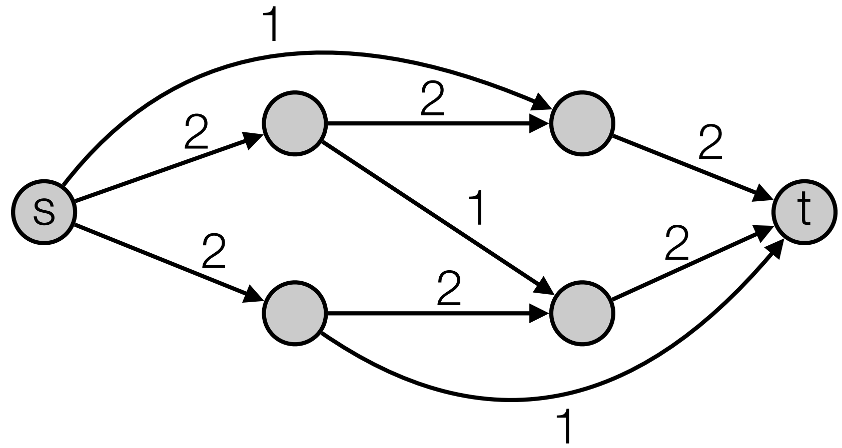
Network Flows

Inge Li Gørtz

Applications

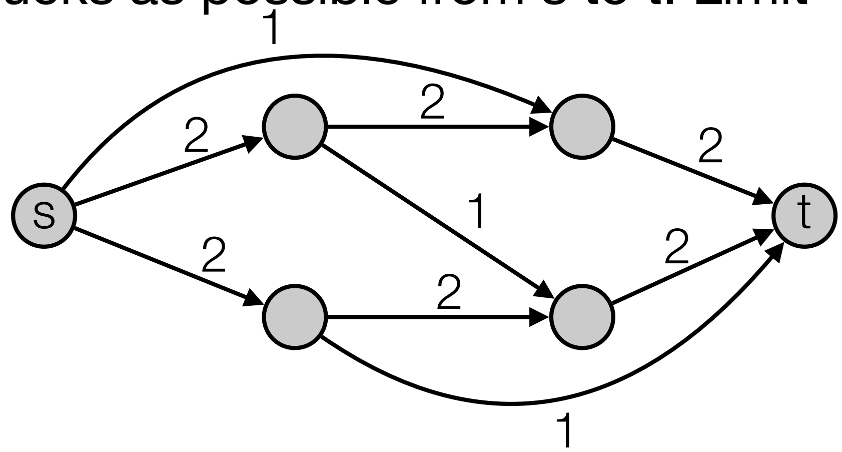
- Matchings
- Job scheduling
- Image segmentation
- Baseball elimination
- Disjoint paths
- Survivable network design

Network Flow



Network Flow

- Truck company: Wants to send as many trucks as possible from s to t. Limit of number of trucks on each road.



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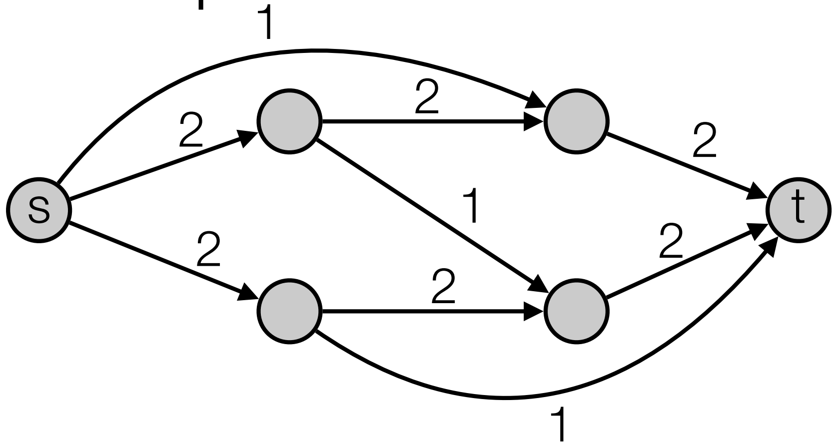

