02417: Time Series Analysis

Week 5 - AR, MA and ARMA processes

Peder Bacher
DTU Compute

Based on material previous material from the course

March 7, 2025

Week 5: Outline of the lecture

- Stochastic processes - 2nd part:

  MA, AR, and ARMA-processes, Sec. 5.5

  Non-stationary models, Sec. 5.6

  Seasonal ARIMA models

  Optimal Prediction, Sec. 5.7

- Estimation of parameters in linear dynamic models, Sec. 6.4

Linear process as a statistical model?

$$Y_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \psi_3 \varepsilon_{t-3} + \dots$$

- Observations: $Y_1, Y_2, Y_3, \dots, Y_N$
- Task: Find an infinite number of parameters from N observations!
- Solution: Restrict the sequence $1, \psi_1, \psi_2, \psi_3, \dots$

MA(q), AR(p), and ARMA(p, q) processes

$$Y_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \ldots + \theta_q\varepsilon_{t-q}$$
$$Y_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} = \varepsilon_t$$
$$Y_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} = \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \ldots + \theta_q\varepsilon_{t-q}$$

$\{\varepsilon_t\}$ is white noise

$$Y_t = \theta(B)\varepsilon_t$$
$$\phi(B)\, Y_t = \varepsilon_t$$
$$\phi(B)\, Y_t = \theta(B)\varepsilon_t$$

where

$$\phi(B) = (1 + \phi_1 B + \phi_1 B^2 + \ldots + \phi_p B^p)$$
$$\theta(B) = (1 + \theta_1 B + \theta_2 B^2 + \ldots + \phi_q B^q)$$

are polynomials in the backward shift operator B, ($BX_t = X_{t-1}$, $B^2 X_t = X_{t-2}$)

Invertibility and Stationarity

- A stochastic process is said to be *invertible* if a finite amount of observations can determine its state.

- A stochastic process is said to be *stationary* if?

- A stochastic process is said to be *stationary* if its distribution does not change over time.

Invertibility and Stationarity of ARMA models

- $MA(q): Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}$

  Always stationary

  Invertible if the roots in $\theta(z^{-1}) = 0$ with respect to z all are within the unit circle

- $AR(p): Y_t + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} = \varepsilon_t$

  Always invertible

  Stationary if the roots of $\phi(z^{-1})$ with respect to z all lie within the unit circle

- $ARMA(p, q)$

  Stationary if the roots of $\phi(z^{-1})$ with respect to z all lie within the unit circle

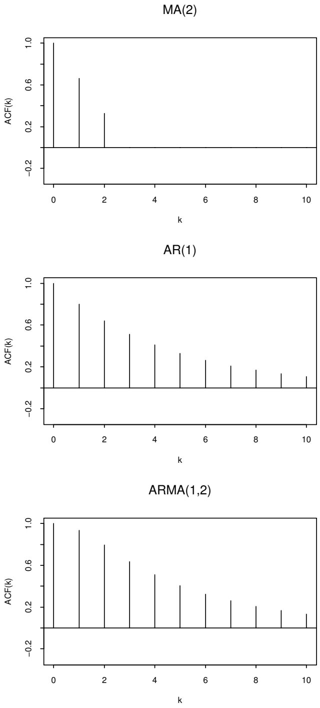
  Invertible if the roots in $\theta(z^{-1})$ with respect to z all are within the unit circle

Autocorrelations

MA(2):
$Y_t = (1 + 0.9B + 0.8B^2)\varepsilon_t$
zero after lag 2

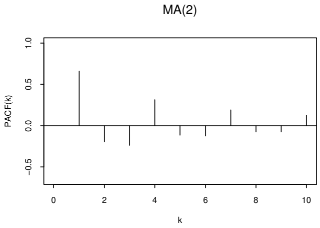
AR(1):
$(1 - 0.8B)\, Y_t = \varepsilon_t$
exponential decay (damped sine in case of complex roots)

ARMA(1,2):
$(1 - 0.8B)\, Y_t = (1 + 0.9B + 0.8B^2)\varepsilon_t$
exponential decay from lag $q + 1 - p = 2 + 1 - 1 = 2$ (damped sine in case of complex roots)

