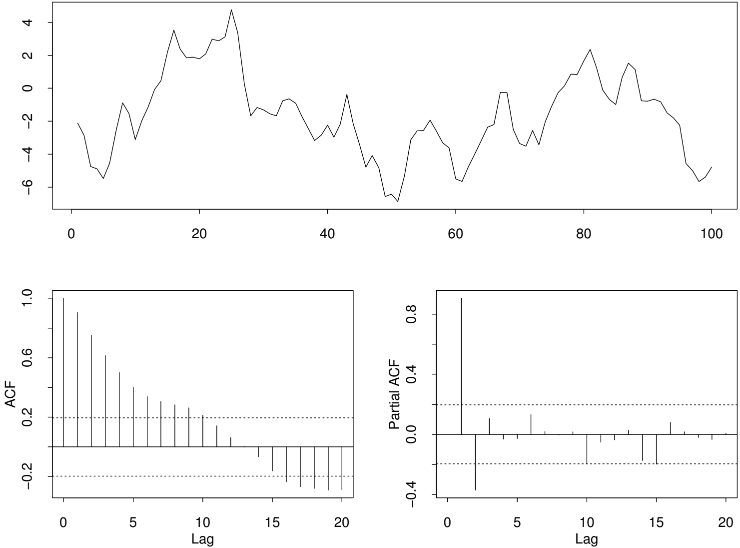

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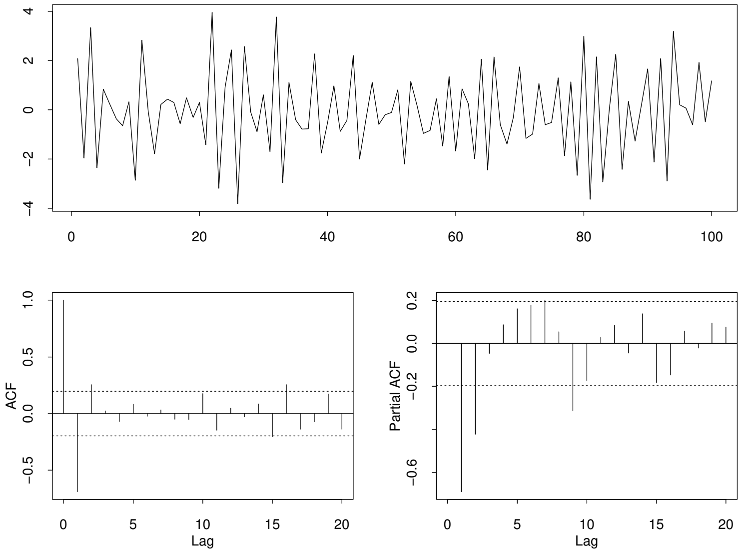

