Suffix Trees

- String Dictionaries
- Tries
- Suffix Trees
- Suffix Sorting

Philip Bille

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- Suffix Sorting

String Dictionaries

- String dictionary problem. Let S be a string of characters from alphabet Σ. Preprocess S into data structure to support:
- search(P): Return the starting positions of all occurrences of P in S.

- Example.
  - S = yabbadabbado
  - search(abba) = {1,6}

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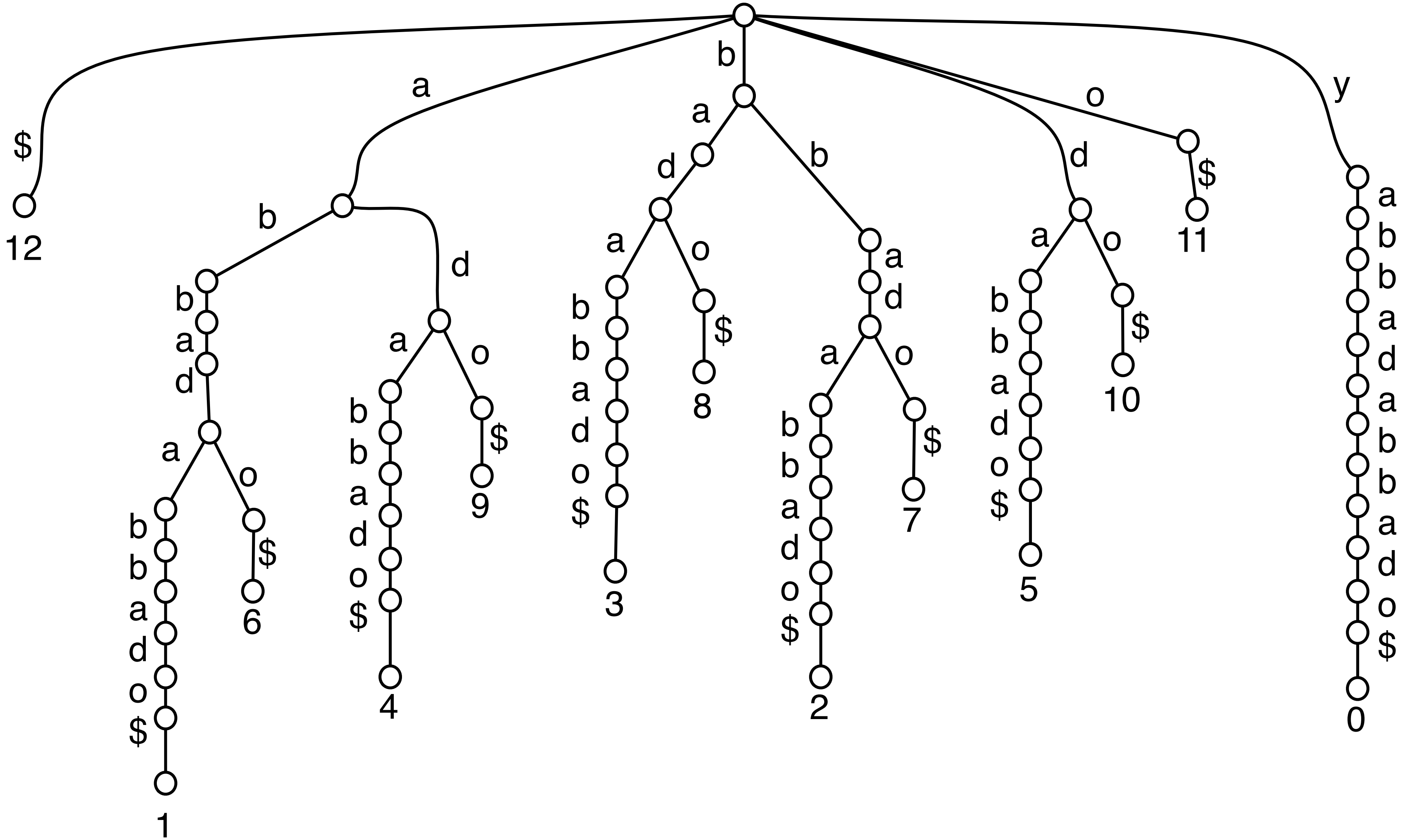
Tries

- Tries [Fredkin 1960]. Retrieval. Store a set of strings in a rooted tree such that:
  - Each edge is labeled by a character. Edges to children of a node are sorted from left-to-right alphabetically.
  - Each root-to-leaf path represents a string in the set. (obtained by concatenating the labels of edges on the path).
  - Common prefixes share same path maximally.
- Prefix free.
  - Append special character $\$ <$ any character in Σ to each string.
  - $\Longrightarrow$ Each leaf correspond to a unique string.
- Suffix trie.
  - Trie of all suffixes of a string.

Tries

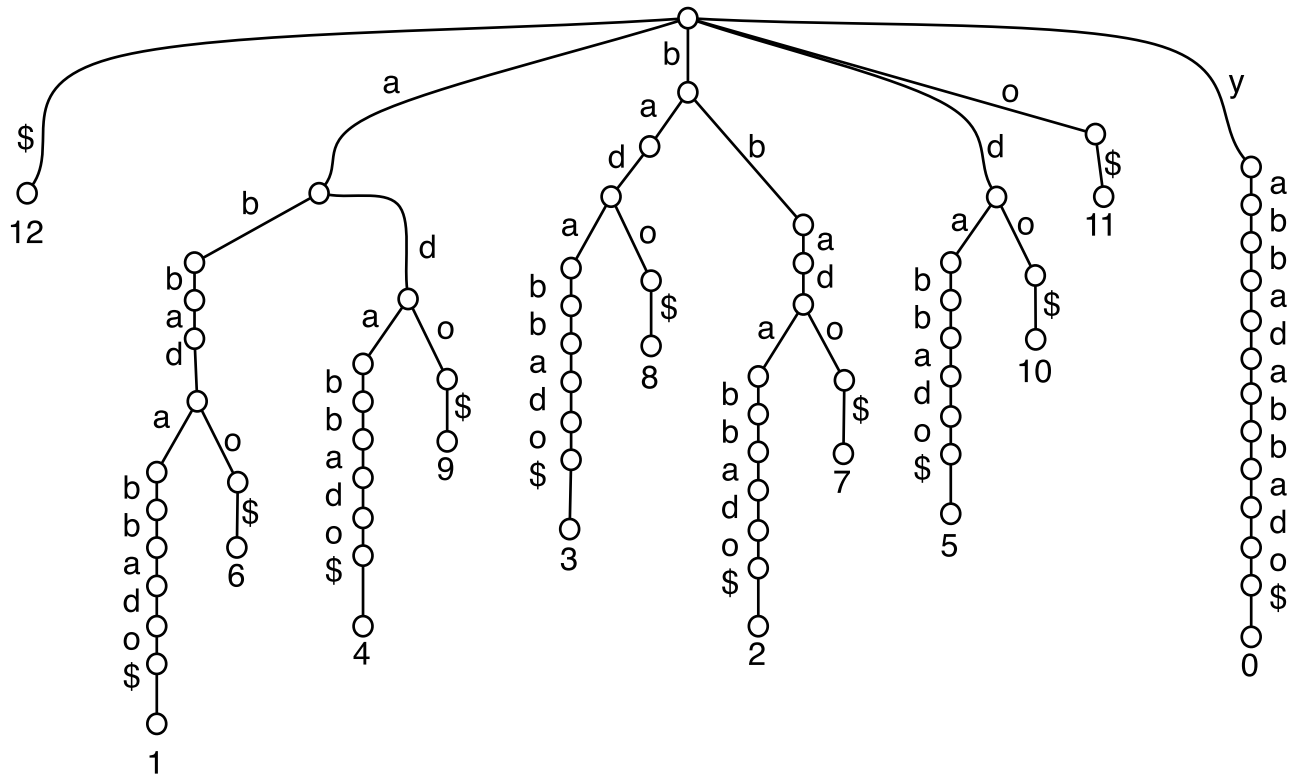


Trie of all suffixes for yabbadabbado$

- Space. $O(n^2)$
- Preprocessing. $O(n^2)$

Tries

- Search(P):
  - Process P from left-to-right while doing top-down search of trie:
    - At each node identify (unique) edge matching next character in P.
    - If no such edge, P is not a substring of S.
  - Report labels of all leaves below final node.
- Example.
  - search(abba) = {1,6}
- Time.
  - Top-down search +
  - Time for reporting leaves
  - ⟹ O(m + occ)



Tries

- Theorem. We can solve the string dictionary problem in
    - $O(n^2)$ space and preprocessing time.
    - $O(m + occ)$ time for queries.

