Hashing

- Dictionaries
- Chained Hashing
- Universal Hashing
- Static Dictionaries and Perfect Hashing

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Dictionaries

- Dictionary problem. Maintain a dynamic set of integers $S \subseteq U$ subject to following operations
  - LOOKUP(x): return true if $x \in S$ and false otherwise.
  - INSERT(x): set $S = S \cup \{x\}$
  - DELETE(x): set $S = S \setminus \{x\}$

- Universe size. Typically $|U| = 2^{64}$ or $|U| = 2^{32}$ and $|S| \ll |U|$.
- Satellite information. Information associated with each integer.

- Goal. A compact data structure with fast operations.

Dictionaries

- Applications.
  - Many!
  - Key component in other data structures and algorithms.

Dictionaries

- Which solutions do we know?

Hashing

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Chained Hashing

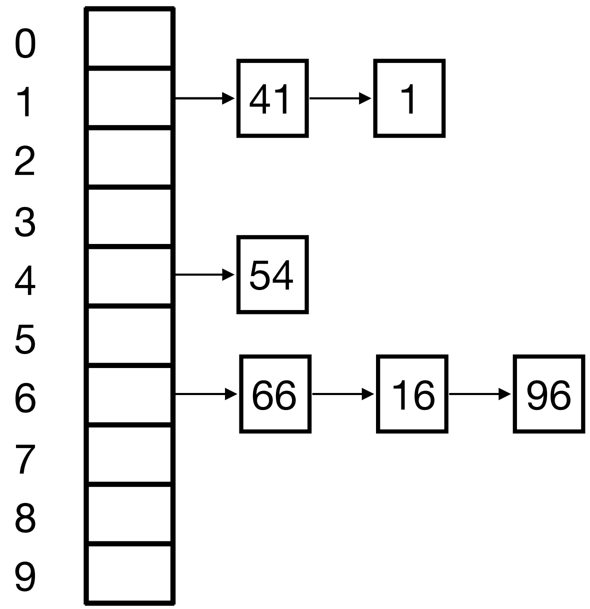
- Chained hashing [Dumey 1956].
  - Hash function. Pick some crazy, chaotic, random function h that maps U to {0, ..., m-1}, where $m = \Theta(n)$.
  - Initialize an array A[0, ..., m-1].
  - A[i] stores a linked list containing the keys in S whose hash value is i.

Chained Hashing

$U = \{0, ..., 99\}$

$S = \{1, 16, 41, 54, 66, 96\}$

$h(x) = x \bmod 10$



- Operations.
  - LOOKUP(x): Compute h(x). Scan through list for h(x). Return true if x is in list and false otherwise.
  - INSERT(x): Compute h(x). Scan through list for h(x). If x is in list do nothing. Otherwise, add x to the front of list.
  - DELETE(x): Compute h(x). Scan through list for h(x). If x is in list remove it. Otherwise, do nothing.
- Time. O(1 + length of linked list for h(x))

Chained Hashing

- Hash functions.
  - h(x) = x mod 10 is not very crazy, chaotic, or random.
  - For any fixed choice of h, there is a set whose elements all map to the same slot.
  - ⇒ We end up with a single linked list.
  - How can we overcome this?
- Use randomness.
  - Assume the input set is random.
  - Choose the hash function at random.

Chained Hashing

- Random hash functions. Assume that:
    1. h is chosen uniformly at random among all functions from U to {0,…, m-1}
    2. We can store h in O(n) space.
    3. We can evaluate h in O(1) time

- What is the expected length of the linked lists?

Chained Hashing

$$
\begin{aligned}
E(\text{length of linked list for } h(x)) &= E\left(|\{y \in S \mid h(y) = h(x)\}|\right) \\
&= E\left(\sum_{y \in S} \begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right) \\
&= \sum_{y \in S} E\left(\begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right) \\
&= \sum_{y \in S} \Pr(h(x) = h(y)) \\
&= 1 + \sum_{y \in S \setminus \{x\}} \Pr(h(x) = h(y))) \\
&= 1 + \sum_{y \in S \setminus \{x\}} \frac{1}{m} \\
&= 1 + (n-1) \cdot \frac{1}{m} = O(1)
\end{aligned}
$$

m^2 choices for pair (h(x), h(y)),
m of which cause collision

Chained Hashing

- Theorem. We can solve the dictionary problem (under assumptions 1+2+3) in
  - O(n) space.
  - O(1) expected time per operation.

