Approximate Near Neighbor Search: Locality Sensitive Hashing

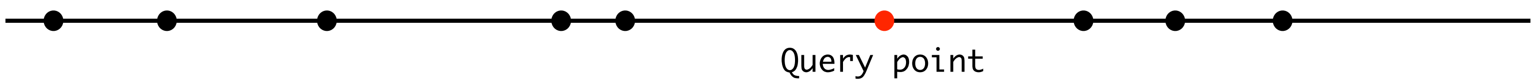
Inge Li Gørtz

Nearest Neighbor

- Nearest Neighbor. Given a set of points P in a metric space, build a data structure which given a query point x returns the point in P closest to x.

- Metric. Distance function d is a metric:
  1. $d(x,y) \geq 0$
  2. $d(x,y) = 0$ if and only if $x = y$
  3. $d(x,y) = d(y,x)$
  4. $d(x,y) \leq d(x,z) + d(z,y)$

- Warmup. 1D: Real line

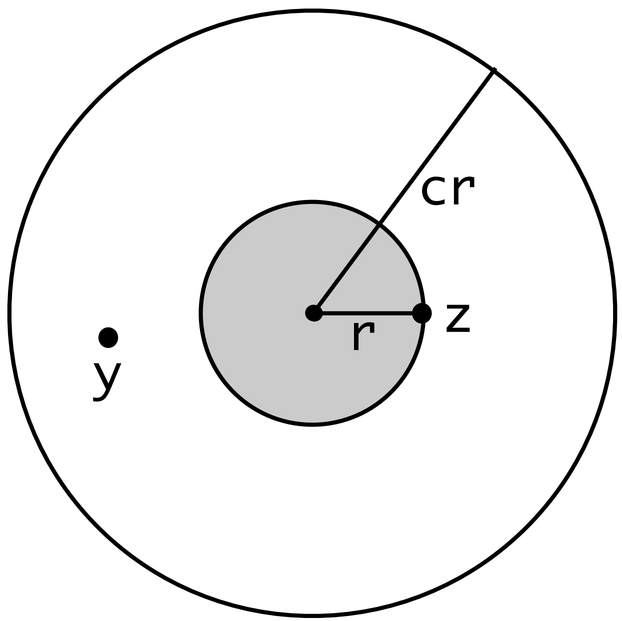


Approximate Near Neighbors

- ApproximateNearNeighbor(x): Return a point y such that $d(x, y) \leq c \cdot \min_{z \in P} d(x, z)$

- c-Approximate r-Near Neighbor: Given a point x if there exists a point z in P $d(x, z) \leq r$ then return a point y such that $d(x, y) \leq c \cdot r$. If no such point z exists return Fail.

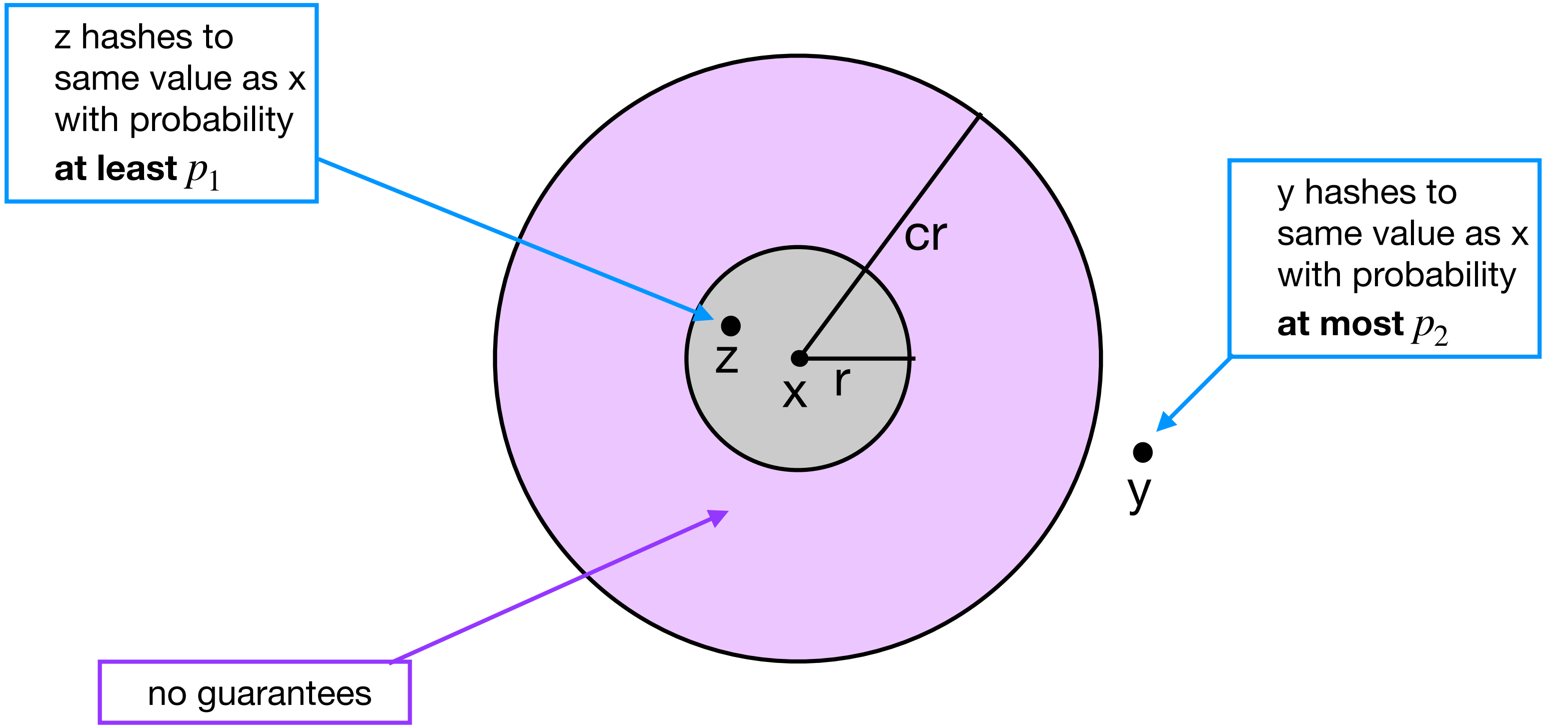
- Randomised version: Return such an y with probability δ.



Locality Sensitive Hashing

- Locality sensitive hashing. A family of hash functions H is (r, cr, p_1, p_2)-sensitive with $p_1 > p_2$ and $c > 1$ if:
  - $d(x, y) \leq r \quad \Rightarrow \quad P[h(x) = h(y)] \geq p_1$ (close points)
  - $d(x, y) \geq cr \quad \Rightarrow \quad P[h(x) = h(y)] \leq p_2$ (distant points)

  for h chosen randomly from H.