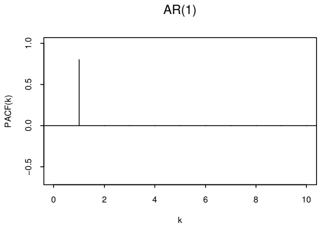


Partial autocorrelations

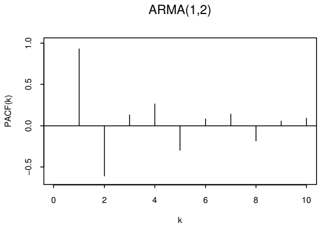
MA(2):
$Y_t = (1 + 0.9B + 0.8B^2)\varepsilon_t$



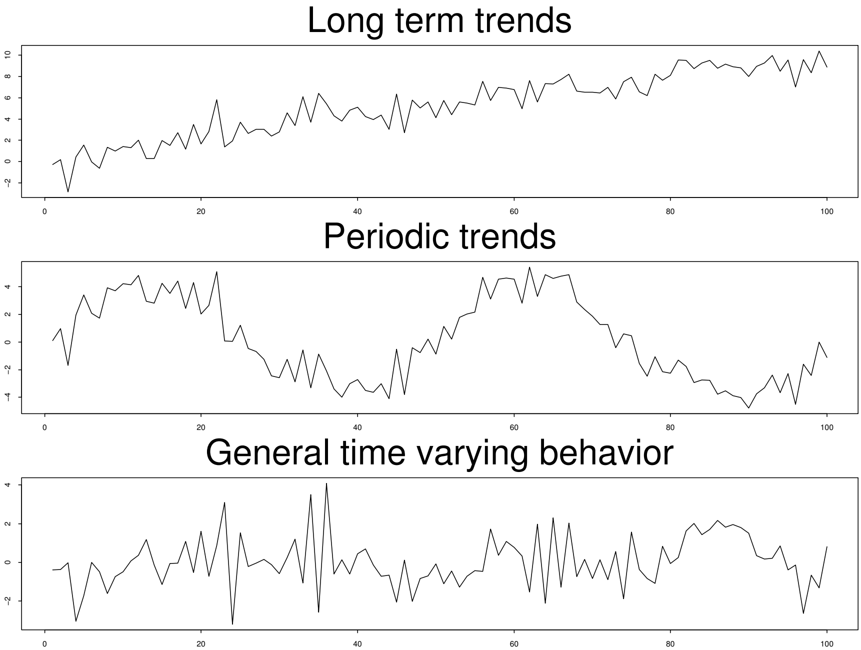
AR(1):
$(1 - 0.8B)\,Y_t = \varepsilon_t$
zero after lag 1



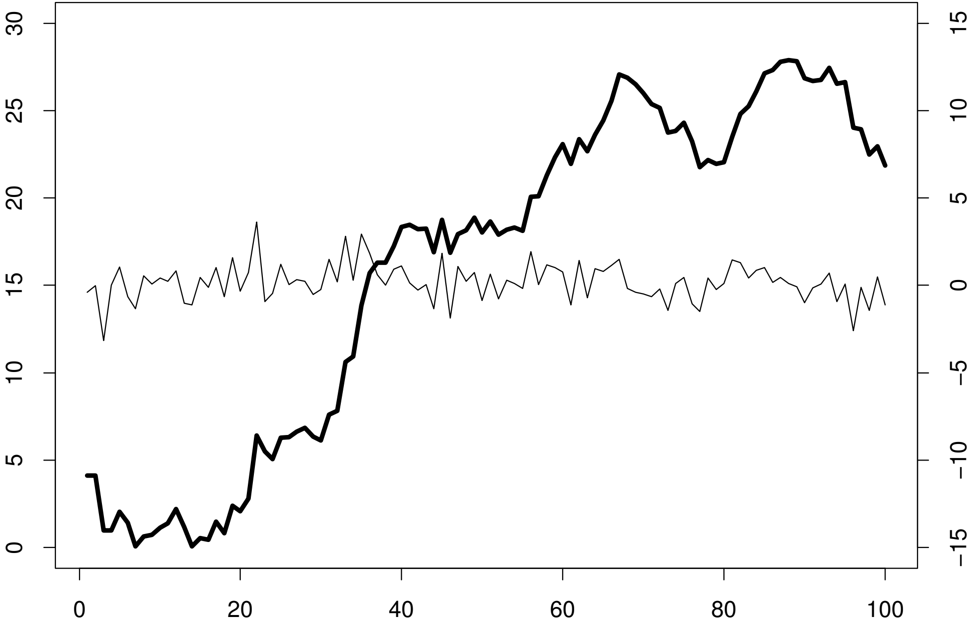
ARMA(1,2):
$(1 - 0.8B)\,Y_t = (1 + 0.9B + 0.8B^2)\varepsilon_t$