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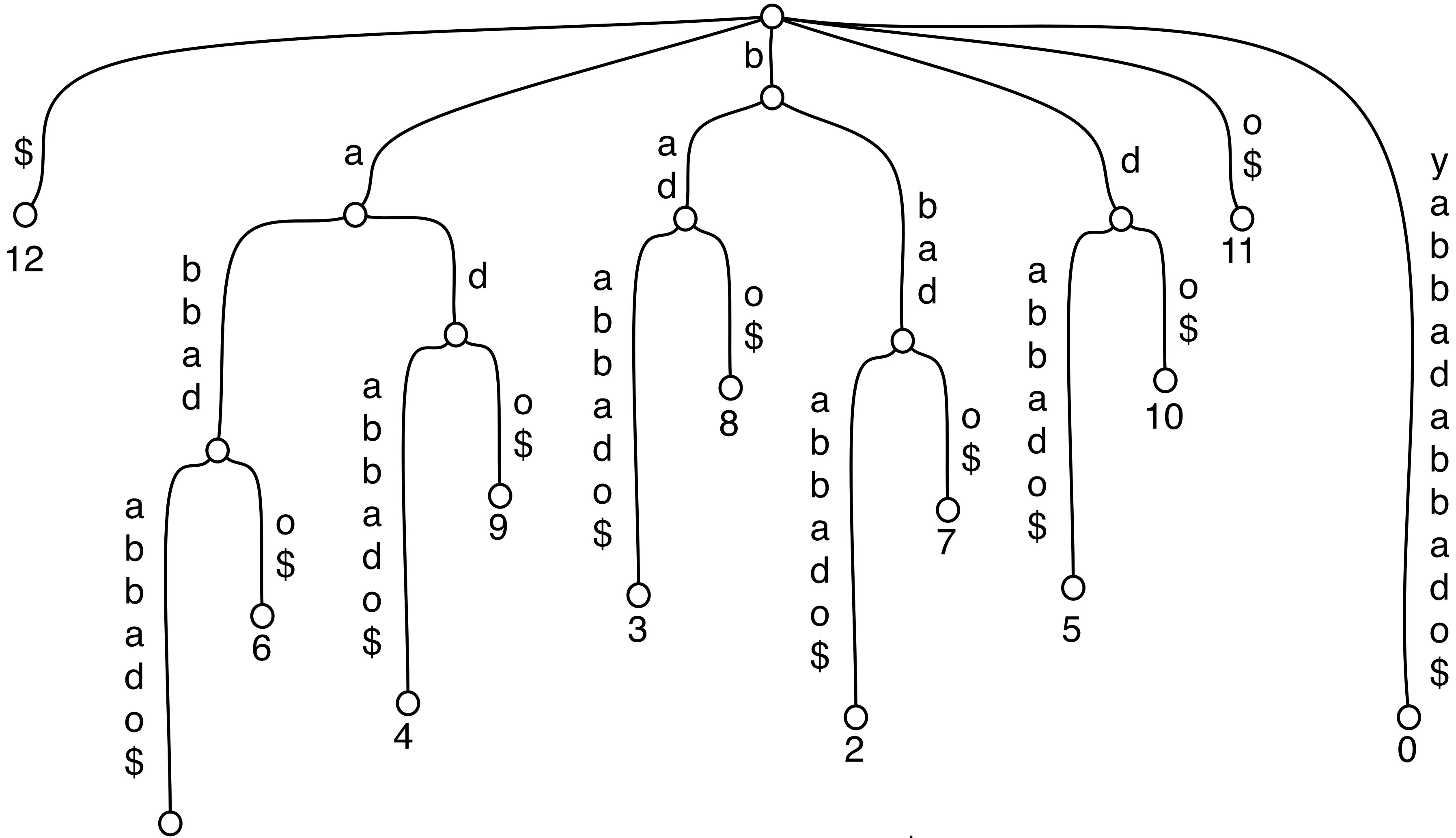
Suffix Trees

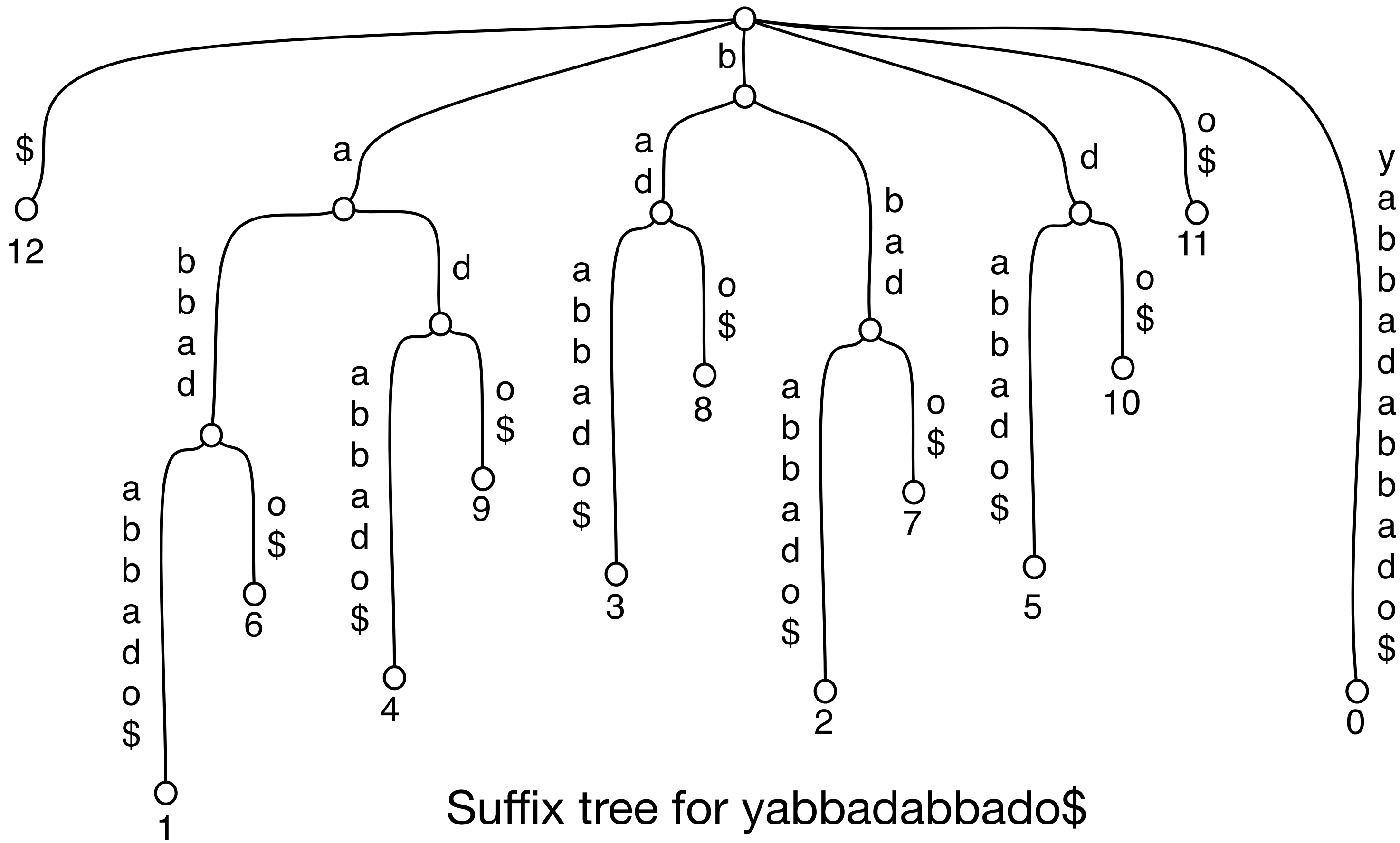
- Suffix trees. The compact trie of all suffixes of S.
  - Chains of nodes with single child are compacted into a single edge.