1: White noise   2: AR(1)   3: AR(2)
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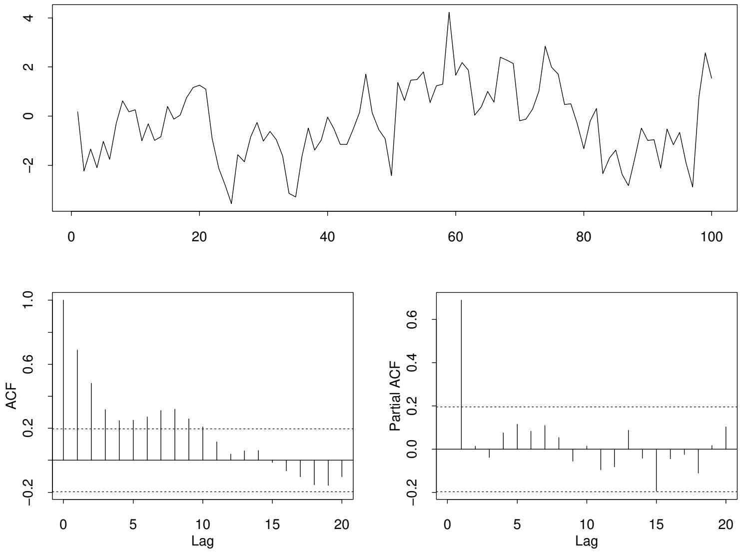
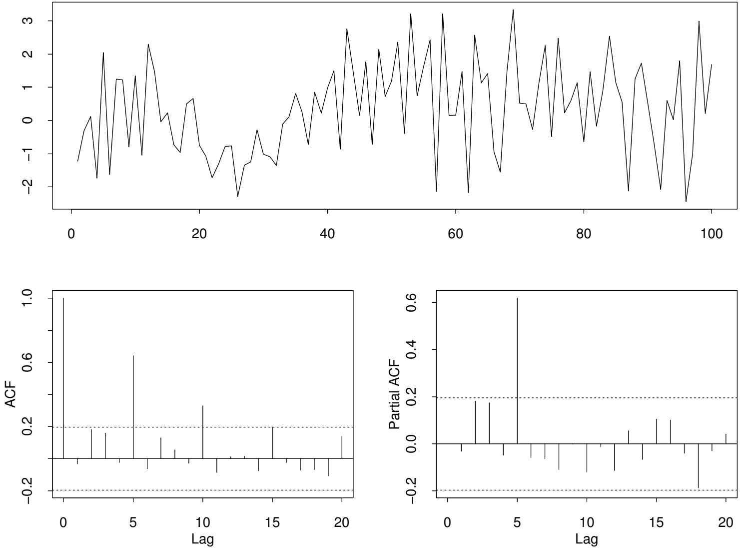
4. What would be an appropriate structure?



1: White noise 2: AR(1) 3: AR(2)
4: MA(1) 5: AR(8) 6: ARMA(1,1)

5. What would be an appropriate structure?



1: AR(1) 2: AR(5) 3: AR(0)x(2)$_5$

4: AR(0)x(1)$_4$ 5: AR(0)x(1)$_5$ 6: AR(10)

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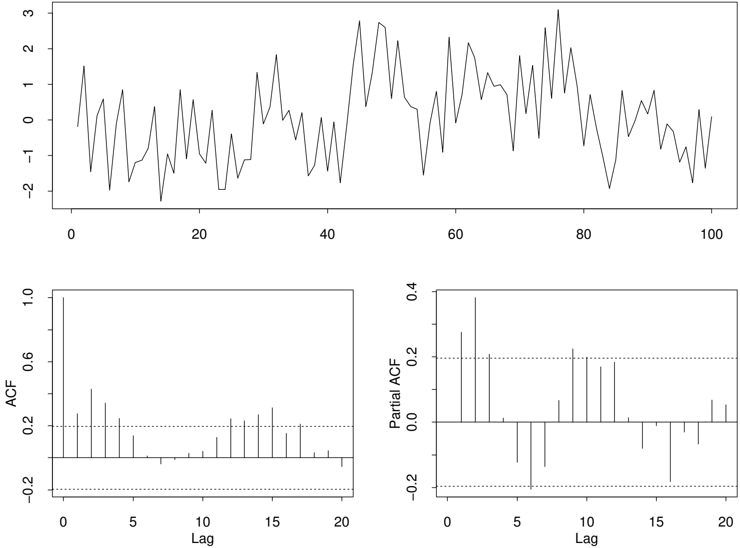
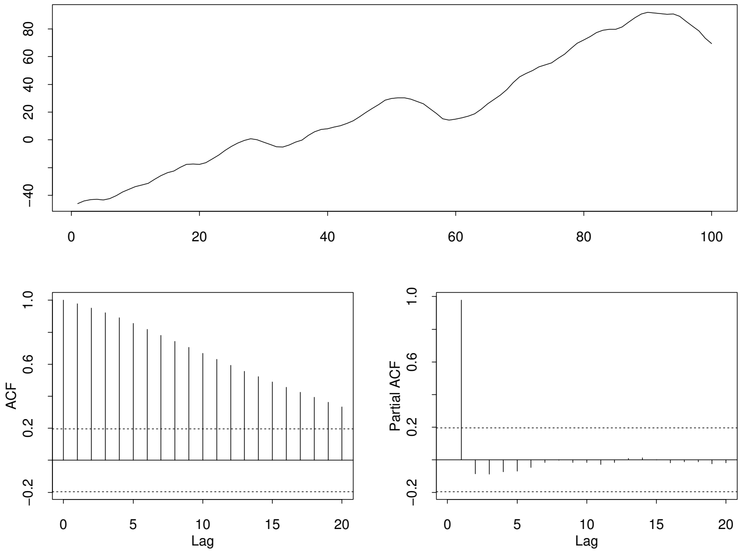