Hamming Distance

- P set of n bit strings each of length d.
- Hamming distance. the number of bits where x and y differ:

$$d(x, y) = |\{i : x_i \neq y_i\}|$$

- Example.

x =	1	0	1	0	0	1	0	0
y =	0	1	1	0	0	1	1	0

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Hamming distance = 3
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- Hash function. Chose $i \in \{1,\ldots,d\}$ uniformly at random and set $h(x) = x_i$.
- What is the probability that h(x) = h(y)?
  - $d(x, y) \leq r \Rightarrow P[h(x) = h(y)] \geq 1 - r/d$
  - $d(x, y) \geq cr \Rightarrow P[h(x) = h(y)] \leq 1 - cr/d$

LSH with Hamming Distance: Solution 1

- Pick random index i uniformly at random. Let $h(x) = x_i$.
- Bucket: Strings with same hash value $h(x)$.
- Insert(x): Insert x in the list $A[h(x)]$
- NearNeighbour(x): Compute Hamming distance from x to all bitstrings in $A[h(x)]$ until find one that is at most cr away. If no such string found return FAIL.

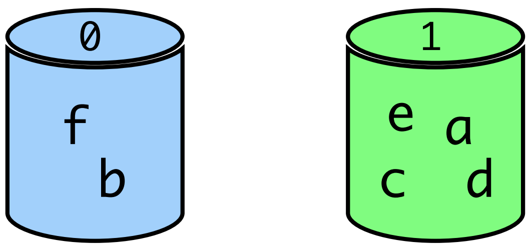
$h(x) = x_3$

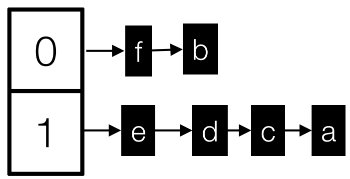
$a = 0011101$ $\quad h(a) = 1$ $\quad d = 0110011$ $\quad h(d) = 1$

$b = 0101001$ $\quad h(b) = 0$ $\quad e = 1011101$ $\quad h(e) = 1$

$c = 0010010$ $\quad h(c) = 1$ $\quad f = 1101101$ $\quad h(f) = 0$

Query time: O(nd).





LSH with Hamming Distance: Solution 2

- Pick k random indexes uniformly and independently at random with replacement:
    - $g(x) = x_{i_1} x_{i_2} \cdots x_{i_k}$

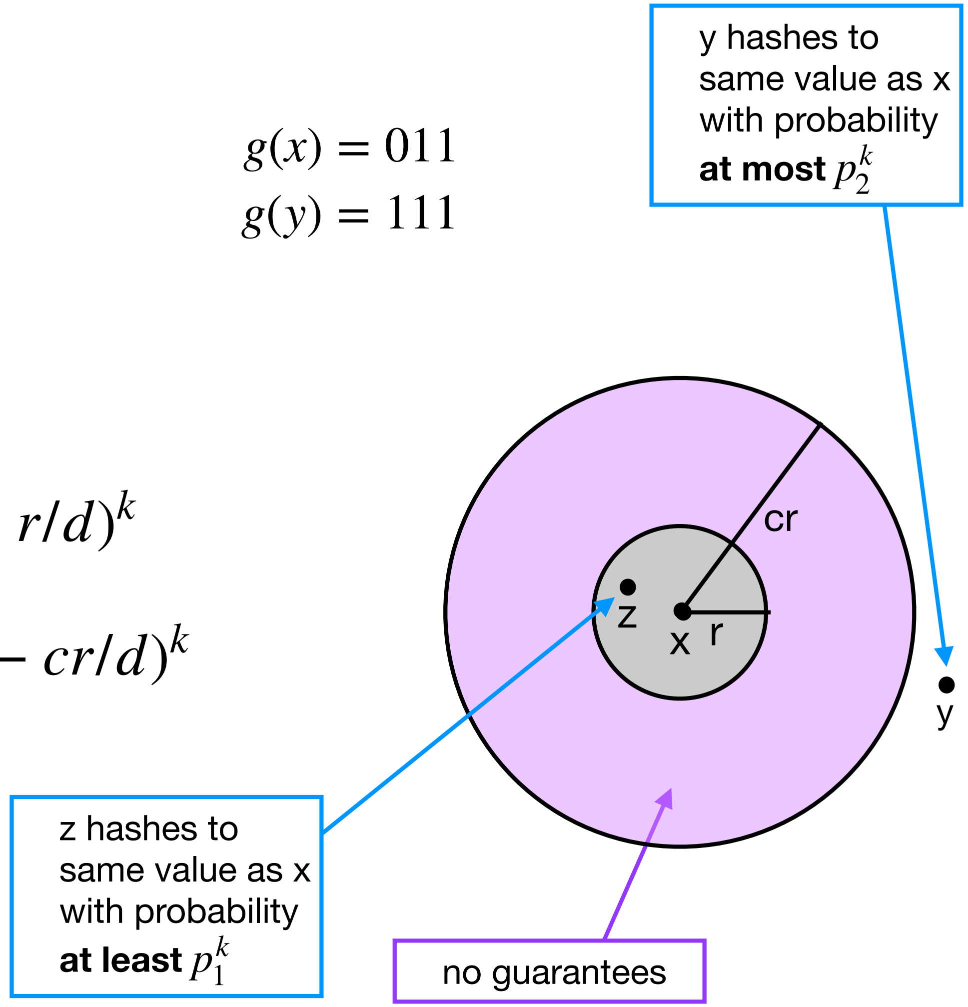
- Example. $k = 3$. $g(x) = x_2 x_3 x_6$

x =	1	0	1	0	0	1	0	0
y =	0	1	1	0	0	1	1	0

$g(x) = 011$

$g(y) = 111$

- Probability that g(x) = g(y)?
    - $d(x, y) \leq r \Rightarrow P[g(x) = g(y)] \geq (1 - r/d)^k$
    - $d(x, y) \geq cr \Rightarrow P[g(x) = g(y)] \leq (1 - cr/d)^k$



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- Bucket: Strings with same hash value $g(x)$.

$g(x) = x_2 x_4 x_7$

$a = 0011101$ $g(a) = 011$ $d = 0110011$ $g(d) = 101$

$b = 0101001$ $g(b) = 111$ $e = 1011101$ $g(e) = 011$

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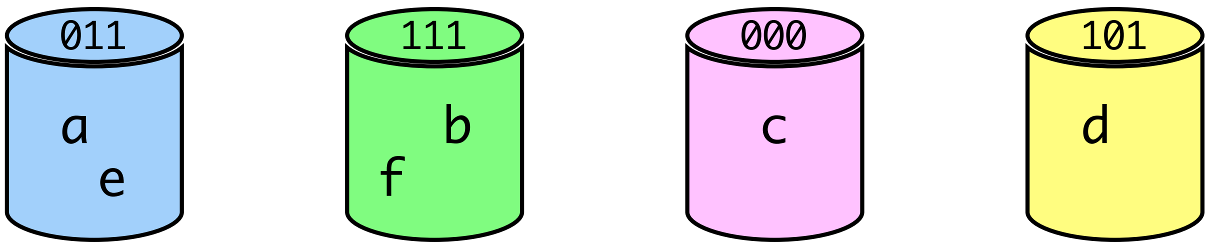

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- Bucket: Strings with same hash value $g(x)$.
- Save buckets in a hash table T with hash function h_T.