Suffix tree for yabbadabbado$

Suffix Trees

- Space.
  - Number of edges + space for edge labels
  - ⟹ O(n) space
- Preprocessing. O(sort(n,|Σ|))
- sort(n,|Σ|) = time to sort n characters from an alphabet Σ.
- Search(P): as before.



Suffix tree for yabbadabbado$

Suffix Trees

- Theorem. We can solve the string dictionary problem in
  - $O(n)$ space and $sort(n,|\Sigma|)$ preprocessing time.
  - $O(m + occ)$ time for queries.

Suffix Trees

- Applications.
  - Approximate string matching problems
  - Compression schemes (Lempel-Ziv family, ...)
  - Repetitive string problems (palindromes, tandem repeats, ...)
  - Information retrieval problems (document retrieval, top-k retrieval, ...)
  - ...

Longest Common Extension

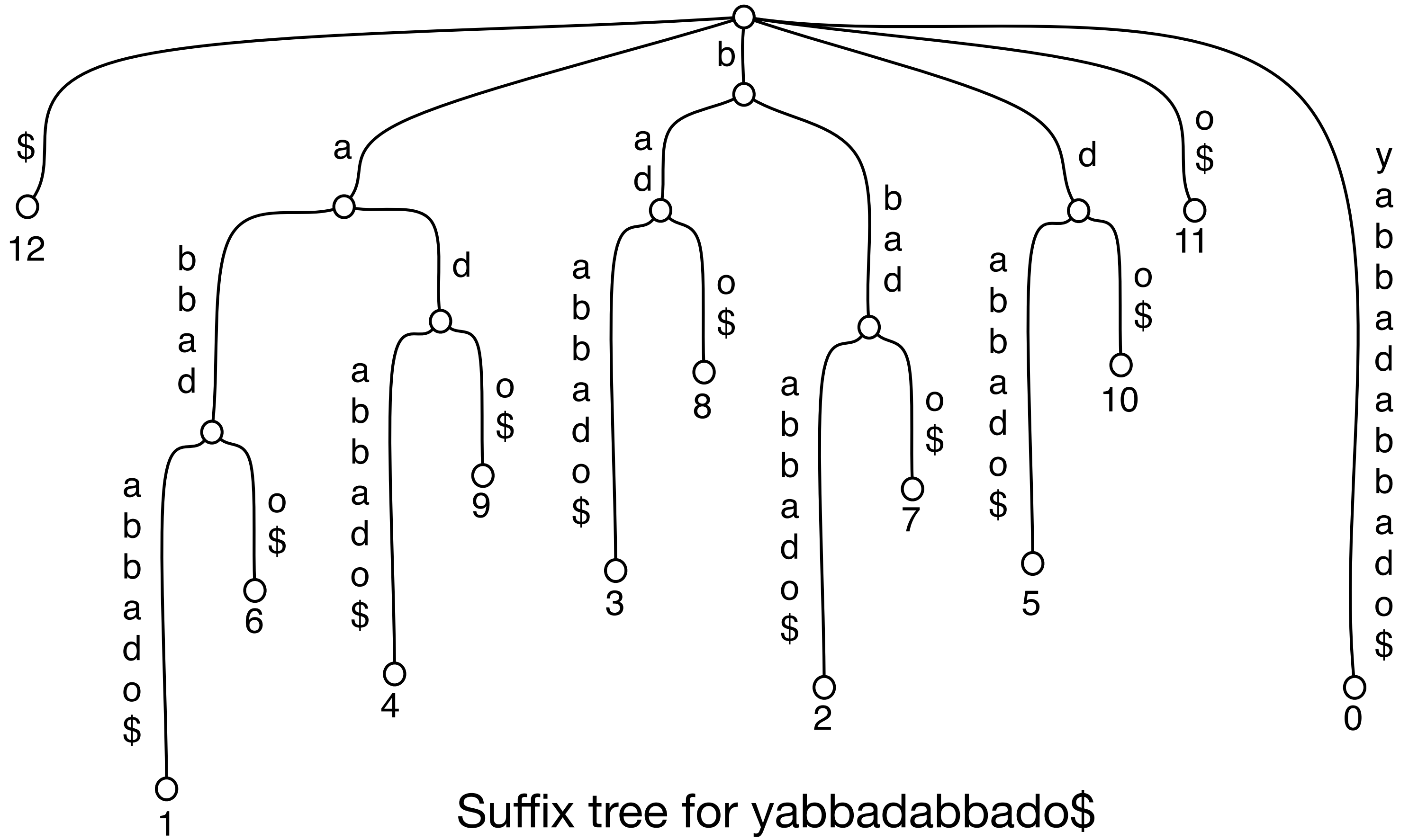
- Longest common extension problem. Let S be a string of characters from alphabet Σ. Preprocess S into data structure to support
  - LCP(i,j): Return the length of the longest common prefix of S[i,n] and S[j,n].

yabbadabbado

LCP(1,6) = 5

Longest Common Extension

- LCP and suffix trees?
- Solution. Suffix tree + string depth of each node + nearest/lowest common ancestor data structure.
- ⟹ O(n) space and O(1) query time.



Suffix tree for yabbadabbado$
LCP(1,6) = 5

Suffix Trees

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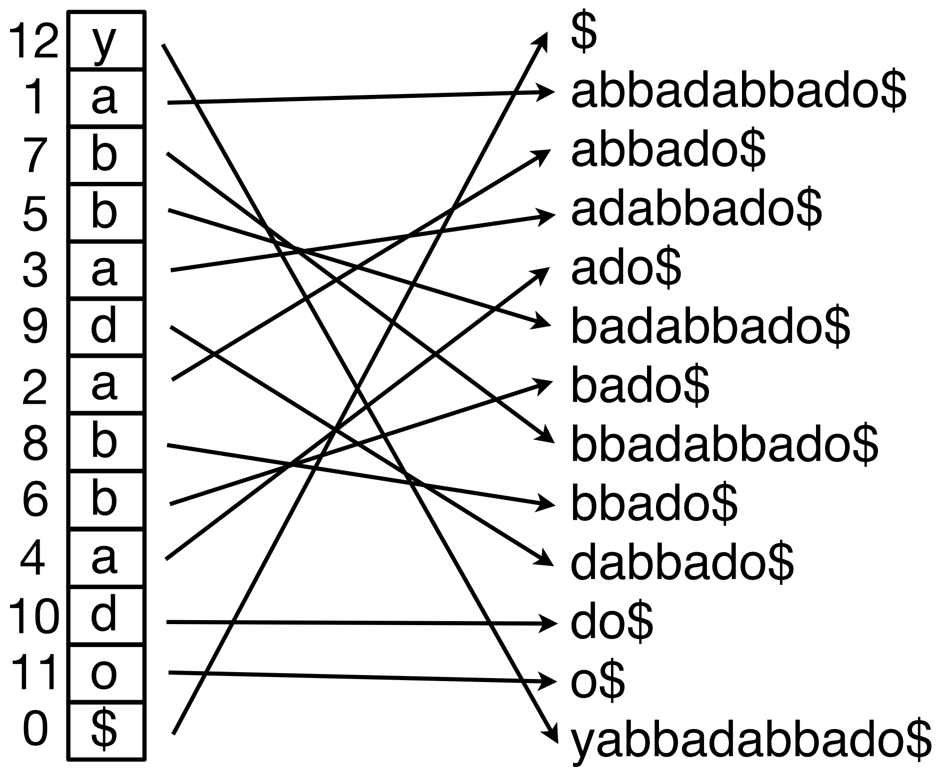
Suffix Sorting

- Suffix sorting. Given string S of length n over alphabet Σ, compute the sorted lexicographic order of all suffixes of S.
- Theorem [Kasai et al. 2001]. Given the sorted lexicographic order of suffixes of S, we can construct the suffix tree for S in linear time.
- How do we sort suffixes?