Non-stationary time series



Differencing



The $ARIMA(p, d, q)$-process

- An $ARMA(p, q)$ model for:

$$W_t = \nabla^d Y_t = (1 - B)^d Y_t$$

  where $\{Y_t\}$ is the series

- That is:

$$\phi(B)\nabla^d Y_t = \theta(B)\varepsilon_t$$

- If we consider stationarity:

$$\phi(z^{-1})(1 - z^{-1})^d = 0$$

  i.e. d roots in $z = 1 + 0i$, and the rest inside the unit circle

Seasonal Models

- In general, would you rather use new or old information in your models, for example would you prefer $Y_t = \theta Y_{t-1} + \epsilon_t$ or $Y_t = \theta Y_{t-2} + \epsilon_t$?
- When and why would it make sense to prefer older information over newer information?

The $(p, d, q) \times (P, D, Q)_s$ seasonal process

- A multiplicative (stationary) $ARMA(p, q)$ model for:

$$W_t = \nabla^d \nabla_s^D Y_t = (1 - B)^d (1 - B^s)^D Y_t$$

  where $\{ Y_t \}$ is the series

- That is:

$$\phi(B)\Phi(B^s)\nabla^d \nabla_s^D Y_t = \theta(B)\Theta(B^s)\varepsilon_t$$

- If we consider stationarity:

$$\phi(z^{-1})\Phi(z^{-s})(1 - z^{-1})^d (1 - z^{-s})^D = 0$$

  i.e. d roots in $z = 1 + 0i$, $D \times s$ roots on the unit circle, and the rest inside the unit circle

The case $d = D = 0$; stationary seasonal process

- General:

$$\phi(B)\Phi(B^s)\, Y_t = \theta(B)\Theta(B^s)\varepsilon_t$$

- Example:

$$(1 - \Phi B^{12})\, Y_t = \varepsilon_t$$

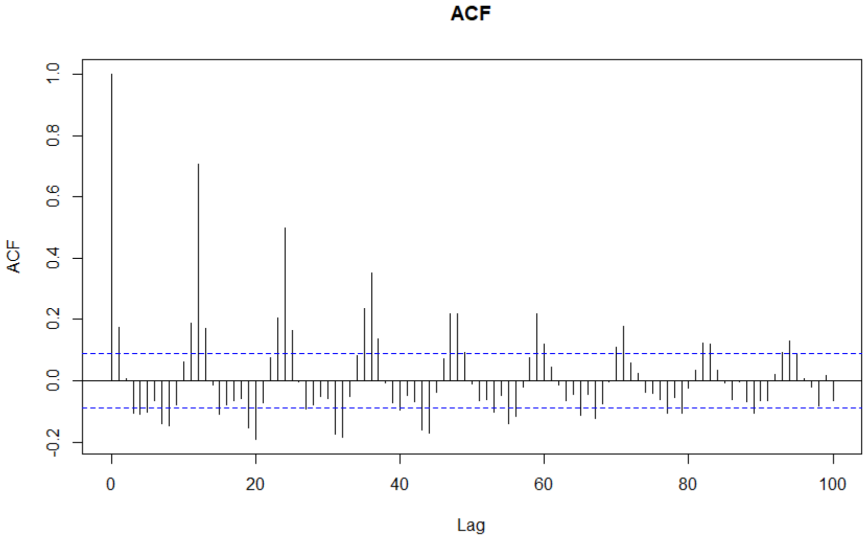
- Which can also be written:

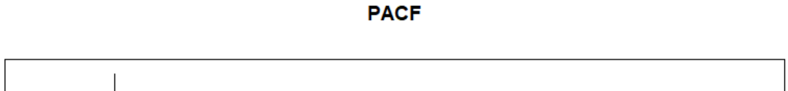
$$Y_t = \Phi\, Y_{t-12} + \varepsilon_t$$

i.e. Y_t depend on Y_{t-12}, Y_{t-24}, ... (thereof the name)

- How would you think that the auto correlation function looks?