$h_T(011_2) = 1$

$h_T(111_2) = 6$

$h_T(000_2) = 9$

$h_T(101_2) = 1$

$g(x) = x_2 x_4 x_7$

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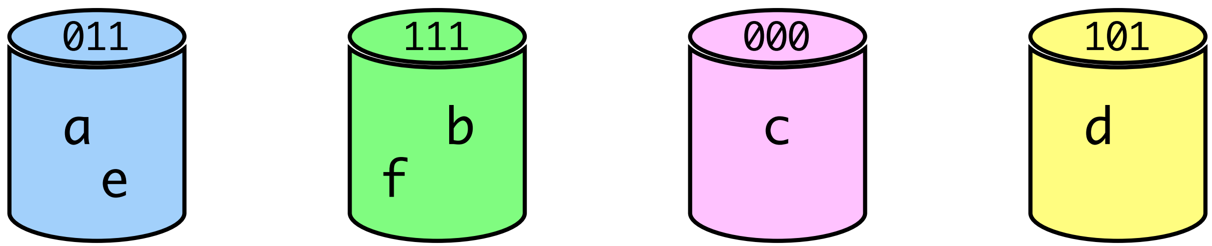

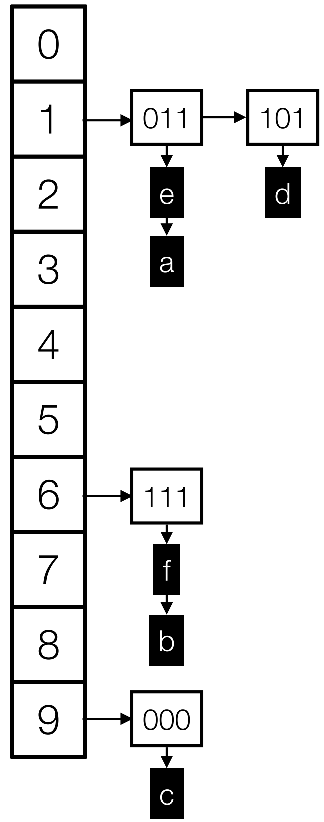

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- Insert(x): Insert x in the list of $g(x)$ in T.
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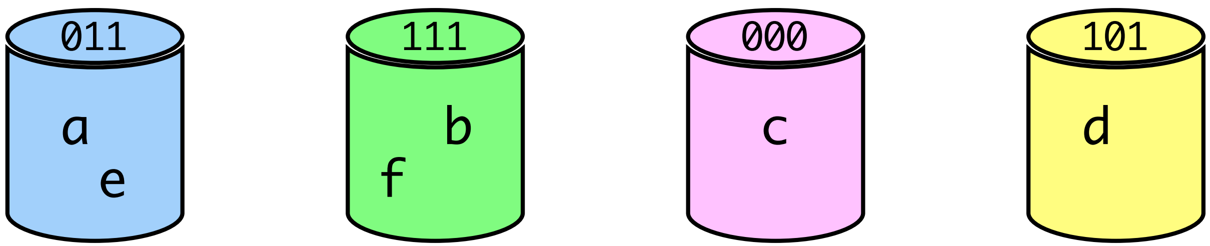

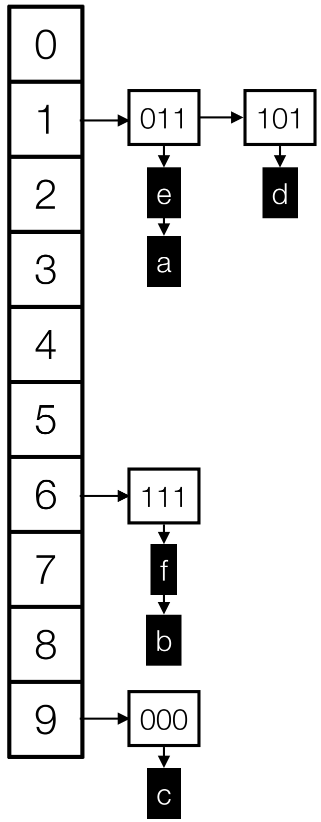

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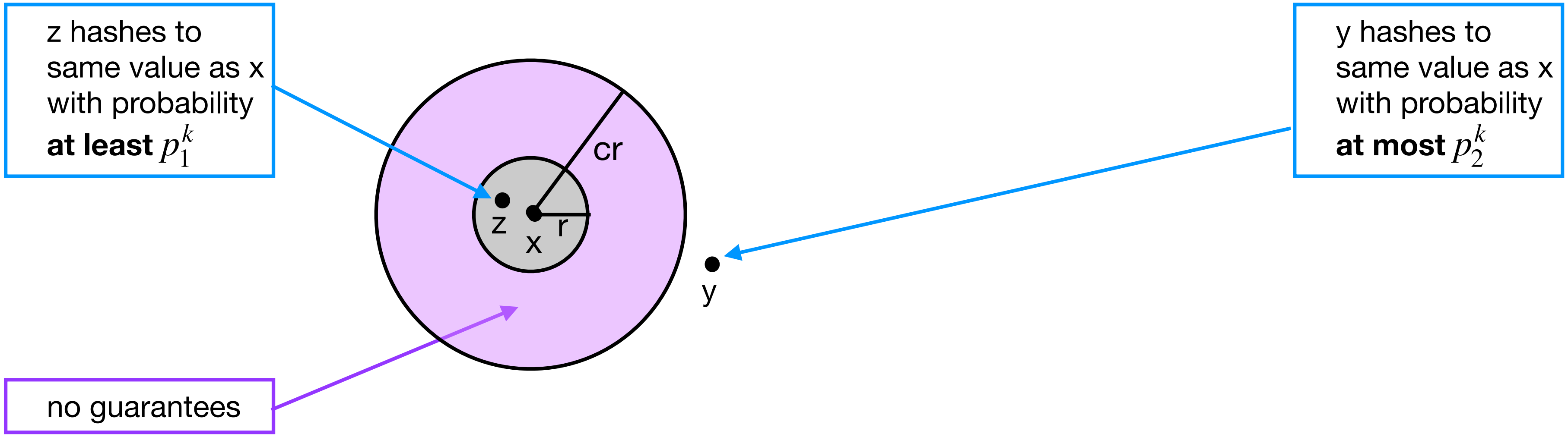
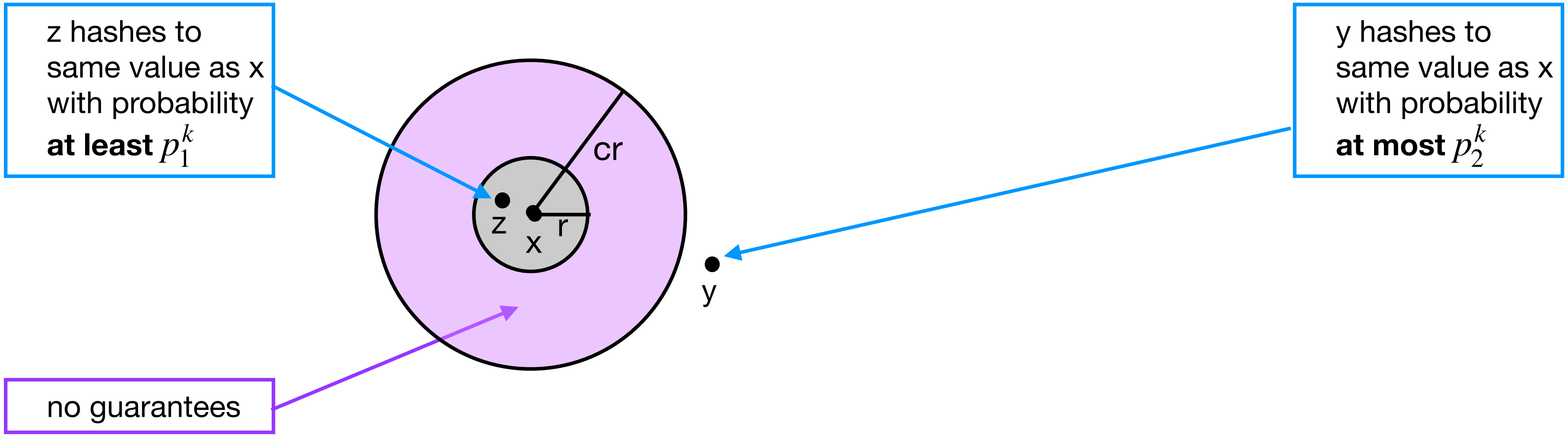