Suffix Sorting

- Goal. Compute the lexicographic order of all suffixes of S fast.
- Warm up. Sorting small universes.
- Solution in 3 steps.
  - Solution 1: Radix sorting
  - Solution 2: Prefix doubling
  - Solution 3: Difference cover sampling

Sorting Small Universes

- Let X be a sequence of n integers from a universe U = {0, 1, ..., u-1}.
- How fast can we sort if the size of the universe is not too big?
  - U = {0,1}?
  - U = {0, ... , n-1}?
  - U = {0, ..., n^3 - 1}?

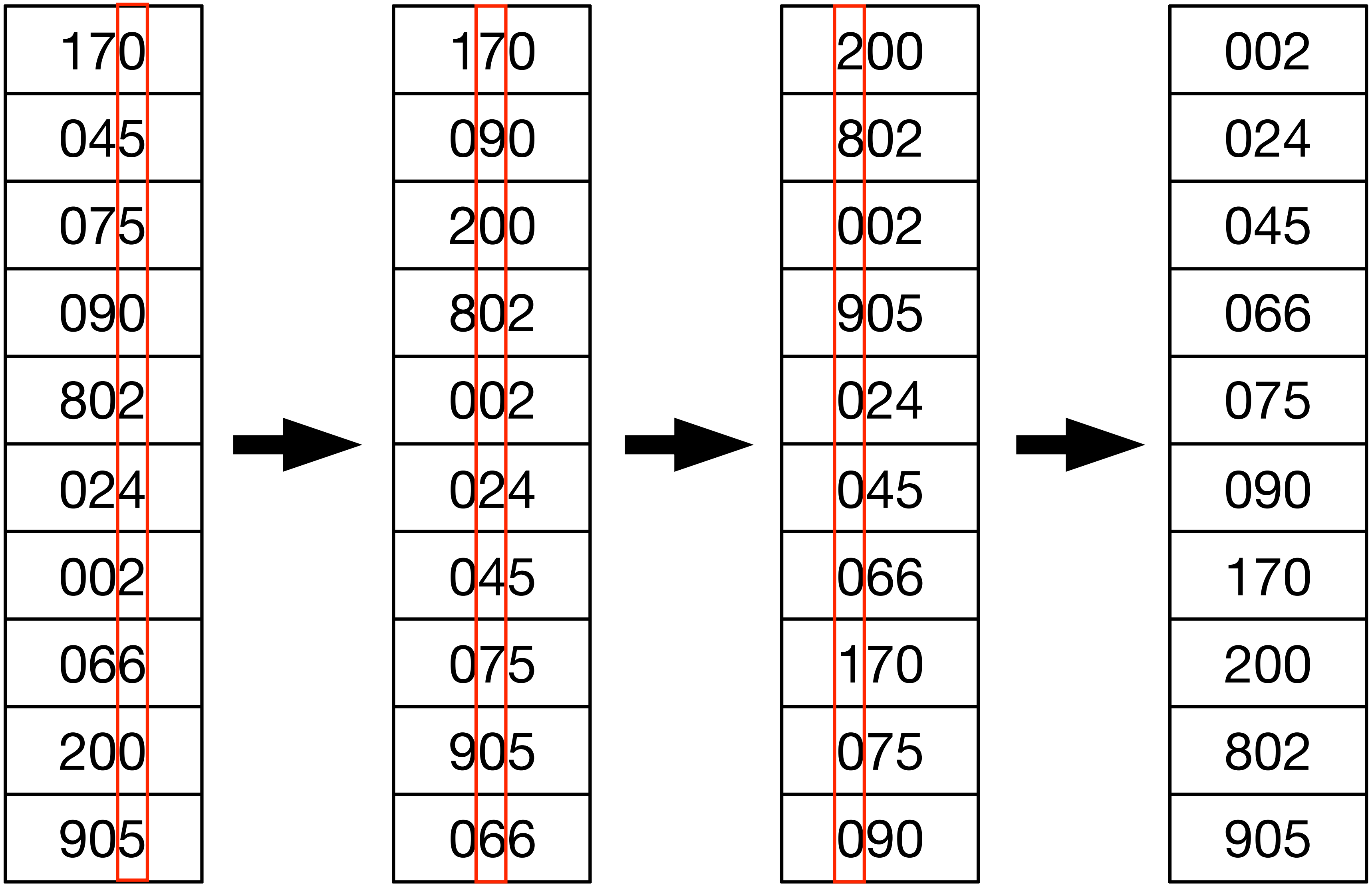
Sorting Small Universes

- Radix Sort [Hollerith 1887]. Sort sequence X of n integers from $U = \{0, ..., n^3-1\}$.
  - Write each $x \in X$ as a base n integer (x_1, x_2, x_3): $x = x_1 \cdot n^2 + x_2 \cdot n + x_3$
  - Sort X according to rightmost (least significant) digit
  - Sort X according to middle digit
  - Sort X according to leftmost (most significant) digit
- Each sort should be stable.
- Final result is the sorted sequence of X.

- Positional number systems. The base-n representation of x is x written in base n.
- Example.
  - $(10)_{10} = (1010)_2$ $(1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0)$
  - $(107)_{10} = (212)_7$ $(2 \cdot 7^2 + 1 \cdot 7^1 + 2 \cdot 7^0)$

Sorting Small Universes

$n = 10, U = \{0, ..., n^3 - 1 = 999\}$



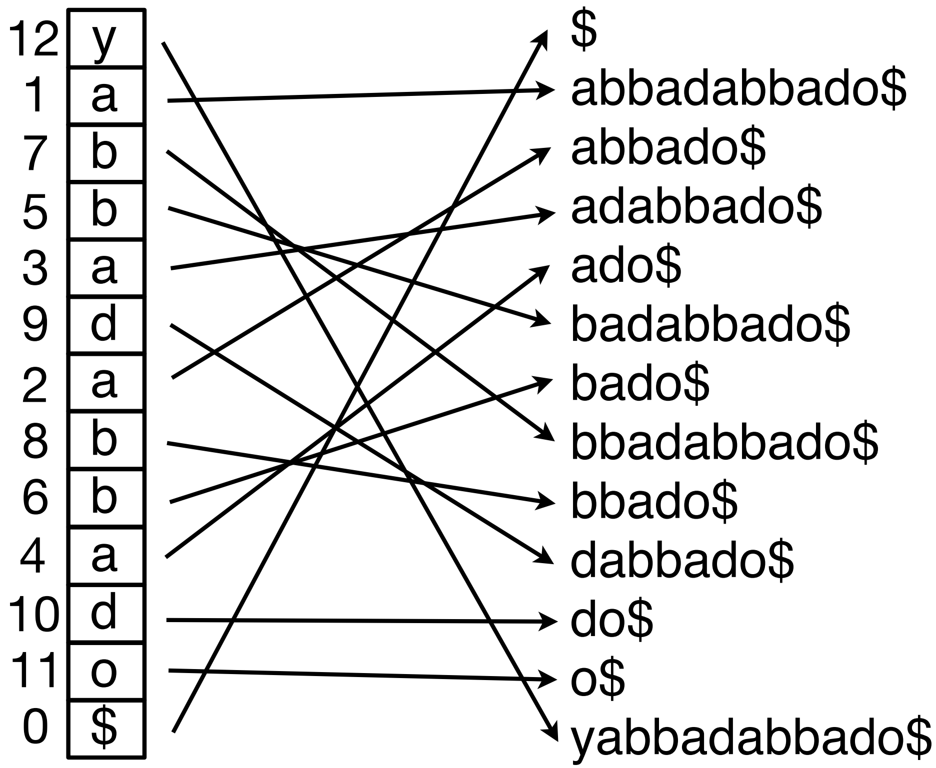
Sorting Small Universes

- Theorem. We can sort n integers from a universe $U = \{0, ..., n^3 - 1\}$ in $O(n)$ time.
- Theorem. We can sort n integers from a universe $U = \{0, ..., n^k - 1\}$ in $O(kn)$ time.

- Larger universes?
- Theorem [Han and Thorup 2002]. We can sort n integers in $O(n \log \log n)$ time or $O(n (\log \log n)^{1/2})$ expected time.