ACF and PACF of seasonal ARMA models





Prediction

- At time t we have observations $Y_t, Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$
- We want a prediction of Y_{t+k}, where $k \geq 1$
- Thus, we want the conditional expectation:

$$\widehat{Y}_{t+k|t} = E[\,Y_{t+k}\,|\,Y_t, Y_{t-1}, Y_{t-2}, \ldots]$$

Example – prediction in the $AR(1)$ model

- We write the model like $Y_{t+1} = \phi Y_t + \varepsilon_{t+1}$ (note the sign on ϕ)

- 1-step prediction:

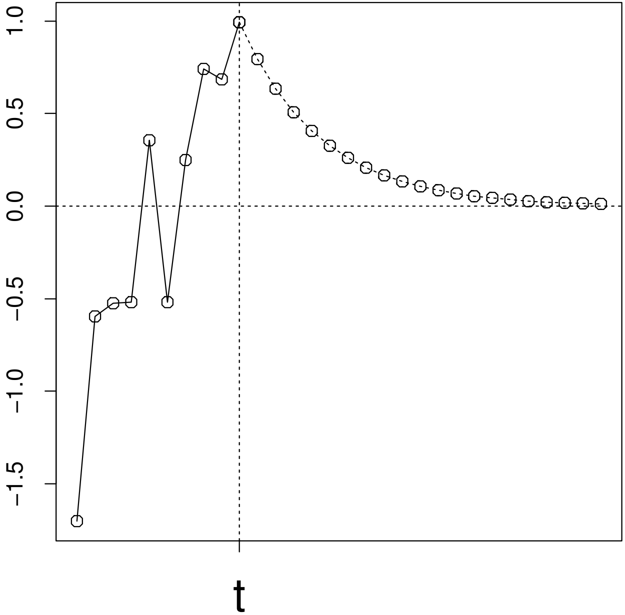
$$\begin{aligned}
\widehat{Y}_{t+1|t} &= E[\,Y_{t+1}|\,Y_t,\,Y_{t-1},\ldots] \\
&= E[\phi Y_t + \varepsilon_{t+1}|\,Y_t,\,Y_{t-1},\ldots] \\
&= \phi Y_t + 0 = \phi Y_t
\end{aligned}$$

- 2-step prediction:

$$\begin{aligned}
\widehat{Y}_{t+2|t} &= E[\,Y_{t+2}|\,Y_t,\,Y_{t-1},\ldots] \\
&= E[\phi Y_{t+1} + \varepsilon_{t+2}|\,Y_t,\,Y_{t-1},\ldots] \\
&= \phi \widehat{Y}_{t+1|t} + 0 \\
&= \phi^2 Y_t
\end{aligned}$$

- k-step prediction: $\boxed{\widehat{Y}_{t+k|t} = \phi^k Y_t}$

Example – prediction in $Y_t = 0.8\,Y_{t-1} + \varepsilon_t$



Variance of prediction error for the $AR(1)$-process

Prediction error:

$$e_{t+k|t} = Y_{t+k} - \widehat{Y}_{t+k|t} = Y_{t+k} - \phi^k Y_t$$

Bring it on psi-form (MA-form):

$$\begin{aligned}
Y_{t+k} &= \phi Y_{t+k-1} + \varepsilon_{t+k} \\
&= \phi(\phi Y_{t+k-2} + \varepsilon_{t+k-1}) + \varepsilon_{t+k} \\
&= \phi^2 Y_{t+k-2} + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k} \\
&= \phi^2(\phi Y_{t+k-3} + \varepsilon_{t+k-2}) + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k} \\
&= \phi^3 Y_{t+k-3} + \phi^2\varepsilon_{t+k-2} + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k} \\
&\vdots \\
&= \phi^k Y_t + \phi^{k-1}\varepsilon_{t+1} + \phi^{k-2}\varepsilon_{t+2} + \ldots + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k}
\end{aligned}$$