- What happens when we increase k?
  - Far away strings:

LSH with Hamming Distance: Solution 2



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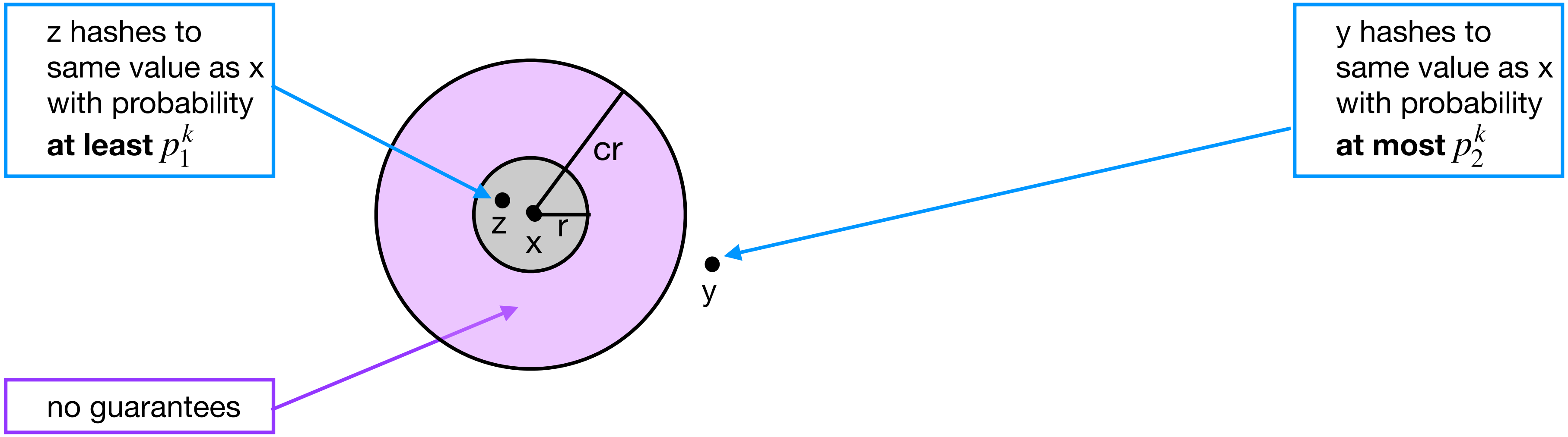

- What happens when we increase k?
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  - Close strings:

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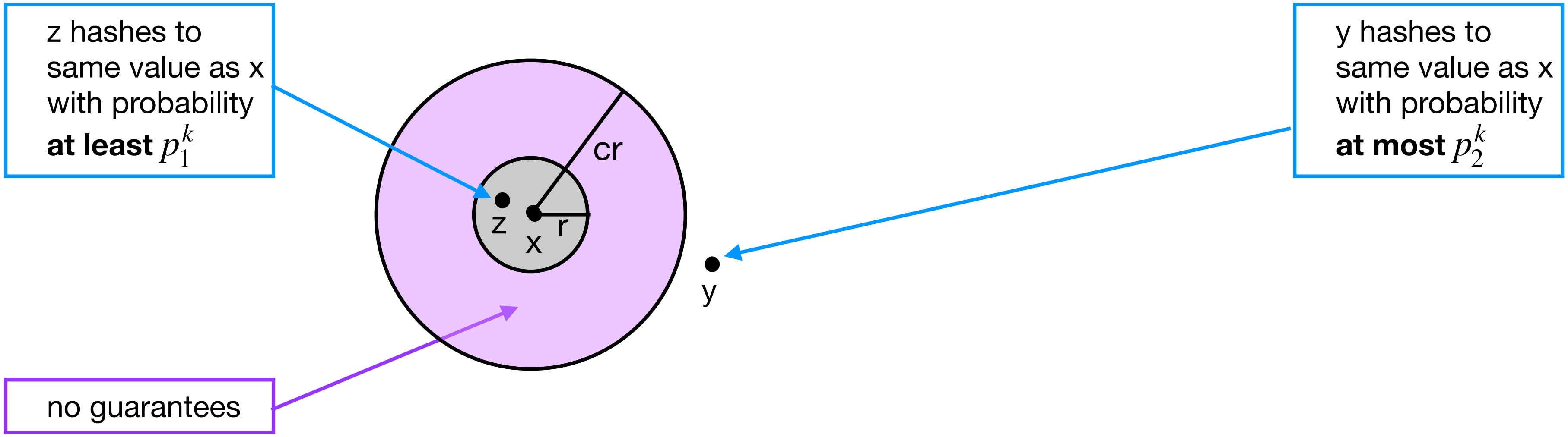
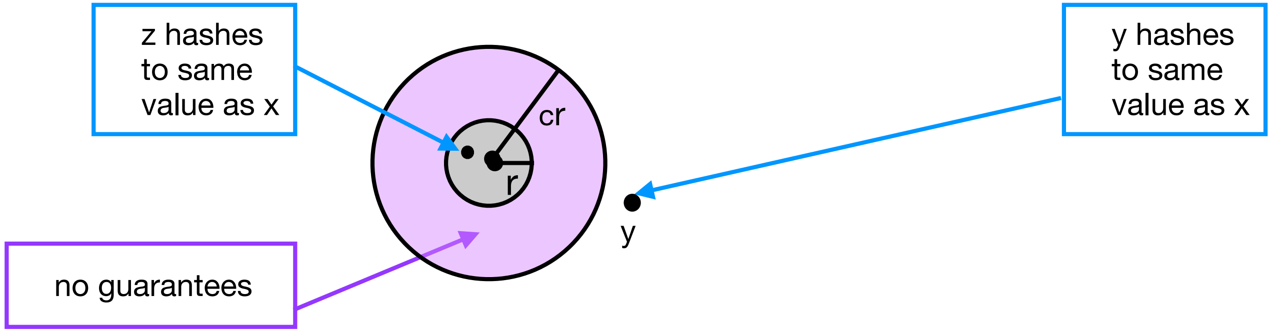