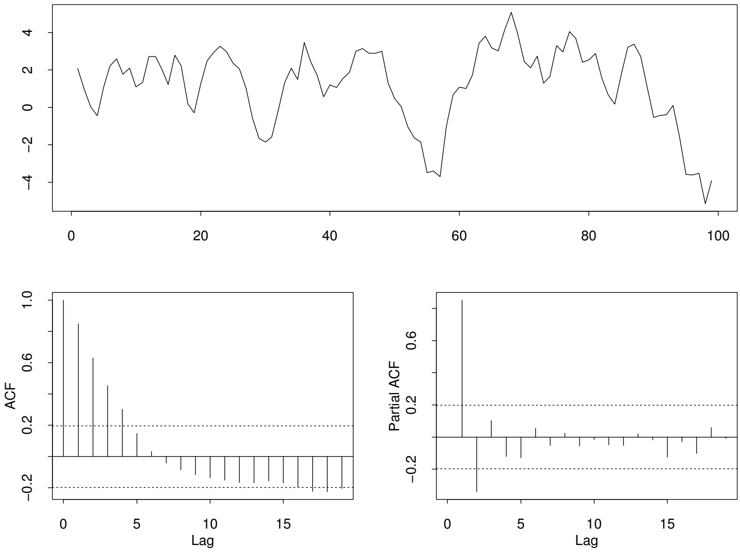
1: AR(4)	2: AR(5)	3: MA(4)
4: ARMA(1,1)	5: AR(2)	6: ARMA(2,2)

7. What would be an appropriate structure?



1: AR(3)	2: AR(2)	3: ARIMA(0,1,0)
4: ARMA(1,1)	5: ARIMA(2,1,0)	6: ARIMA(0,2,2)

8. Same series; analysing $\nabla Y_t = (1 - B)Y_t = Y_t - Y_{t-1}$



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How does $C(k)$ (i.e. auto-covariance) behave for non-stationary series?

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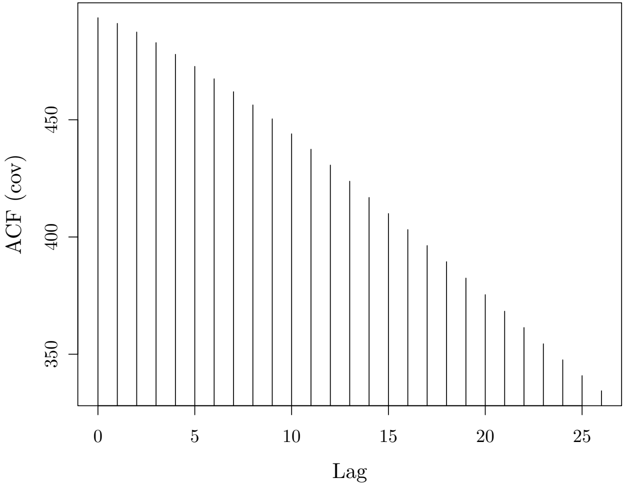

Identification of the order of differencing

- Select the order of differencing d as the first order for which the autocorrelation decreases sufficiently fast towards 0
- In practice d is 0, 1, or maybe 2
- Sometimes a periodic difference is required, e.g. $Y_t - Y_{t-12}$