Suffix Sorting

- Suffix sorting. Given string S of length n over alphabet Σ, compute the sorted lexicographic order of all suffixes of S.
- For simplicity assume |Σ| = O(n)



Solution 1: Radix Sort

- Radix Sort.
  - Generate all suffixes (pad with $).
  - Radix sort.

```
yabbadabbado$
abbadabbado$$
bbadabbado$$$
badabbado$$$$
adabbado$$$$$
dabbado$$$$$$
abbado$$$$$$$
bbado$$$$$$$$
bado$$$$$$$$$
ado$$$$$$$$$$
do$$$$$$$$$$$
o$$$$$$$$$$$$
$$$$$$$$$$$$$
```

- Time. $O(n^2)$

Solution 2: Prefix Doubling

- Prefix doubling [Manber and Myers 1990]. Sort substrings (padded with $) of lengths 1, 2, 4, 8, ..., n. Each step uses radix sort on pair from previous step.

5	y	8	51	ya	10	84	yabb	
1	a	1	12	ab	1	13	abba	
2	b	4	22	bb	6	42	bbad	
2	b	3	21	ba	4	35	bada	
1	a	2	13	ad	2	21	adab	
3	d	5	31	da	7	54	dabb	
1	a	1	12	ab	1	13	abba	....
2	b	4	22	bb	6	42	bbad	
2	b	3	21	ba	5	36	bado	
1	a	2	13	ad	3	27	ado$	
3	d	6	34	do	8	60	do$$	
4	o	7	40	o$	9	70	o$$$	
0	$	0	00	$$	0	00	$$$$	

- Time. O(n log n)

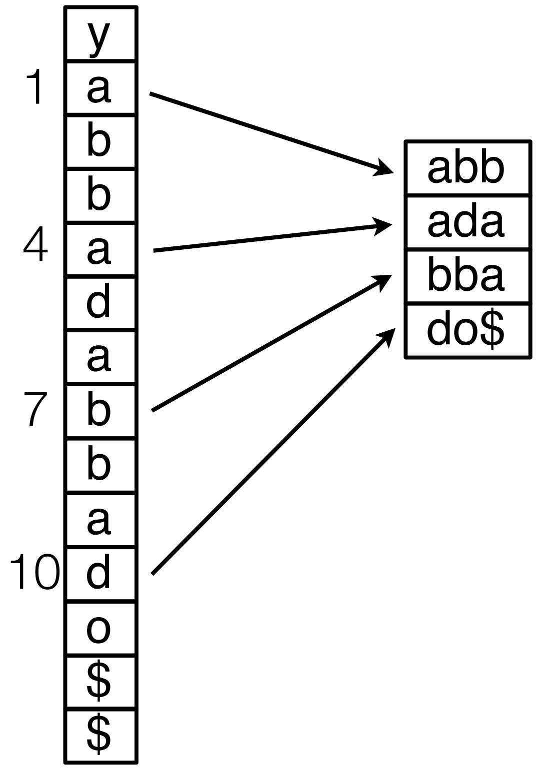
Solution 3: Difference Cover Sampling

- DC3 Algorithm [Karkkainen et al. 2003]. Sort suffixes in three steps:
  - Step 1. Sort sample suffixes.
    - Sample all suffixes starting at positions i = 1 mod 3 and i = 2 mod 3.
    - Recursively sort sample suffixes.
  - Step 2. Sort non-sample suffixes.
    - Sort the remaining suffixes (starting at positions i = 0 mod 3).
  - Step 3. Merge.
    - Merge sample and non-sample suffixes.