Variance of prediction error for the $AR(1)$-process

Variance of prediction error:

$$\begin{aligned} V[e_{t+k|t}] &= V[\phi^{k-1}\varepsilon_{t+1} + \phi^{k-2}\varepsilon_{t+2} + \ldots + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k}] \\ &= (\phi^{2(k-1)} + \phi^{2(k-2)} + \ldots + \phi^2 + 1)\sigma_\varepsilon^2 \end{aligned}$$

$(1-\alpha) \times 100\%$ prediction interval:

$$\widehat{Y}_{t+k|t} \pm u_{\alpha/2}\sqrt{V[e_{t+k|t}]}$$

$u_{\alpha/2}$ is the $\alpha/2$-quantile in the standard normal distribution

Estimation

- Assume that we have an appropriate model structure $AR(p)$, $MA(q)$, $ARMA(p, q)$, $ARIMA(p, d, q)$ with p, d, and q known
- **Task**: Based on the observations find appropriate values of the parameters
- The book describes many methods:
  - Moment estimates
  - LS-estimates
  - Prediction error estimates
    - Conditioned
    - Unconditioned
  - ML-estimates
    - Conditioned
    - Unconditioned (exact)

Estimation in AR(2) model

- Observations: $y_1, \; y_2, \; \ldots, \; y_N$
- Model: $y_t + \phi_1 y_{t-1} + \phi_2 y_{t-2} = \epsilon_t$

$$
\begin{aligned}
y_3 &= \phi_1 y_2 + \phi_2 y_1 + e_3 \\
y_4 &= \phi_1 y_3 + \phi_2 y_2 + e_4 \\
y_5 &= \phi_1 y_4 + \phi_2 y_3 + e_5 \\
&\vdots \\
y_N &= \phi_1 y_{N-1} + \phi_2 y_{N-2} + e_N
\end{aligned}
$$

$$
\begin{bmatrix} y_3 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} -y_2 & -y_1 \\ \vdots & \vdots \\ -y_{N-1} & -y_{N-2} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} e_{3|2} \\ \vdots \\ e_{N|N-1} \end{bmatrix}
$$

Or just:

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\theta} + \boldsymbol{\varepsilon}$$

Solution

To minimize the sum of the squared 1-step prediction errors $\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}$ we use the result for the General Linear Model from Chapter 3:

$$\widehat{\boldsymbol{\theta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}$$

With

$$\boldsymbol{X} = \begin{bmatrix} -y_2 & -y_1 \\ \vdots & \vdots \\ -y_{N-1} & -y_{N-2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{Y} = \begin{bmatrix} y_3 \\ \vdots \\ y_N \end{bmatrix}$$

- Asymptotically: $V(\hat{\theta}) = \sigma_\epsilon^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}$
- How does it generalize to AR(p)-models?
- How about ARMA(p,q)-models?

Least squares for AR

```
# Test it by comparing
model <- list(ar=c(0.4))
set.seed(12)
sim(model, 10, nburnin=100)
set.seed(12)
x <- arima.sim(model, 100)

X <- lagdf(x, 0:3)
summary(lm(k0 ~ k1, X))
summary(lm(k0 ~ k1 + k2, X))
summary(lm(k0 ~ k1 + k2 + k3, X))
```

Maximum Likelihood estimates

- $ARMA(p, q)$-process:

$$Y_t + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}$$

- Notation:

$$\begin{aligned} \boldsymbol{\theta}^T &= (\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q) \\ \mathbf{Y}_t^T &= (Y_t, Y_{t-1}, \ldots, Y_1) \end{aligned}$$

- The Likelihood function is the joint probability distribution function for all observations for given values of $\boldsymbol{\theta}$ and σ_ε^2:

$$L(\mathbf{Y}_N; \boldsymbol{\theta}, \sigma_\varepsilon^2) = f(\mathbf{Y}_N | \boldsymbol{\theta}, \sigma_\varepsilon^2)$$

- Given the observations $\mathbf{Y}_N$ we estimate $\boldsymbol{\theta}$ and σ_ε^2 as the values for which the likelihood is maximized.