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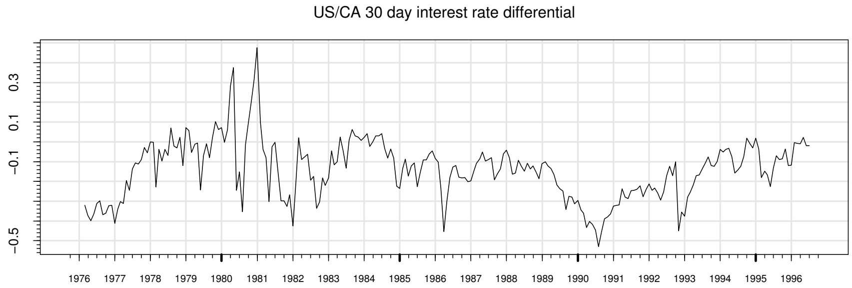
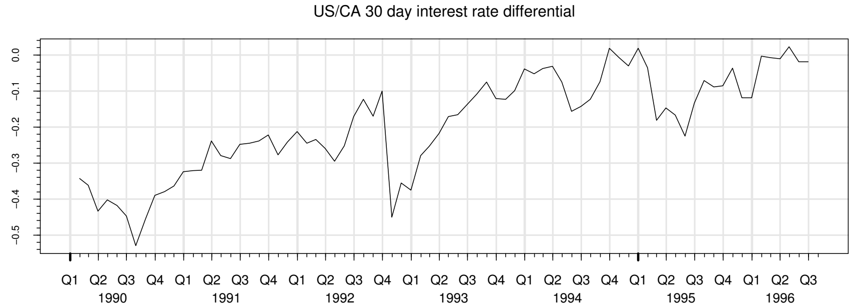
- Remember to consider the practical application. E.g. it may be that the system is stationary, but you measured over a too short period

Stationarity vs. length of measuring period





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- then try a small model and analyse the residuals
- and/or Consider transformations
  Typically sqrt, log, square or inverse.

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- (1: Identification step): Construct a model for your data:

$$\phi(B)\,Y_t = \theta(B)\varepsilon_t$$

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- (4): If the model residuals do not resemble white noise, then what do they look like?

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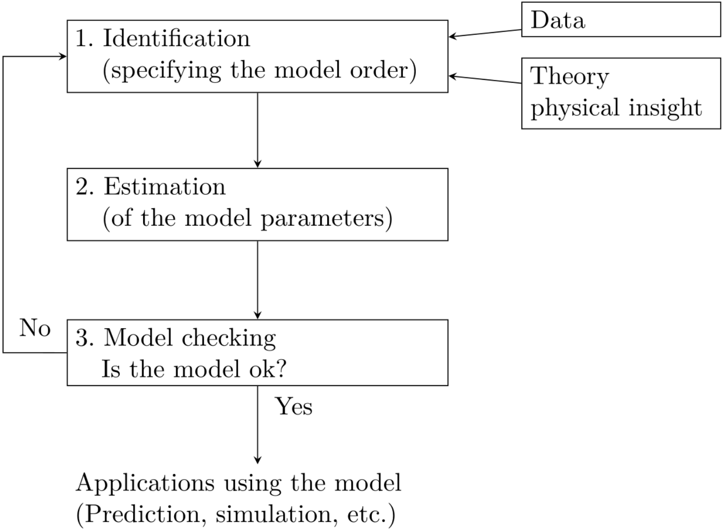
- (3): Estimate the parameters in the model above with coefficients in $\phi^* \cdot \phi$, $\theta \cdot \theta^*$ varying freely, and proceed to model check.