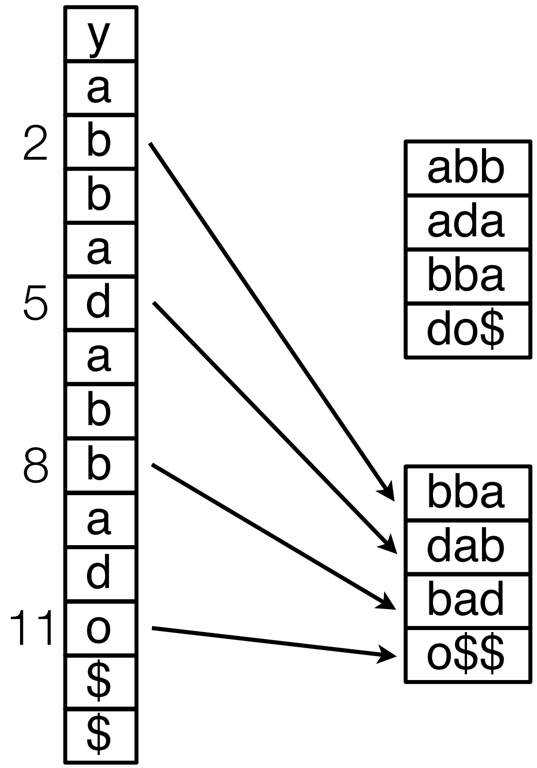
Step 1: Sort Sample Suffixes

y
a
b
b
a
d
a
b
b
a
d
o
$
$

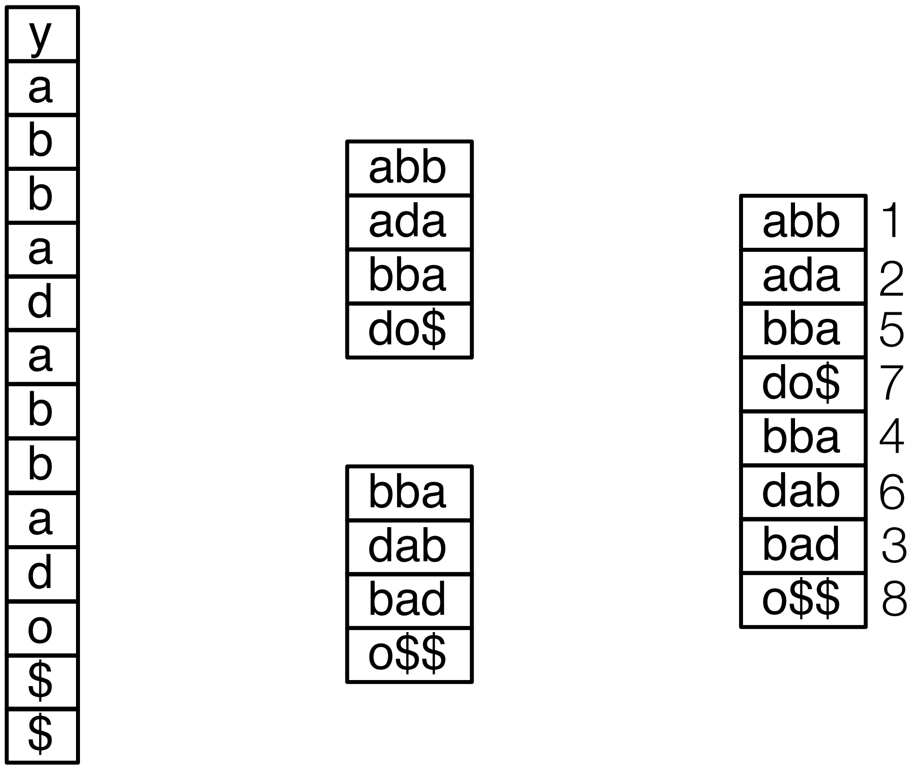
Step 1: Sort Sample Suffixes



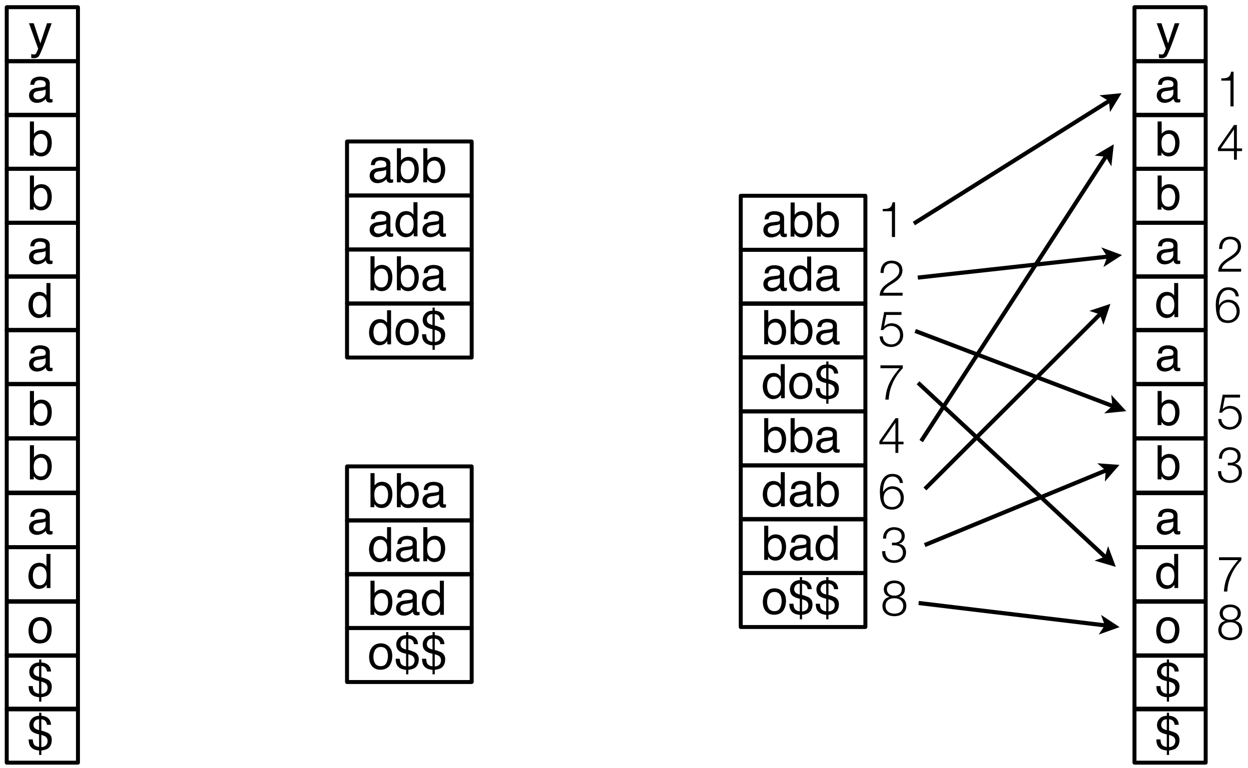
Step 1: Sort Sample Suffixes



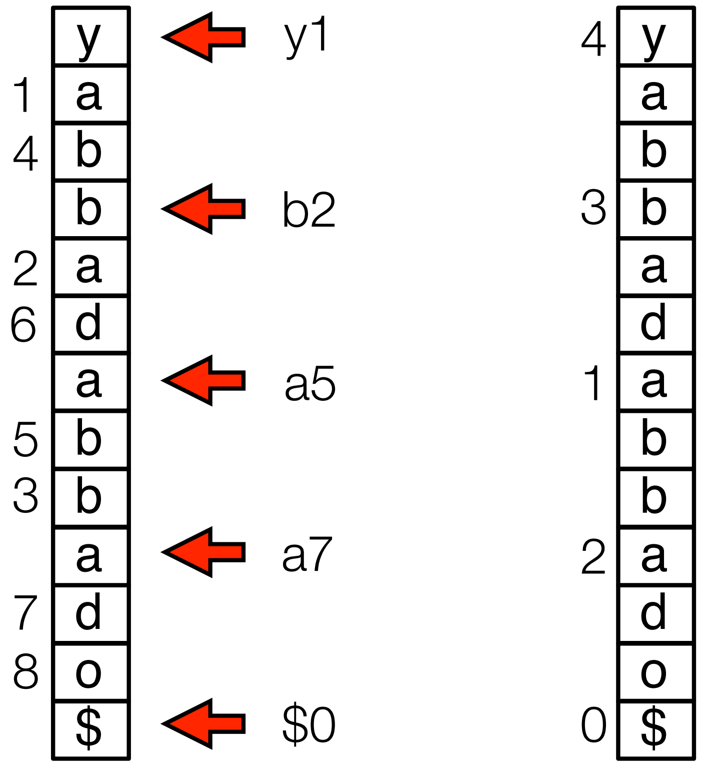
Step 1: Sort Sample Suffixes



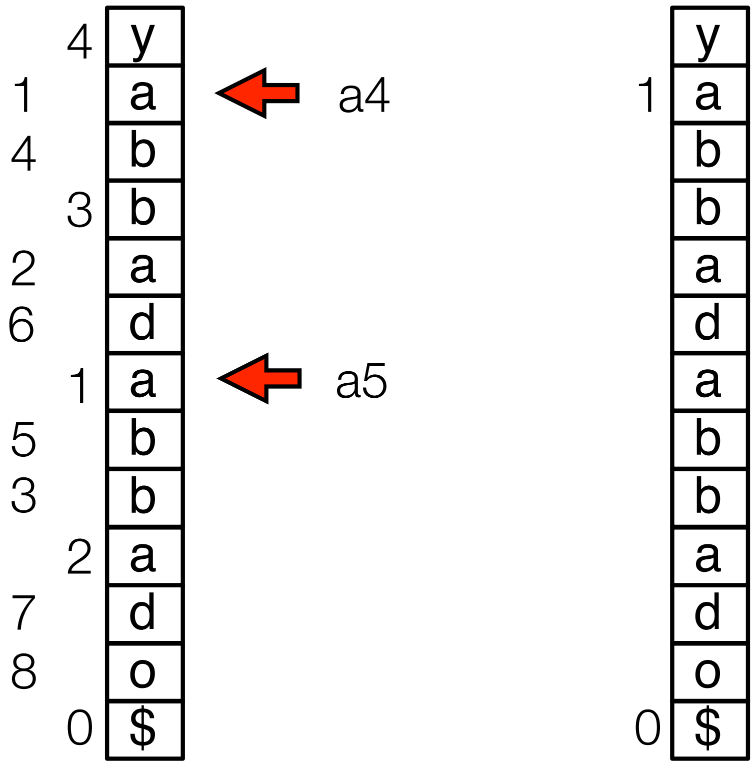
Step 1: Sort Sample Suffixes



Step 2: Sort Non-Sample Suffixes



Step 3: Merge



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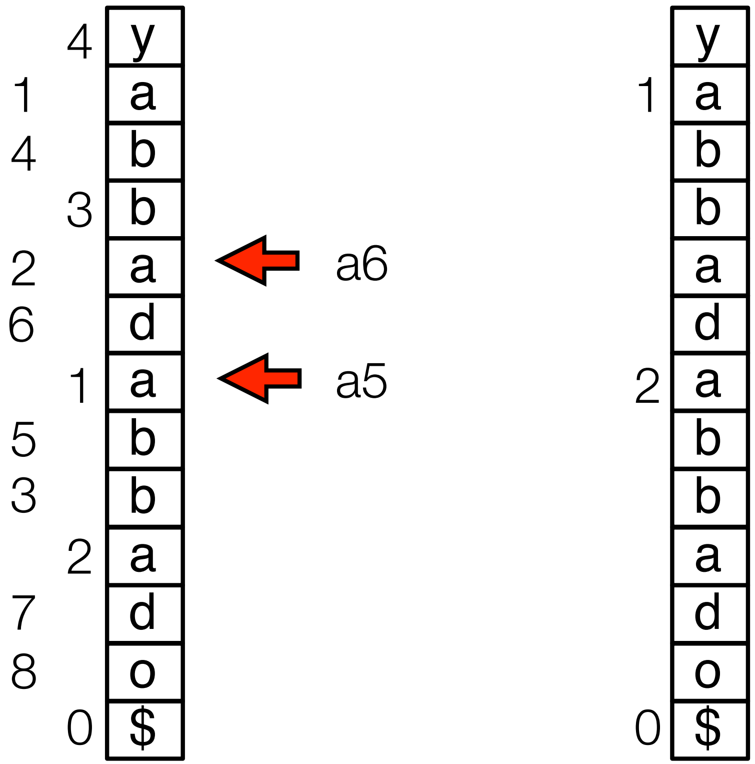