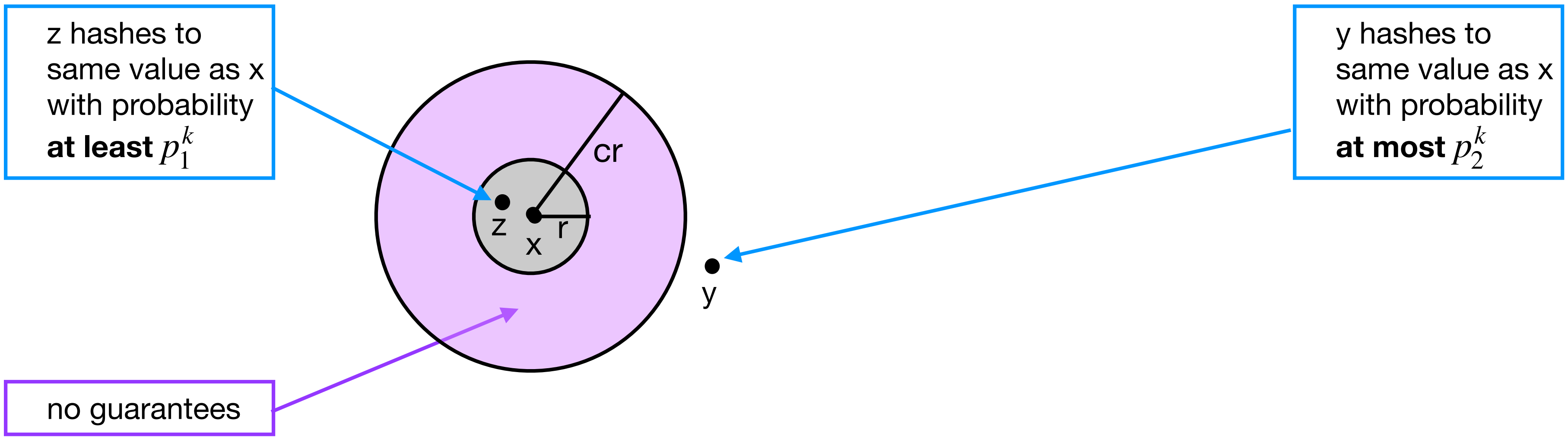
- What happens when we increase k?
    - Far away strings: Probability that a far away string hashes to the same bucket as x decrease.
    - Close strings: Probability that a close string hashes to the same as x decrease.

LSH with Hamming Distance: Solution 2



- Expected number of far away strings that hashes to same bucket as x:
  - $F = \{y : d(x, y) > cr\}$.
- For $y \in F$ we want $P[g(y) = g(x)] \leq 1/n$:
  - Set $k = \lg n / \lg(1/p_2)$
- $X_y = \begin{cases} 1 & y \text{ collides with } x \\ 0 & \text{otherwise} \end{cases}$
- #far away strings colliding with x: $X = \sum_{y \in F} X_y$
- $E[X] = \sum_{y \in F} E[X_y] = \sum_{y \in F} 1/n \leq 1.$
- Markov: $P[X > 6] < E[X]/6 \leq 1/6.$