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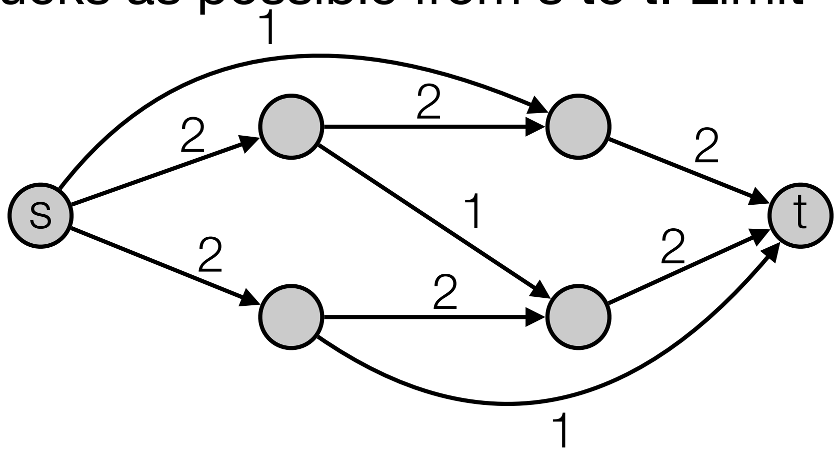

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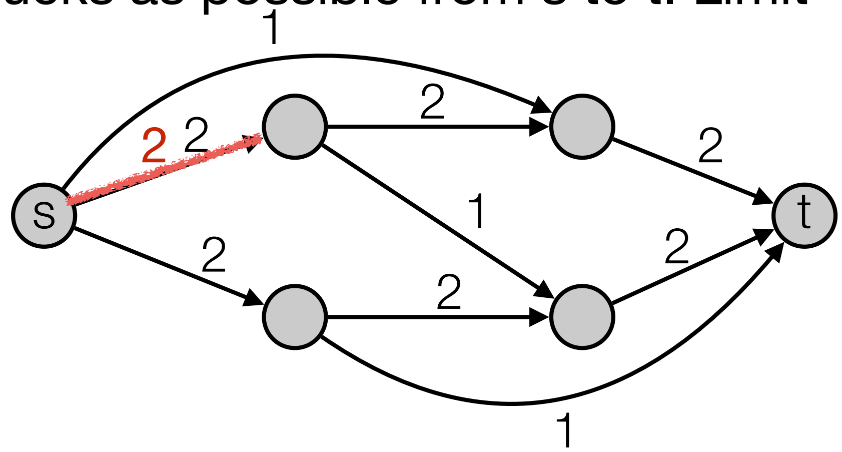

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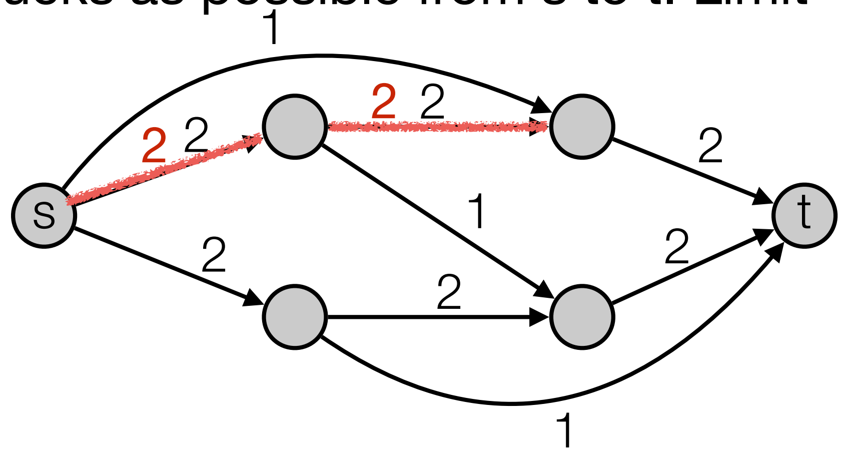