- Expectation is over the choice of hash function.
- Independent of the input set.

Chained Hashing

- Random hash functions assumptions.
  1. h is chosen uniformly at random among all functions from U to {0,…, m-1}
  2. We can store h in O(n) space.
  3. We can evaluate h in O(1) time

- Random hash functions. Can we efficiently compute and store a random function?
  - We need $\Theta(u \log m)$ bits to store an arbitrary function h: {0,…, u-1} → {0,..., m-1}
  - We need a lot of random bits to generate the function.
  - We need a lot of time to generate the function.

- Do we need a truly random hash function?
- When did we use the fact that h was random in our analysis?

Chained Hashing

$$
\begin{aligned}
E(\text{length of linked list for } h(x)) &= E\left(|\{y \in S \mid h(y) = h(x)\}|\right) \\
&= E\left(\sum_{y \in S} \begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right) \\
&= \sum_{y \in S} E\left(\begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right) \\
&= \sum_{y \in S} \Pr(h(x) = h(y)) \\
&= 1 + \sum_{y \in S \setminus \{x\}} \Pr(h(x) = h(y))) \\
&= 1 + \sum_{y \in S \setminus \{x\}} \frac{1}{m} \qquad \text{For all } x \neq y,\ \Pr(h(x) = h(y)) \leq 1/m \\
&= 1 + (n-1) \cdot \frac{1}{m} = O(1)
\end{aligned}
$$

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Universal Hashing

- Universel hashing [Carter and Wegman 1979].
  - Let H be a family of functions mapping U to {0, ..., m-1}.
  - H is universal if for any $x \neq y$ in U and h chosen uniformly at random in H,

$$\Pr(h(x) = h(y)) \leq 1/m$$

Universal Hashing

- Positional number systems. For integers x and p, the base-p representation of x is x written in base p.
- Example.
  - $(10)_{10} = (1010)_2$ $(1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0)$
  - $(107)_{10} = (212)_7$ $(2 \cdot 7^2 + 1 \cdot 7^1 + 2 \cdot 7^0)$

Universal Hashing

- Hash function. Given a prime $m < p < 2m$ and $a = (a_1 a_2 \ldots a_r)_p$, define

$$h_a(x = (x_1 x_2 \ldots x_r)_p) = a_1 x_1 + a_2 x_2 + \ldots + a_r x_r \bmod p$$

- Example.
  - $p = 7$
  - $a = (107)_{10} = (212)_7$
  - $x = (214)_{10} = (424)_7$
  - $h_a(x) = 2 \cdot 4 + 1 \cdot 2 + 2 \cdot 4 \bmod 7 = 18 \bmod 7 = 4$

- Universal family.
  - $H = \{h_a \mid a = (a_1 a_2 \ldots a_r)_p \in \{0, \ldots, p-1\}^r\}$
  - Choose random hash function from H ~ choose random a.
  - H is universal (analysis next).
  - O(1) time evaluation.
  - O(1) space.
  - Fast construction.