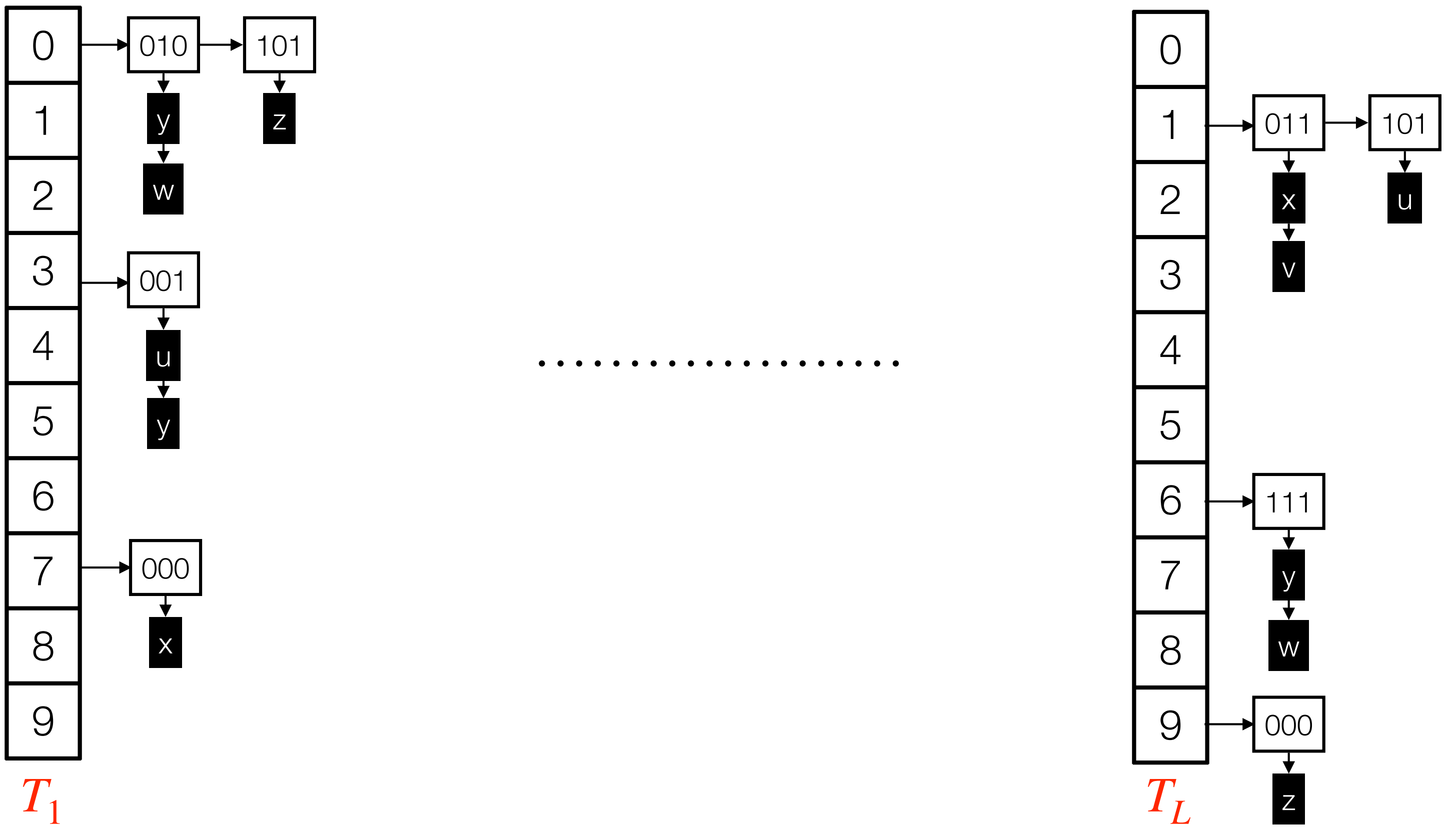
LSH with Hamming Distance: Solution 2



- What happens when we increase k?
  - Probability that a far away string hashes to the same bucket as x decrease.
    - $k = \lg n / \lg(1/p_2)$ $\Rightarrow$ with probability ≥ 5/6 at most 6 far away strings hashes to x's bucket.
  - Probability that a close string hashes to the same as x decrease. 😕

LSH with Hamming Distance: Solution 3 (Amplification)

- Construct L hash tables T_j . Each table T_j has its own independently chosen hash function h_j and its own independently chosen locality sensitive hash function g_j.
- **Insert(x):** For all $1 \leq j \leq L$ insert x in the list of $g_j(x)$ in T_j.
- **Query(x):** For all $1 \leq j \leq L$ check each element in bucket $g_j(x)$ in T_j. Return the one closest to x.



LSH with Hamming Distance

Let $k = \dfrac{\lg n}{\lg(1/p_2)}$, $\rho = \dfrac{\lg(1/p_1)}{\lg(1/p_2)}$, and $L = \lceil 2n^\rho \rceil$, where $p_1 = 1 - r/d$ and $p_2 = 1 - cr/d$.

- Claim 1. *If there exists a string z* in P with d(x,z*) ≤ r then with probability at least 5/6 we will return some z in P for which d(x,z) ≤ r.*

- Probability that z* collides with x:
  - $P[\exists i : g_i(x) = g_i(z^*)] \ = 1 - P[g_i(x) \neq g_i(z^*)$ for all $i]$

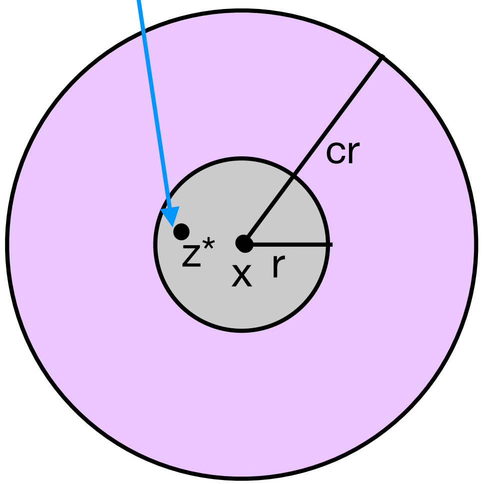
$$= 1 - \prod_{i=1}^{L} P[g_i(x) \neq g_i(z^*)]$$

$$= 1 - \prod_{i=1}^{L} \left(1 - P[g_i(x) = g_i(z^*)]\right)$$

$$\geq 1 - \prod_{i=1}^{L} (1 - p_1^k) \ = 1 - (1 - p_1^k)^L \ \geq 1 - e^{-Lp_1^k}$$

$$\geq 1 - \frac{1}{e^2} \geq 1 - 1/6 = 5/6$$

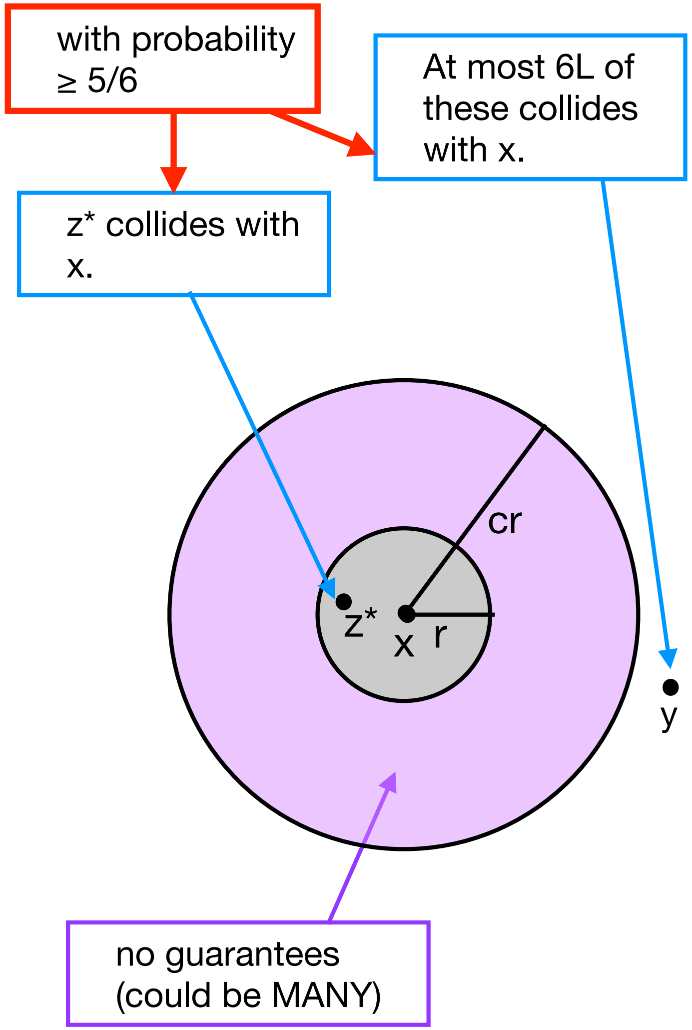
z* hashes to same value as x with probability **at least** p_1^k



LSH with Hamming Distance

- Expected query time is O(L): Can show that the expected number of far away strings that collides with x is L.