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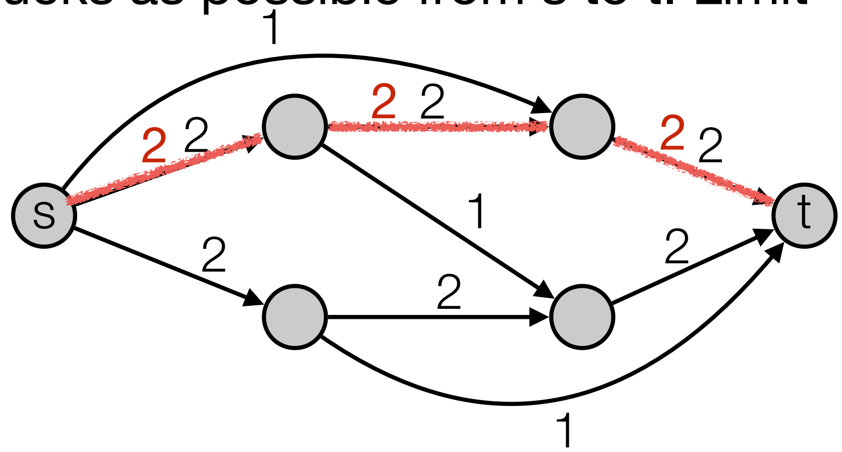

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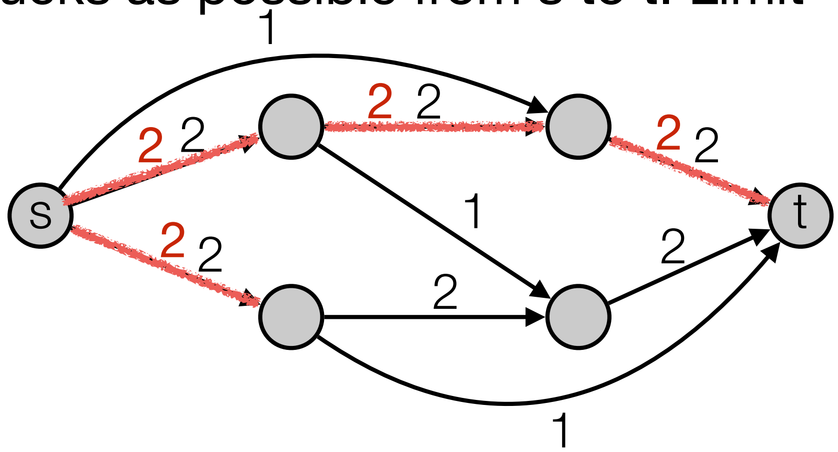

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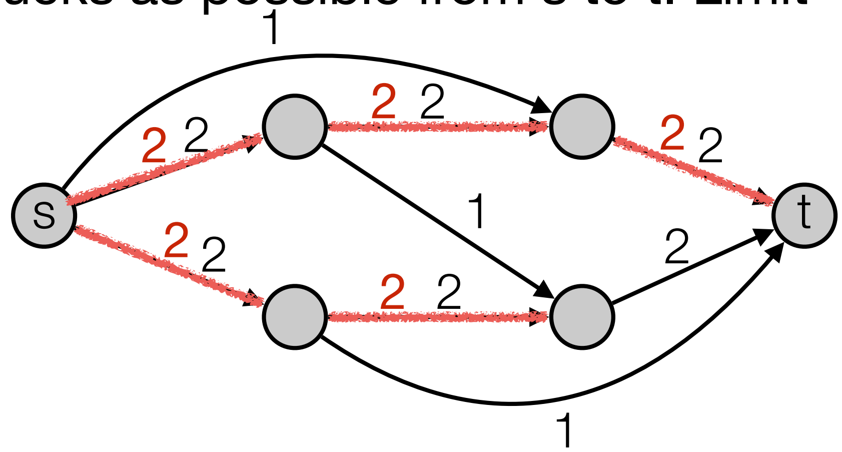