Model building in general



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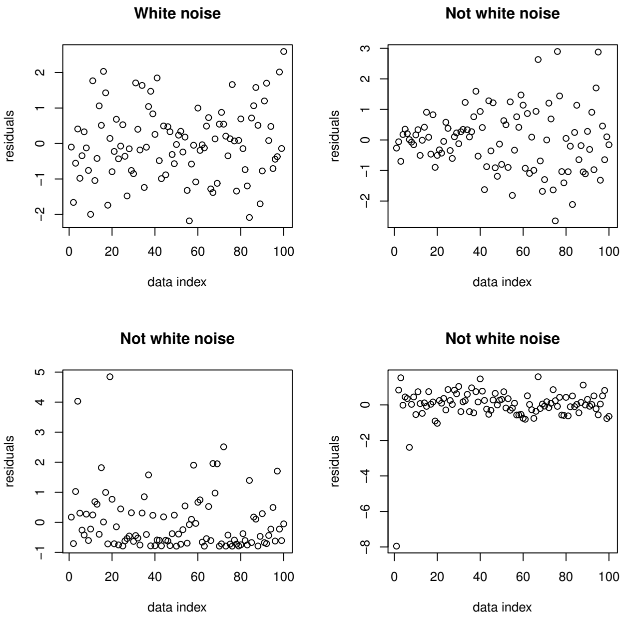
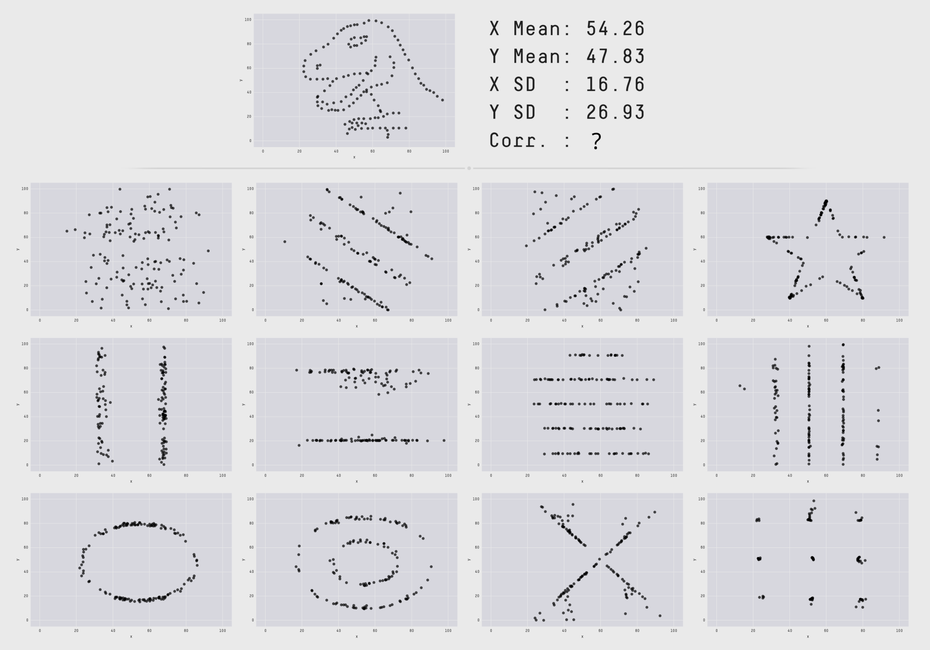
- How can we check that the model errors resemble white noise?
- First and most important - plot the data.

Residual analysis – Plot the data



Residual analysis – Plot the data II: What is the correlation?