- Claim. *The expected number of far away strings that collides with x is L.*



Locality Sensitive Hashing

- Locality sensitive hash function. A family of hash functions $\mathscr{H}$ is (r, cr, p_1, p_2) -sensitive with $p_1 > p_2$ and $c > 1$ if:
  - $d(x, y) \leq r \quad \Rightarrow \quad P[h(x) = h(y)] \geq p_1$ (close points)
  - $d(x, y) \geq cr \quad \Rightarrow \quad P[h(x) = h(y)] \leq p_2$ (distant points)

- Amplification.
  - Choose L hash functions $g_j(x) = h_{1,j}(x) \cdot h_{2,j}(x) \ \cdots \ h_{k,j}(x)$, where $h_{i,j}$ is chosen independently and uniformly at random from $\mathscr{H}$.

- Locality sensitive hashing scheme.
  - Construct L hash tables T_j.
  - **Insert(x):** For all $1 \leq j \leq L$ insert x in the list of $g_j(x)$ in T_j.
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Jaccard distance and Min Hash

- Jaccard distance. Jaccard similarity: Jsim$(A, B) = \dfrac{|A \cap B|}{|A \cup B|}$
    - Jaccard distance: 1- Jsim(A,B).
    - Hash function: *Min Hash*. (exercise)

Exercises

Angular Distance and Sim Hash

- Collection of vectors.
- Distance between two vectors is the angular distance between them dist$(u, v) = \angle(u, v)/\pi$.
  - Assume u and v are unit vectors. Then $u \cdot v = \cos(\angle(u, v))$
- Hash function: Sim Hash.
  - Random projection: Take a random vector r and set $h_r(u) = \text{sign}(r \cdot u)$

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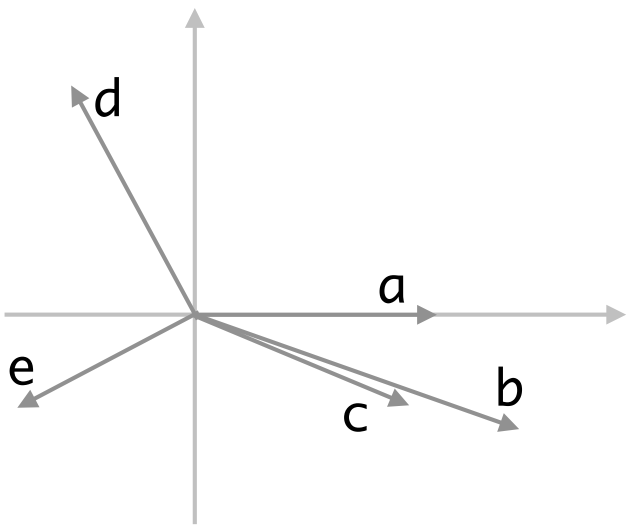

a					
b					
c					
d					
e					

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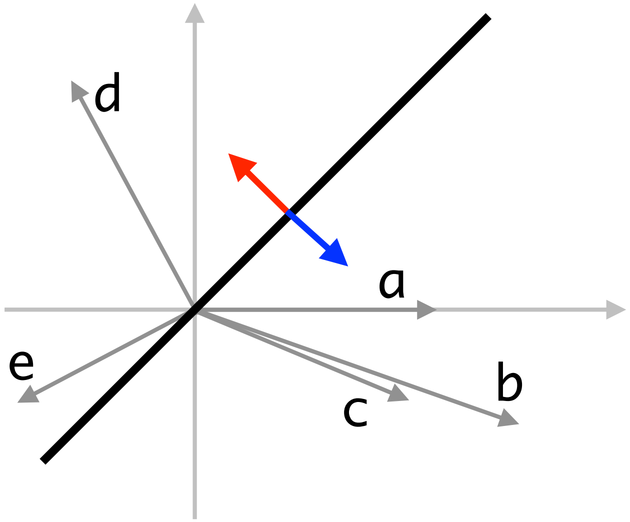

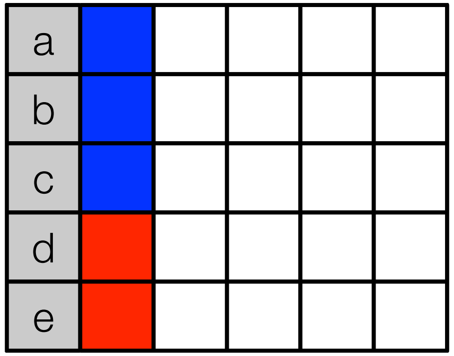

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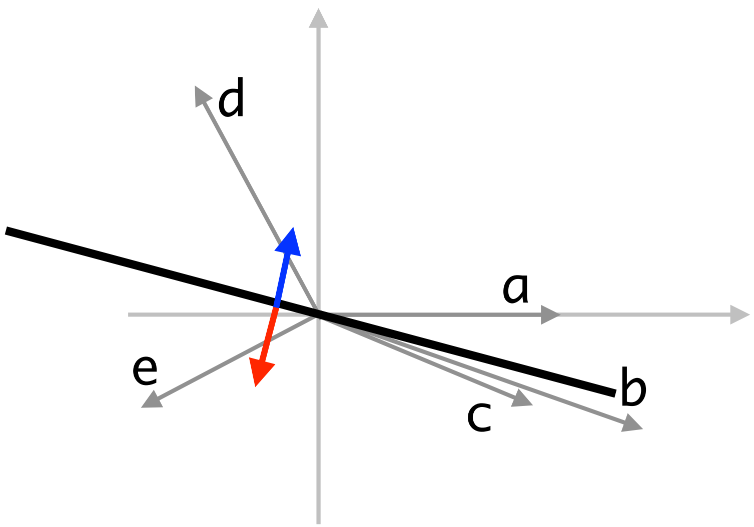

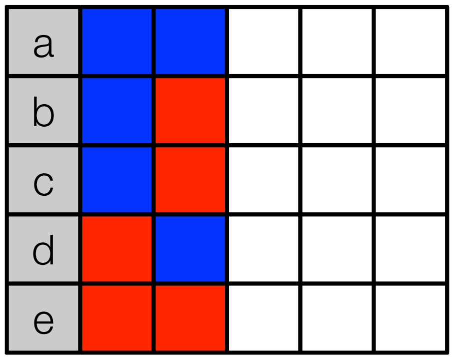

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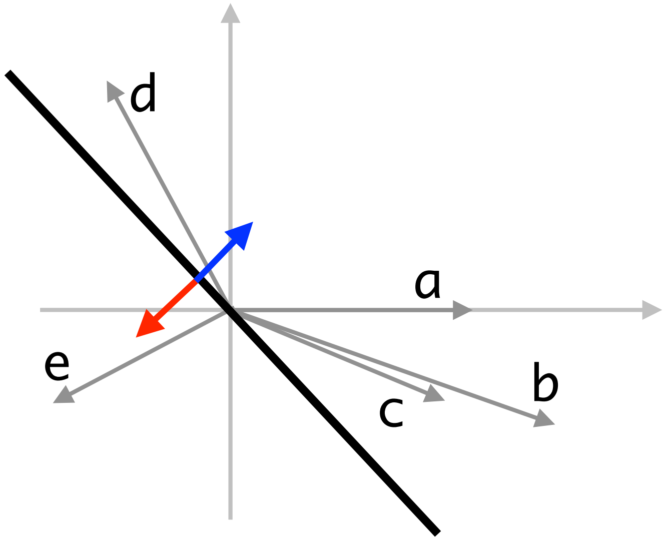