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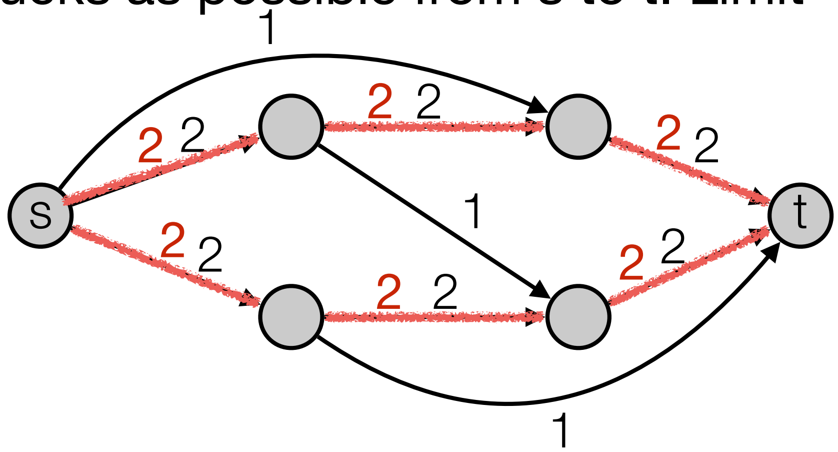
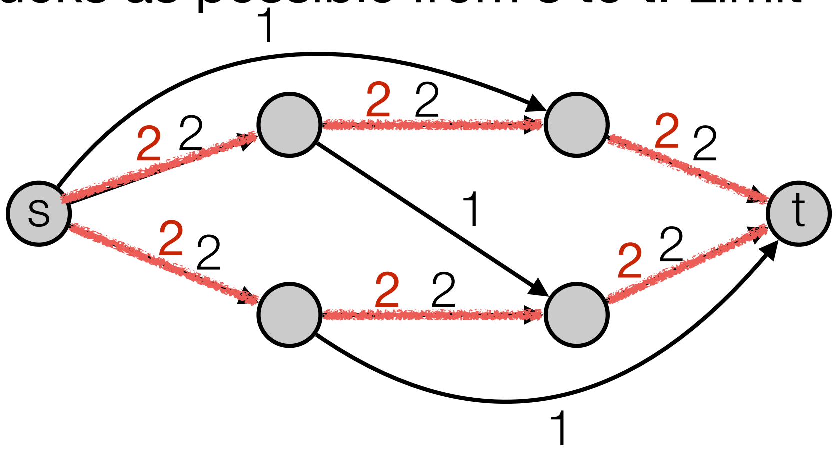


Network Flow

- Truck company: Wants to send as many trucks as possible from s to t. Limit of number of trucks on each road.

- Example 1:
  - Solution 1: 4 trucks



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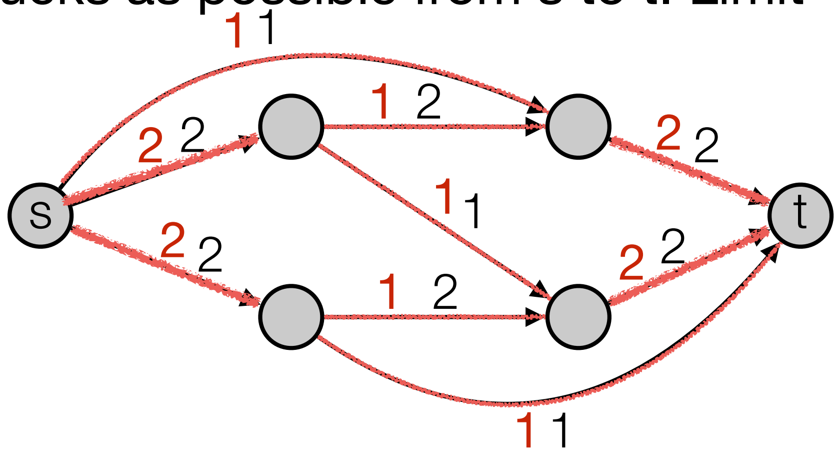