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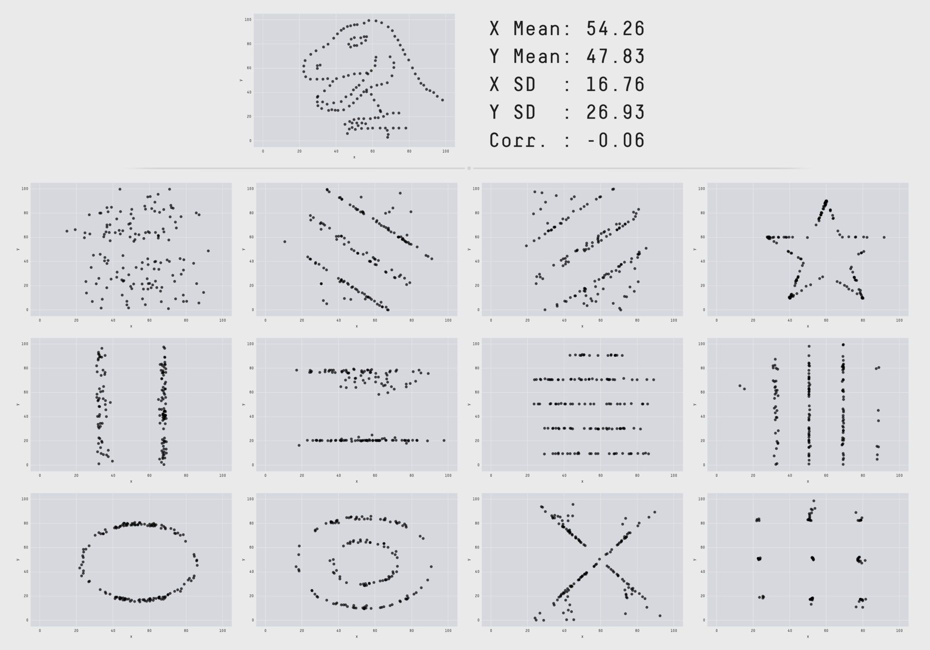
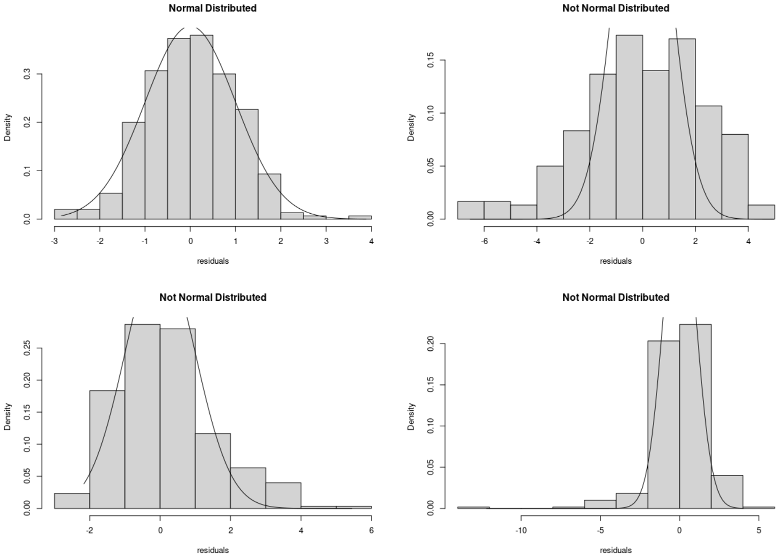
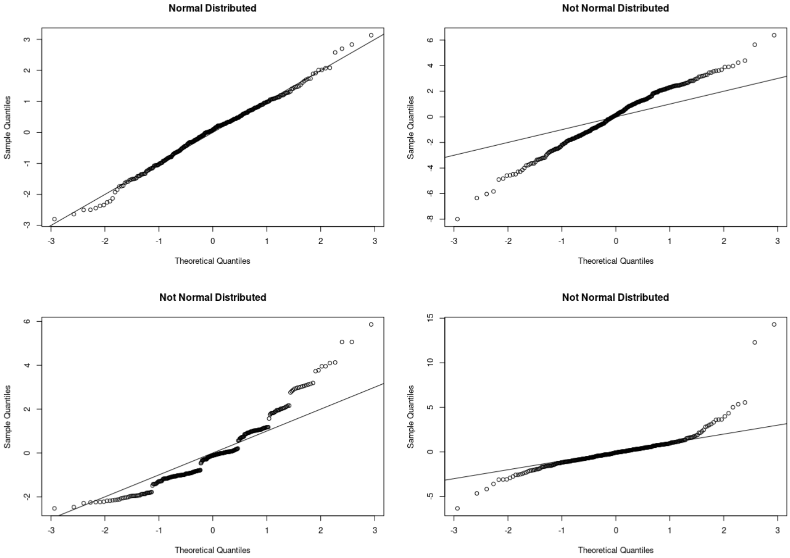