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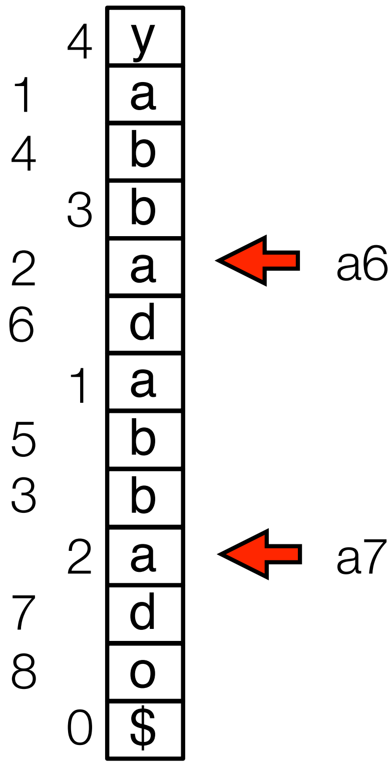

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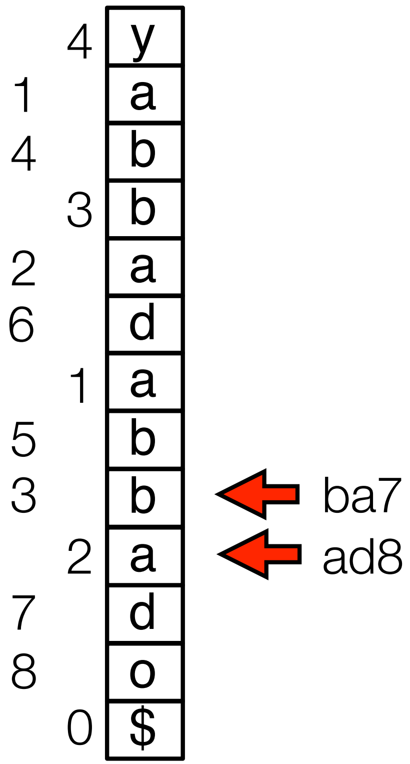
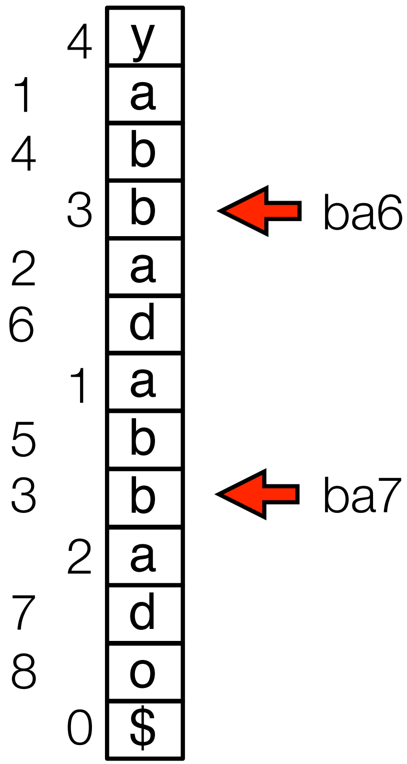
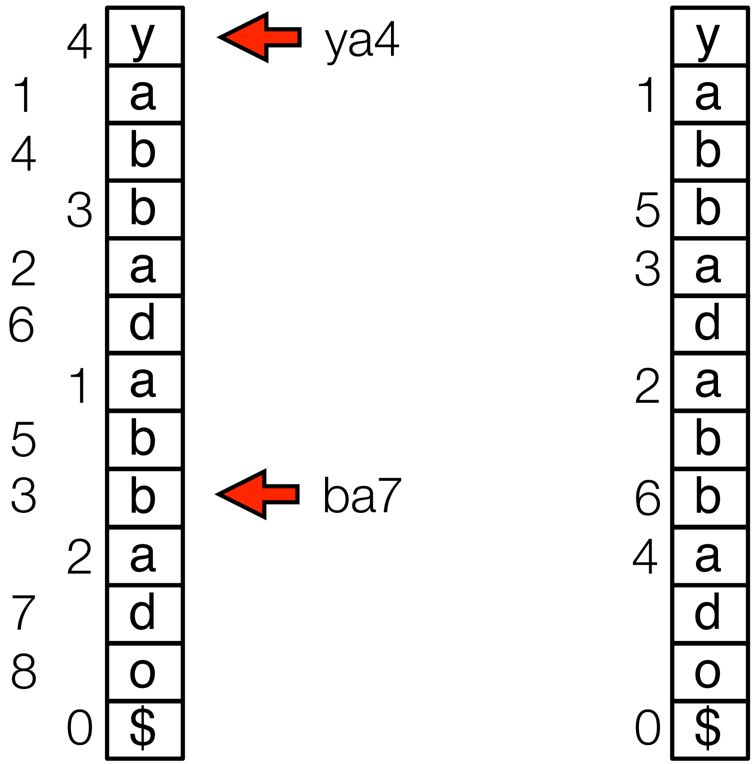


Step 3: Merge

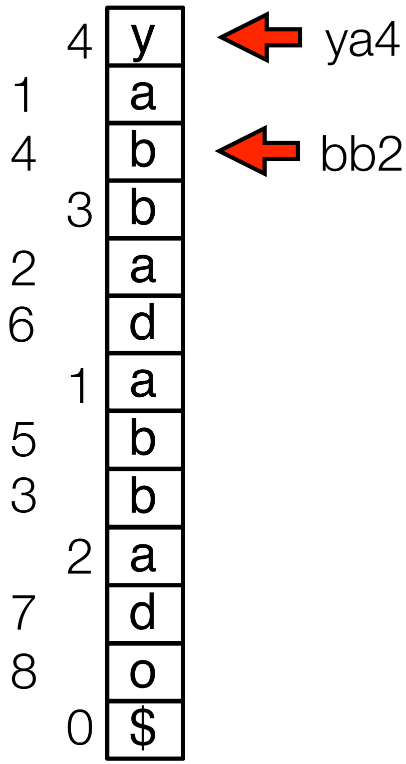


Step 3: Merge

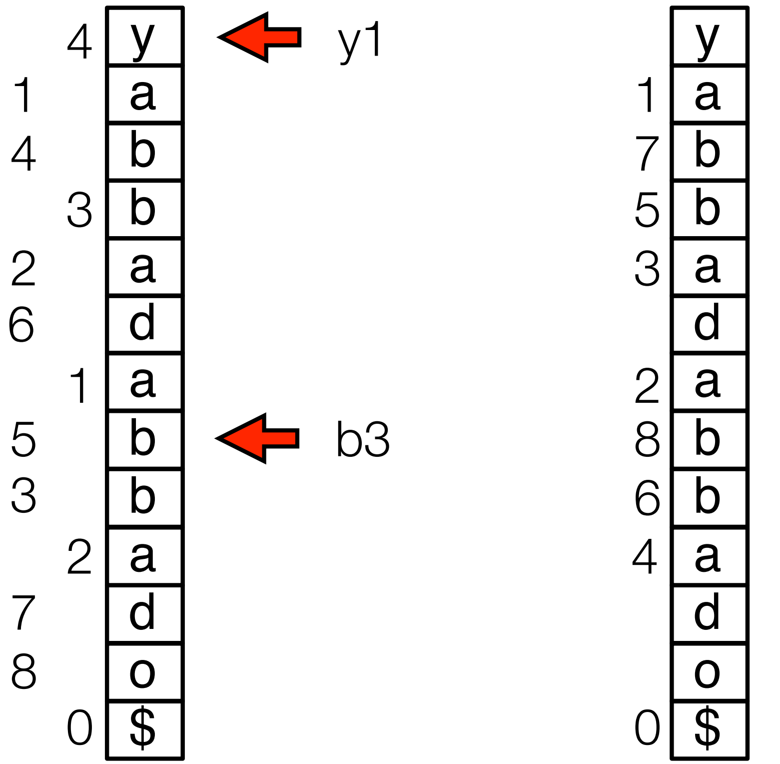


4 y
1 a
4 b
3 b
2 a
6 d
1 a
5 b
3 b
2 a
7 d
8 o
0 $
ya4
ba7
y
1 a
b
5 b
3 a
d
2 a
b
6 b
4 a
d
o
0 $