- Example 1:
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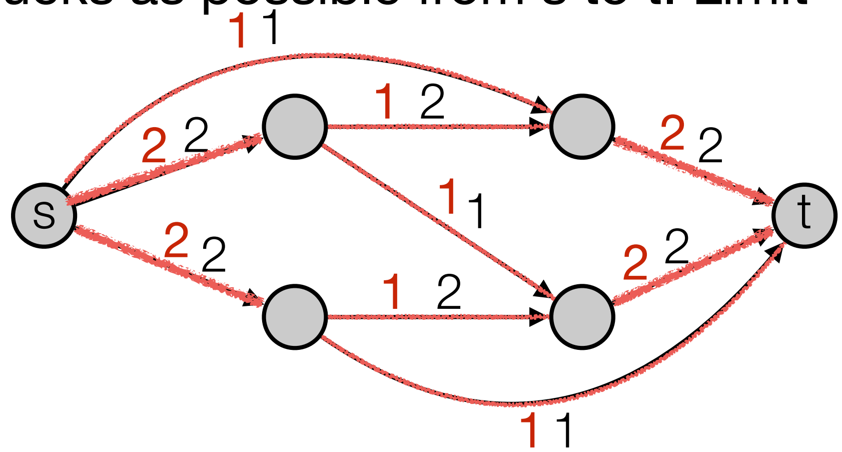
Network Flow

- Truck company: Wants to send as many trucks as possible from s to t. Limit of number of trucks on each road.

- Example 1:
    - Solution 1: 4 trucks
    - Solution 2: 5 trucks



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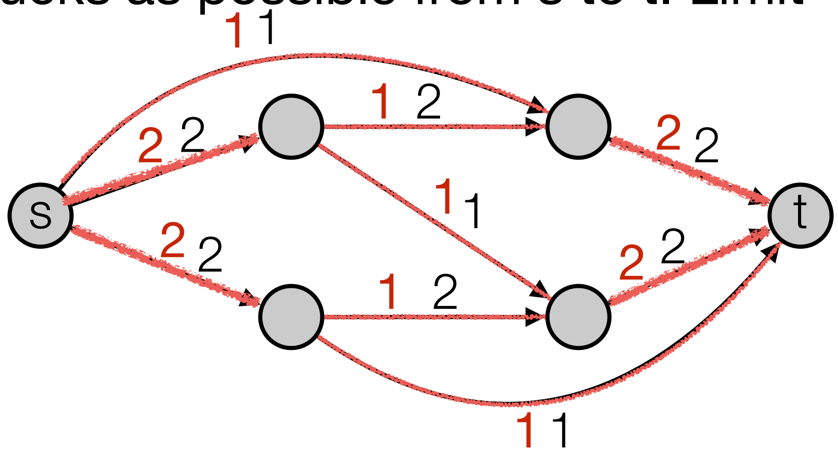

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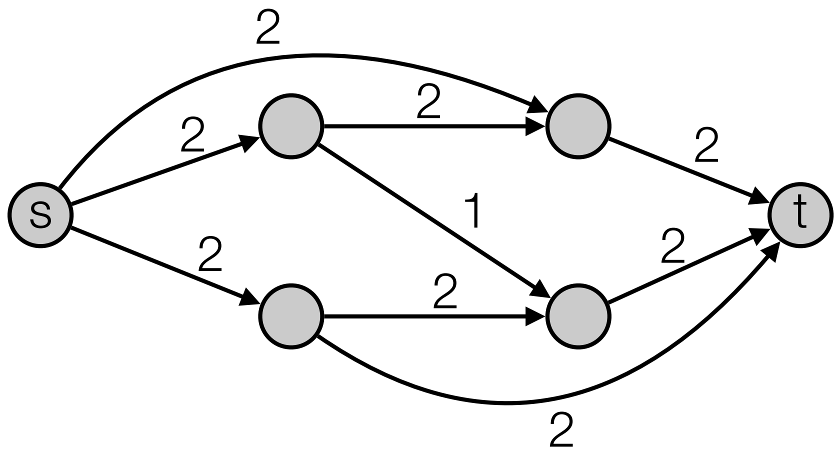