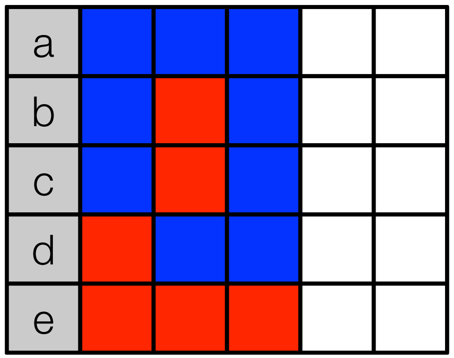

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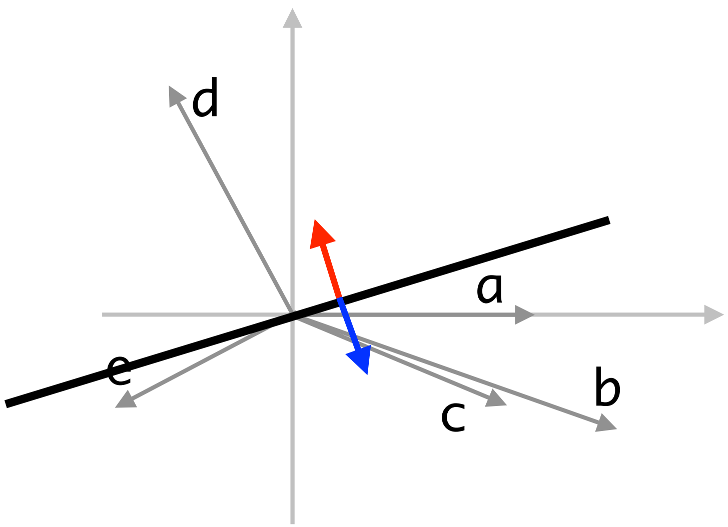

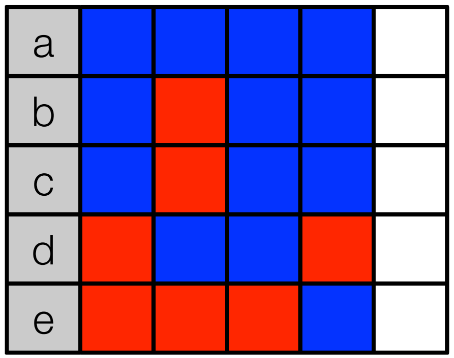

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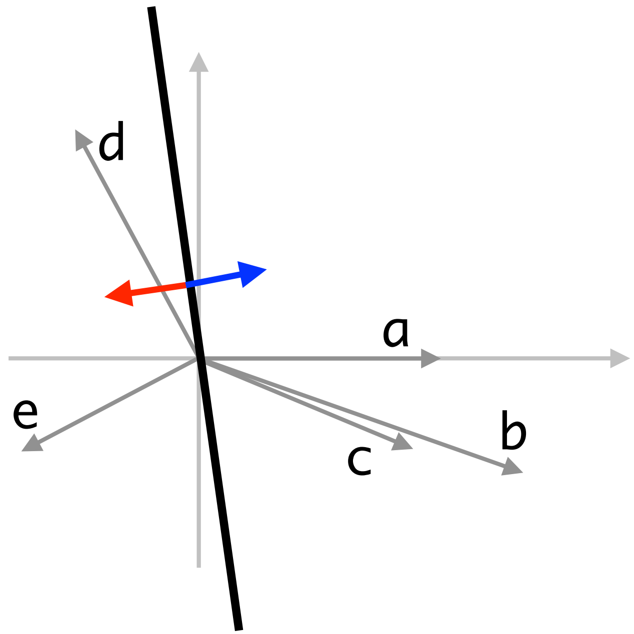

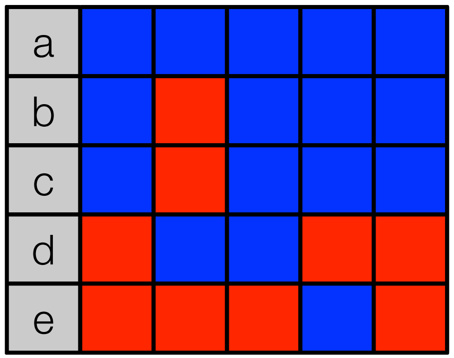

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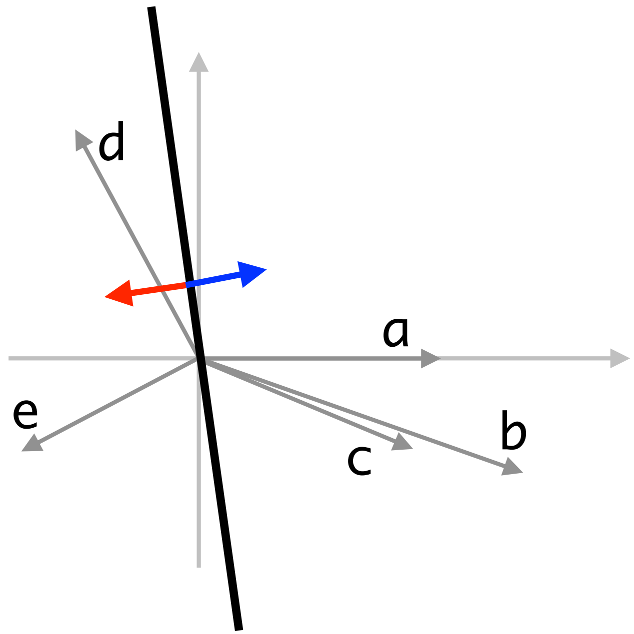

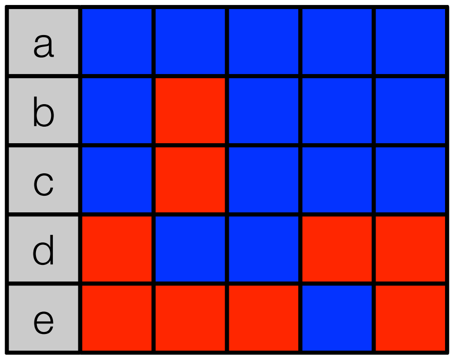

- Can show that $P[h(u) = h(v)] = 1 - \angle(u, v)/\pi$.