Step 3: Merge

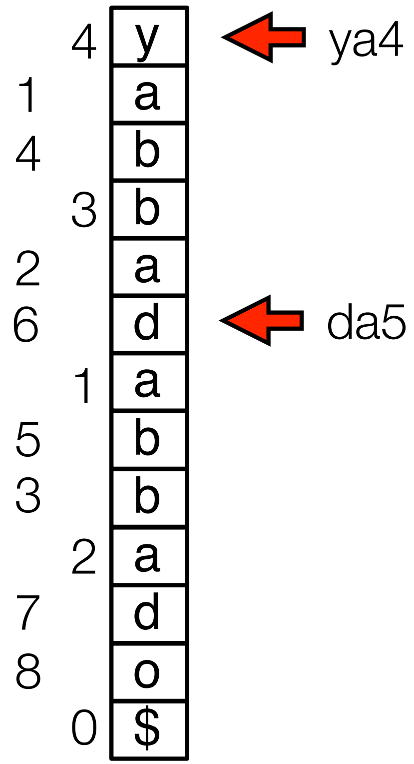

4 y
1 a
4 b
3 b
2 a
6 d
1 a
5 b
3 b
2 a
7 d
8 o
0 $
ya4
bb2
y
1 a
7 b
5 b
3 a
d
2 a
b
6 b
4 a
d
o
0 $

Step 3: Merge



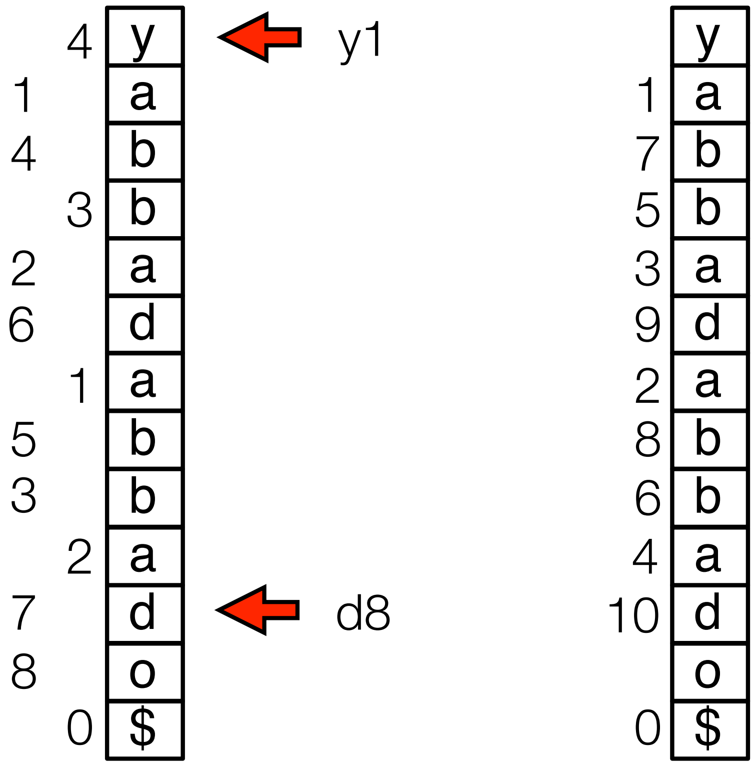
4 y
1 a
4 b
3 b
2 a
6 d
1 a
5 b
3 b
2 a
7 d
8 o
0 $
y1
b3
y
1 a
7 b
5 b
3 a
d
2 a
8 b
6 b
4 a
d
o
0 $

Step 3: Merge

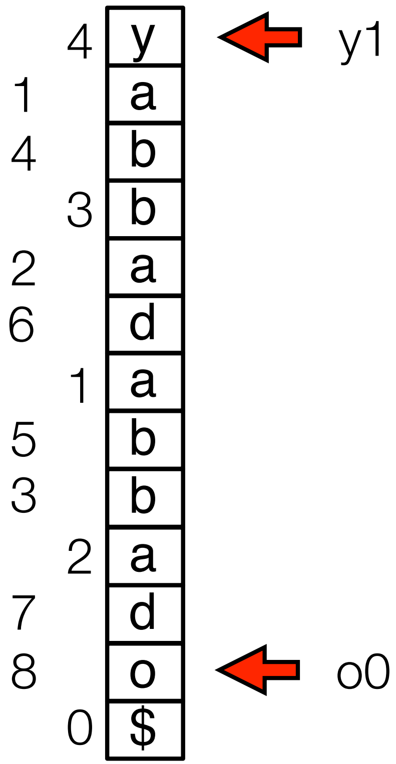


4 y ya4
1 a
4 b
3 b
2 a
6 d da5
1 a
5 b
3 b
2 a
7 d
8 o
0 $
y
1 a
7 b
5 b
3 a
9 d
2 a
8 b
6 b
4 a
d
o
0 $

Step 3: Merge

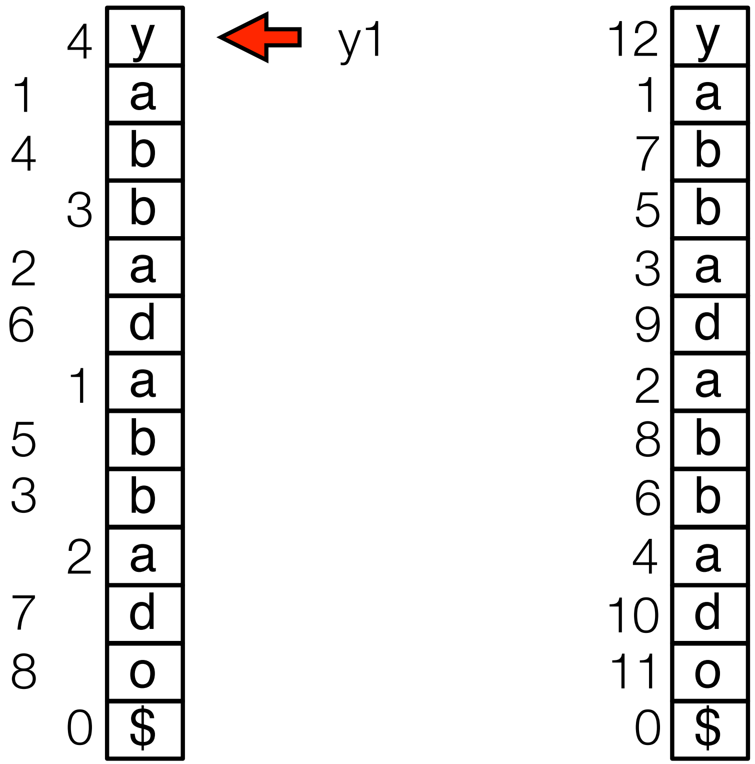


Step 3: Merge



4 y
1 a
4 b
3 b
2 a
6 d
1 a
5 b
3 b
2 a
7 d
8 o
0 $
y1
o0
y
1 a
7 b
5 b
3 a
9 d
2 a
8 b
6 b
4 a
10 d
11 o
0 $

Step 3: Merge


4 y
1 a
4 b
3 b
2 a
6 d
1 a
5 b
3 b
2 a
7 d
8 o
0 $
y1
12 y
1 a
7 b
5 b
3 a
9 d
2 a
8 b
6 b
4 a
10 d
11 o
0 $

Solution 3: Difference Cover Sampling

- DC3 Algorithm. Sort suffixes in three steps:
  - Step 1. Sort sample suffixes.
    - Sample all suffixes starting at positions i = 1 mod 3 and i = 2 mod 3. O(n)
    - Recursively sort sample suffixes. T(2n/3)
  - Step 2. Sort non-sample suffixes.
    - Sort the remaining suffixes (starting at positions i = 0 mod 3). O(n)
  - Step 3. Merge.
    - Merge sample and non-sample suffixes. O(n)
- T(n) = time to suffix sort a string of length n over alphabet of size n

- Time. T(n) = T(2n/3) + O(n) = O(n)

Solution 3: Difference Cover Sampling

- Theorem. We can suffix sort a string of length n over alphabet Σ of size n in time O(n).
- Larger alphabets?
- Theorem. We can suffix sort a string of length n over alphabet Σ O(sort(n, |Σ|)) time.
- Bound is optimal.

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