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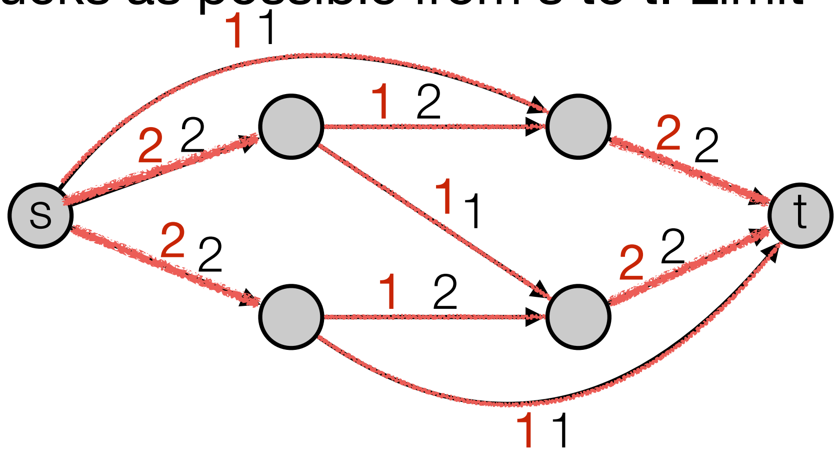

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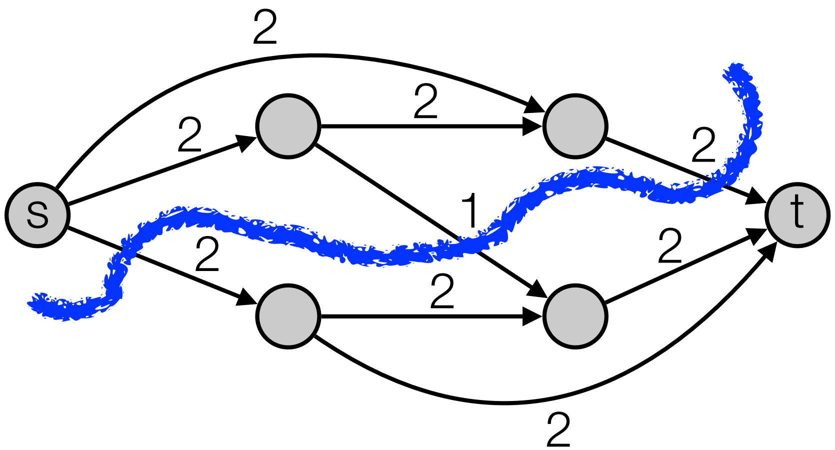