Universal Hashing

- Lemma. Let p be a prime. For any $a \in \{1, ..., p-1\}$ there exists a unique inverse a^{-1} such that $a^{-1} \cdot a \equiv 1 \mod p$. ($Z_p$ is a field)
- Example. p = 7

a	1	2	3	4	5	6
a^{-1}						

a	1	2	3	4	5	6
a^{-1}	1	4	5	2	3	6

Universal Hashing

- Goal. For random $a = (a_1 a_2 \ldots a_r)_p$, show that if $x = (x_1 x_2 \ldots x_r)_p \neq y = (y_1 y_2 \ldots y_r)_p$ then $\Pr[h_a(x) = h_a(y)] \leq 1/m$
- $(x_1 x_2 \ldots x_r)_p \neq (y_1 y_2 \ldots y_r)_p \Longrightarrow x_i \neq y_i$ for some i. Assume wlog. that $x_r \neq y_r$.

$$
\begin{aligned}
&\Pr(h_a((x_1 \ldots x_r)_p) = h_a((y_1 \ldots, y_r)_p)) \\
&= \Pr\left(a_1 x_1 + \cdots + a_r x_r \equiv a_1 y_1 + \cdots + a_r y_r \bmod p\right) \\
&= \Pr\left(a_r x_r - a_r y_r \equiv a_1 y_1 - a_1 x_1 + \cdots + a_{r-1} y_{r-1} - a_{r-1} x_{r-1} \bmod p\right) \\
&= \Pr\left(a_r (x_r - y_r) \equiv a_1 (y_1 - x_1) + \cdots + a_{r-1}(y_{r-1} - x_{r-1}) \bmod p\right) \\
&= \Pr\left(a_r (x_r - y_r)(x_r - y_r)^{-1} \equiv (a_1 (y_1 - x_1) + \cdots + a_{r-1}(y_{r-1} - x_{r-1}))(x_r - y_r)^{-1} \bmod p\right) \\
&= \Pr\left(a_r \equiv (a_1 (y_1 - x_1) + \cdots + a_{r-1}(y_{r-1} - x_{r-1}))(x_r - y_r)^{-1} \bmod p\right) = \frac{1}{p} \leq \frac{1}{m}
\end{aligned}
$$

existence of inverses

for any choice of $a_1, a_2, \ldots, a_{r-1}$, the RH defines a unique a_r that matches (uniqueness of inverses).
Of the p^r choices for $a_1, a_2, \ldots, a_r$ exactly p^{r-1} cause a collision ⇒ probability is $p^{r-1}/p^r = 1/p$

Universal Hashing

- Lemma. H is universal with O(1) time evaluation and O(1) space.

- Theorem. We can solve the dictionary problem (without special assumptions) in:
  - O(n) space.
  - O(1) expected time per operation (lookup, insert, delete).

Universal Hashing

- Other universal families.
  - For prime p > 0, a ∈ {1, .., p-1}, b ∈ {0, ..., p-1}

$$h_{a,b}(x) = (ax + b \bmod p) \bmod m$$
$$H = \{h_{a,b} \mid a \in \{1, \ldots, p-1\}, b \in \{0, \ldots, p-1\}\}$$

  - Hash function from k-bit numbers to l-bit numbers. a is an odd k-bit integer.

l most significant bits of the k least significant bits of ax

$$h_a(x) = (ax \bmod 2^k) \gg (k - l)$$
$$H = \{h_a \mid a \text{ is an odd integer in } \{1, \ldots, 2^k - 1\}\}$$

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Static Dictionaries and Perfect Hashing

- Static dictionary problem. Given a set $S \subseteq U = \{0,..,u-1\}$ of size n for preprocessing support the following operation
  - lookup(x): return true if $x \in S$ and false otherwise.

- As the dictionary problem with no updates (insert and deletes).
- Set given in advance.

Static Dictionaries and Perfect Hashing

- Dynamic solution. Use chained hashing with a universal hash function as before $\Longrightarrow$ solution with O(n) space and O(1) expected time per lookup.

- Can we do better?

- Perfect Hashing. A perfect hash function for S is a collision-free hash function on S.
  - Perfect hash function in O(n) space and O(1) evaluation time $\Longrightarrow$ solution with O(n) space and O(1) worst-case lookup time.
  - Do perfect hash functions with O(n) space and O(1) evaluation time exist for any set S?