Residual analysis – Plot the data III

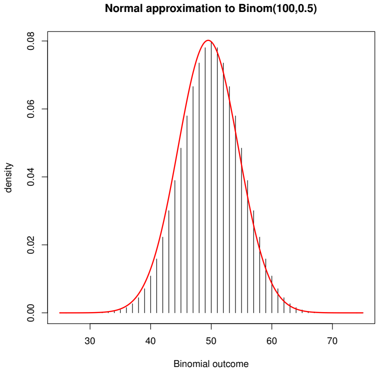


Residual analysis – Plot the data IV

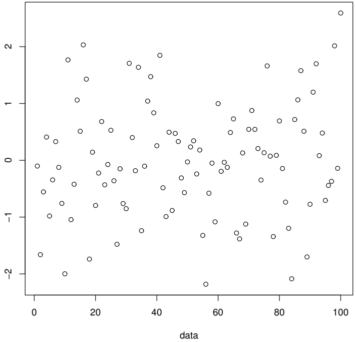


Residual analysis – sign test

- If (ε_t) is white noise, the probability that a new value has a different sign than the previous is $\frac{1}{2}$.
- Number of sign changes: $Binom(N-1, \frac{1}{2})$.
- Approx. normal distribution; $N((N-1)/2, (N-1)/4)$:



Residual analysis – sign test II



- 95% confidence interval for sign changes within 100 white noise residuals: [40; 59]. Actual sign changes from the 100 data: 47.

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Sign tests detects both asymmetry and correlation.

- Too few may indicate positive one-step correlation;
- Too many may indicate negative one-step correlation;
- Too few or too many may indicate that P(being above the mean) $\neq \frac{1}{2}$ with no correlation.

Residual analysis - other tests

- There is a bunch of other tests out there, sometimes it's actually not clear which the best in a given case!
- You can use them in assignments, if not covered by the book, then you must introduce them.

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- Perform a couple of statistical tests to get some quantitative measures of whether your residuals are alright.
- Finally, see whether parameters are significant and if not, remove them (you do not need to redo residuals analysis after this).

Information criteria

When considering multiple non-nested candidate models, information criteria can be used:

- Select the model which minimizes an information criterion.

  Akaike's Information Criterion:

$$AIC = -2\log(L(Y_N; \hat{\theta}, \hat{\sigma}_\varepsilon^2)) + 2n_{\text{par}}$$

  Bayesian Information Criterion (preferred):

$$BIC = -2\log(L(Y_N; \hat{\theta}, \hat{\sigma}_\varepsilon^2)) + \log(N)n_{\text{par}}$$

- AIC is most commonly used, but BIC yields a consistent estimate of the model order.

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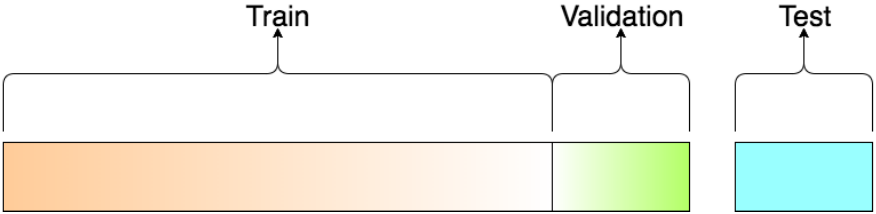

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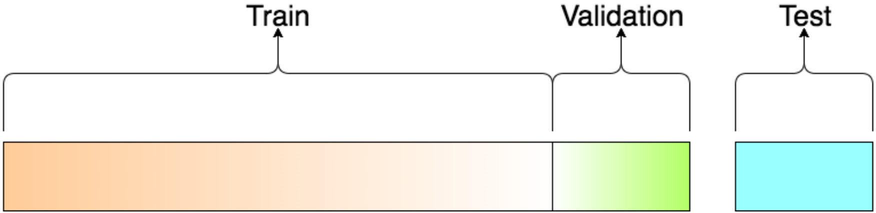

- So we must gradually move the part used for training forward in time, it's called "rolling horizon" cross-validation
- Mainly used for forecasting applications
- Remember a burn-in period and then step forward from there