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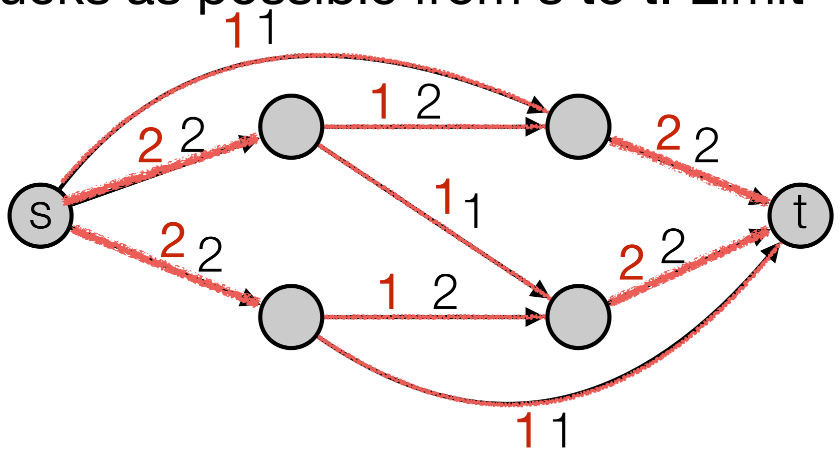
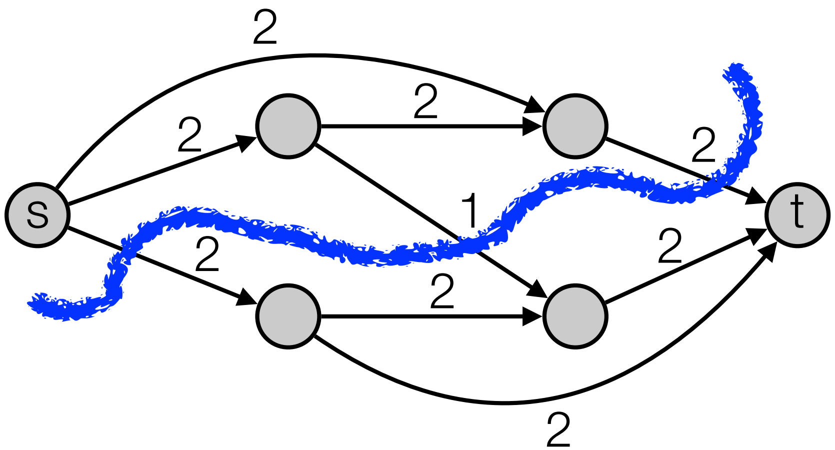


- Example 2:
    - 5 trucks (need to cross river).



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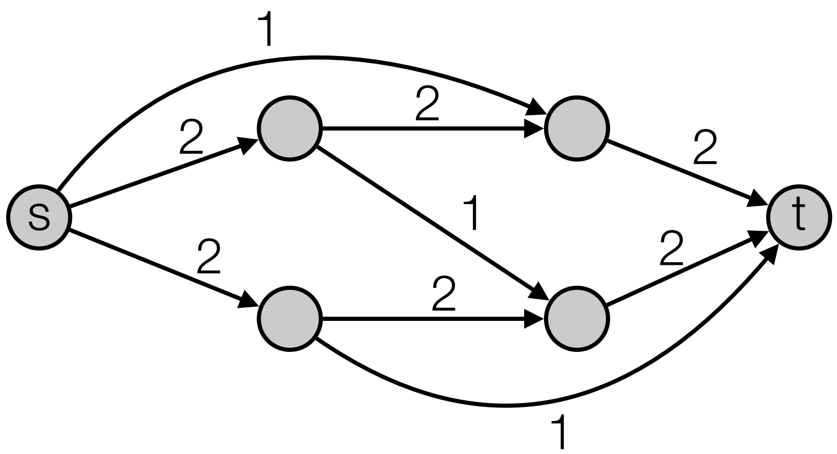

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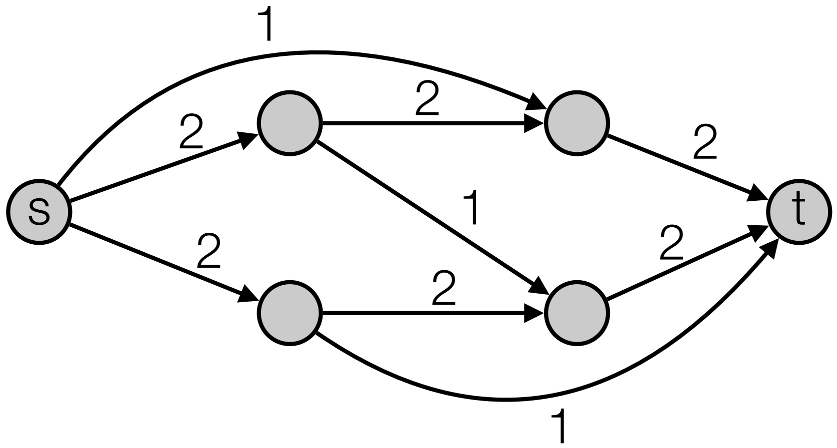