Static Dictionaries and Perfect Hashing

- Goal. Perfect hashing in linear space and constant worst-case time.
- Solution in 3 steps.
  - Solution 1. Collision-free but with too much space.
  - Solution 2. Many collisions but linear space.
  - Solution 3: FKS scheme [Fredman, Komlós, Szemerédi 1984]. Two-level solution. Combines solution 1 and 2.
    - At level 1 use solution with lots of collisions and linear space.
    - Resolve collisions at level 1 with collision-free solution at level 2.
    - lookup(x): look-up in level 1 to find the correct level 2 dictionary. Lookup in level 2 dictionary.

Static Dictionaries and Perfect Hashing

- Solution 1. Collision-free but with too much space.
- Use a universal hash function to map into an array of size n^2. What is the expected total number of collisions in the array?

$$E(\#\text{collisions}) = E\left(\sum_{x,y\in S, x\neq y} \begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right)$$

$$= \sum_{x,y\in S, x\neq y} E\left(\begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right)$$

$$= \sum_{x,y\in S, x\neq y} \Pr(h(x) = h(y)) = \binom{n}{2}\frac{1}{n^2} \leq \frac{n^2}{2}\cdot\frac{1}{n^2} = 1/2$$

#distinct pairs universal hashing into n^2 range

- With probability 1/2 we get perfect hashing function. If not perfect try again.
- $\Longrightarrow$ Expected number of trials before we get a perfect hash function is O(1).
- $\Longrightarrow$ For a static set S we can support lookups in O(1) worst-case time using O(n^2) space.

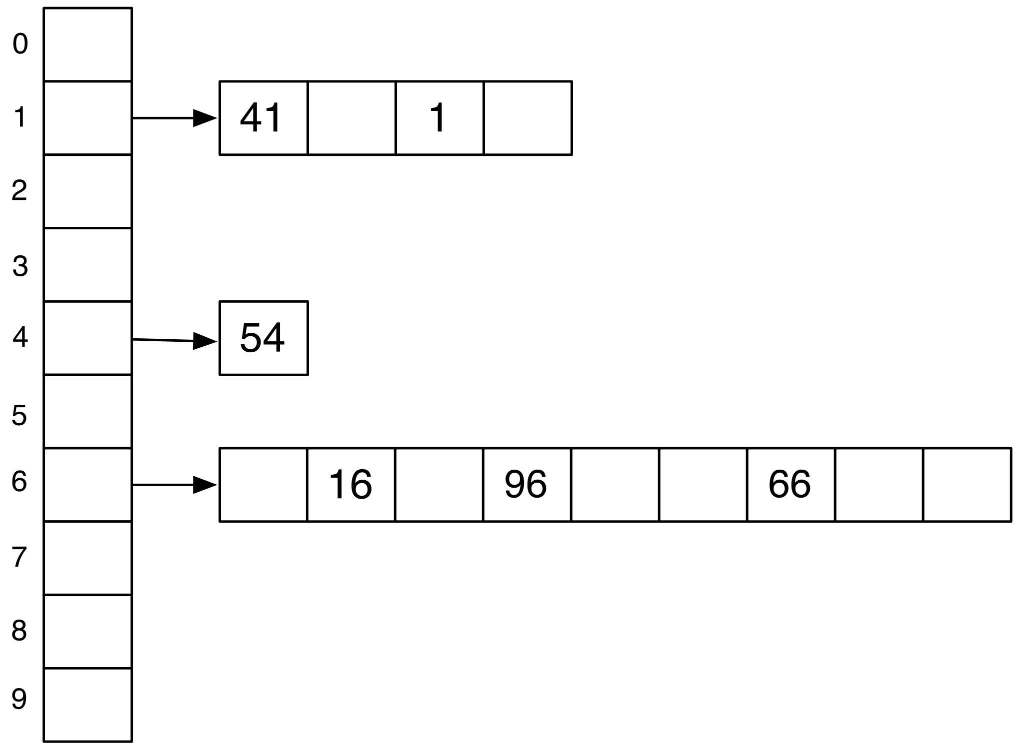
Static Dictionaries and Perfect Hashing

- Solution 2. Many collisions but linear space.
- As solution 1 but with array of size n. What is the expected total number of collisions in the array?