The likelihood function for $ARMA(p, q)$-models

- The random variable $Y_N|\mathbf{Y}_{N-1}$ only contains ε_N as a random component
- $\{\varepsilon_t\}$ is a white noise process and therefore does not depend on anything
- Thus we know that the random variables $Y_N|\mathbf{Y}_{N-1}$ and $\mathbf{Y}_{N-1}$ are independent, hence:

$$f(\mathbf{Y}_N|\boldsymbol{\theta}, \sigma_\varepsilon^2) = f(Y_N|\mathbf{Y}_{N-1}, \boldsymbol{\theta}, \sigma_\varepsilon^2) f(\mathbf{Y}_{N-1}|\boldsymbol{\theta}, \sigma_\varepsilon^2)$$

- Repeating these arguments:

$$L(\mathbf{Y}_N; \boldsymbol{\theta}, \sigma_\varepsilon^2) = \left(\prod_{t=p+1}^{N} f(Y_t|\mathbf{Y}_{t-1}, \boldsymbol{\theta}, \sigma_\varepsilon^2) \right) f(\mathbf{Y}_p|\boldsymbol{\theta}, \sigma_\varepsilon^2)$$

Evaluating the conditional likelihood function

- **Task**: Find the conditional 1-step densities, $f(Y_t|\mathbf{Y}_{t-1}, \boldsymbol{\theta}, \sigma_\varepsilon^2)$, given specified values of the parameters $\boldsymbol{\theta}$ and σ_ε^2
- The mean of the random variable $Y_t|\mathbf{Y}_{t-1}$ is the the 1-step forecast $\widehat{Y}_{t|t-1}$
- The prediction error $\varepsilon_t = Y_t - \widehat{Y}_{t|t-1}$ has variance σ_ε^2
- We assume that the process is Gaussian:

$$f(Y_t|\mathbf{Y}_{t-1}, \boldsymbol{\theta}, \sigma_\varepsilon^2) = \frac{1}{\sigma_\varepsilon\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma_\varepsilon^2}(Y_t - \widehat{Y}_{t|t-1}(\boldsymbol{\theta}))^2\right)$$

- And therefore:

$$L(\mathbf{Y}_N; \boldsymbol{\theta}, \sigma_\varepsilon^2) = (\sigma_\varepsilon^2 2\pi)^{-\frac{N-p}{2}} \exp\left(-\frac{1}{2\sigma_\varepsilon^2}\sum_{t=p+1}^{N} \varepsilon_t^2(\boldsymbol{\theta})\right)$$

ML-estimates

- The (conditional) ML-estimate $\widehat{\boldsymbol{\theta}}$ is a prediction error estimate since it is obtained by minimizing

$$S(\boldsymbol{\theta}) = \sum_{t=p+1}^{N} \varepsilon_t^2(\boldsymbol{\theta})$$

- By differentiating w.r.t. σ_ε^2 it can be shown that the ML-estimate of σ_ε^2 is (remember that p is the order of the AR part):

$$\widehat{\sigma}_\varepsilon^2 = S(\widehat{\boldsymbol{\theta}})/(N-p)$$

- The estimate $\widehat{\boldsymbol{\theta}}$ is asymptotically unbiased and efficient, and the variance-covariance matrix is approximately

$$2\sigma_\varepsilon^2 \boldsymbol{H}^{-1}$$

where $\boldsymbol{H}$ contains the 2nd order partial derivatives of $S(\boldsymbol{\theta})$ at the minimum

Finding the ML-estimates using the PE-method

- 1-step predictions:

$$\widehat{Y}_{t+1|t} = -\phi_1 Y_t - \ldots - \phi_p Y_{t-p+1} + \theta_1 \varepsilon_t + \ldots + \theta_q \varepsilon_{t-q+1}$$

- If we use (Condition on) $\varepsilon_p = \varepsilon_{p-1} = \ldots = \varepsilon_{p+1-q} = 0$ we can find:

$$\widehat{Y}_{p+1|p} = -\phi_1 Y_p - \ldots - \phi_p Y_1 + \theta_1 \varepsilon_p + \ldots + \theta_q \varepsilon_{p-q+1}$$

- Which will give us $\varepsilon_{p+1} = Y_{p+1} - \widehat{Y}_{p+1|p}$ and we can then calculate $\widehat{Y}_{p+2|p+1}$ and ε_{p+2} . . . and so on until we have all the 1-step prediction errors we need.

- We use numerical optimization to find the parameters which minimize the sum of squared prediction errors