$$
\begin{aligned}
E(\#\text{collisions}) &= E\left(\sum_{x,y \in S, x \neq y} \begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right) \\
&= \sum_{x,y \in S, x \neq y} E\left(\begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right) \\
&= \sum_{x,y \in S, x \neq y} \Pr(h(x) = h(y)) = \binom{n}{2}\frac{1}{n} \leq \frac{n^2}{2} \cdot \frac{1}{n} = \frac{1}{2}n
\end{aligned}
$$

Static Dictionaries and Perfect Hashing

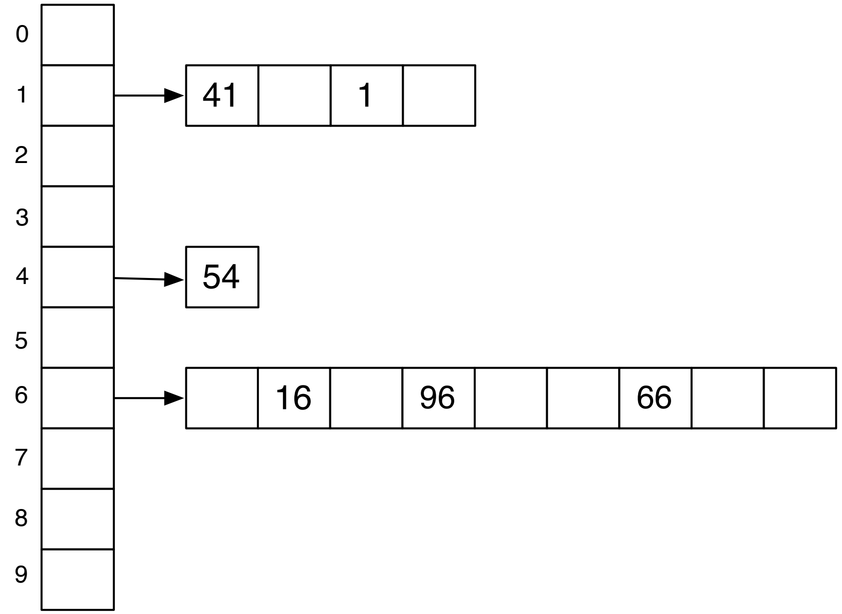
- Solution 3. Two-level solution.
  - At level 1 use solution with lots of collisions and linear space.
  - Resolve each collisions at level 1 with collision-free solution at level 2.
  - lookup(x): look-up in level 1 to find the correct level 2 dictionary. Lookup in level 2 dictionary.
- Example.
  - $S = \{1, 16, 41, 54, 66, 96\}$
  - Level 1 collision sets:
    - $S_1 = \{1, 41\}$,
    - $S_4 = \{54\}$,
    - $S_6 = \{16, 66, 96\}$
  - Level 2 hash info stored with subtable.
    - (size of table, multiplier a, prime p)
- Time. O(1)
- Space?



Static Dictionaries and Perfect Hashing

- Space. What is the the total size of level 1 and level 2 hash tables?

$$\#\text{collisions} = \sum \binom{|S_i|}{2} = O(n)$$



$a^2 = a + 2\binom{a}{2}$, for any integer a

$$\text{space} = O\left(n + \sum_i |S_i|^2\right) = O\left(n + \sum_i \left(|S_i| + 2\binom{|S_i|}{2}\right)\right)$$

$$= O\left(n + \sum_i |S_i| + 2\sum_i \binom{|S_i|}{2}\right) = O(n + n + 2n) = O(n)$$

Static Dictionaries and Perfect Hashing

- FKS scheme.
  - O(n) space and O(n) expected preprocessing time.
  - Lookups with two evaluations of a universal hash function.

- Theorem. We can solve the static dictionary problem for a set S of size n in:
  - O(n) space and O(n) expected preprocessing time.
  - O(1) worst-case time per lookup.

- Multilevel data structures.
  - FKS is example of multilevel data structure technique. Combine different solutions for same problem to get an improved solution.

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