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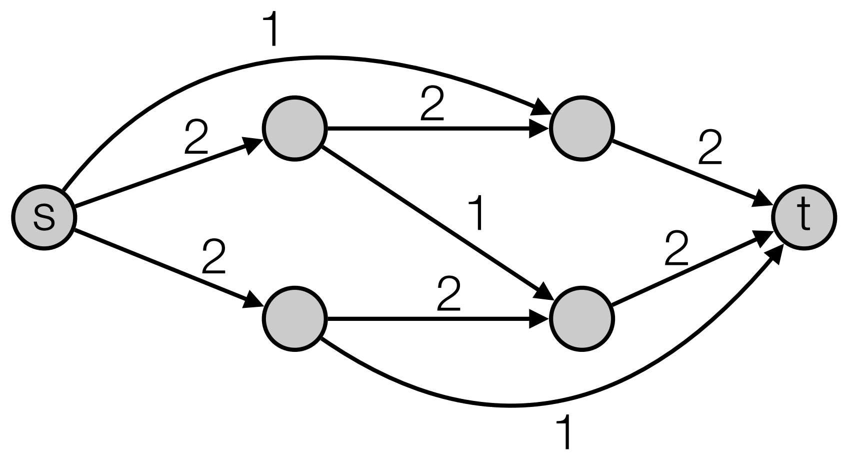

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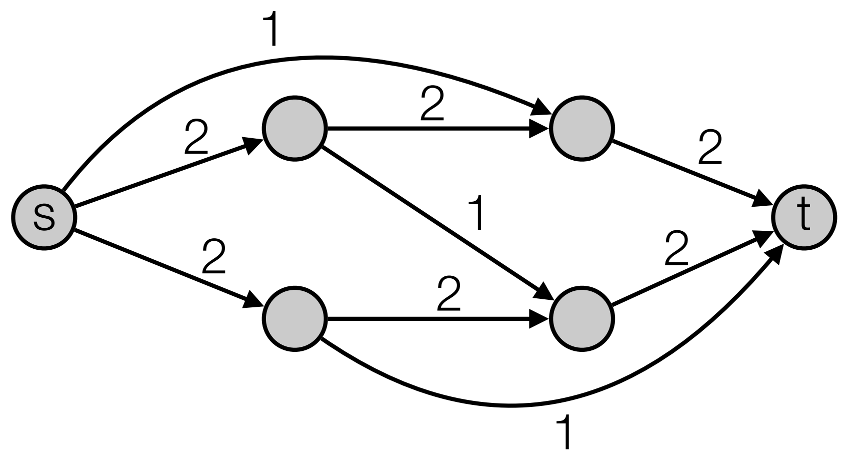

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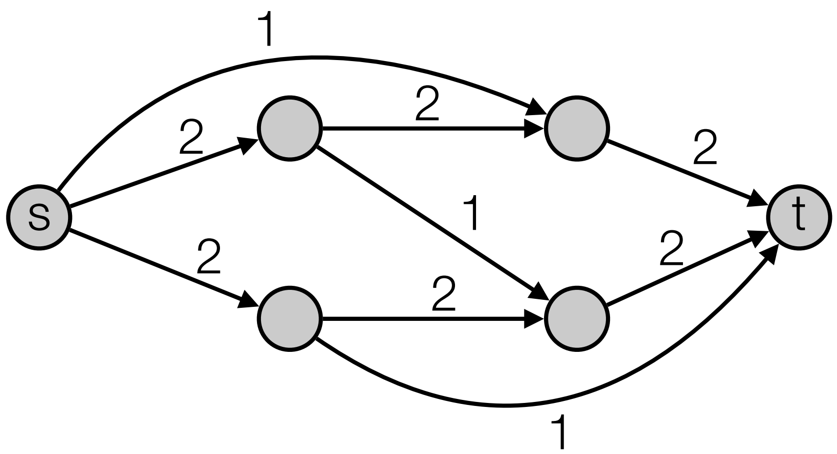

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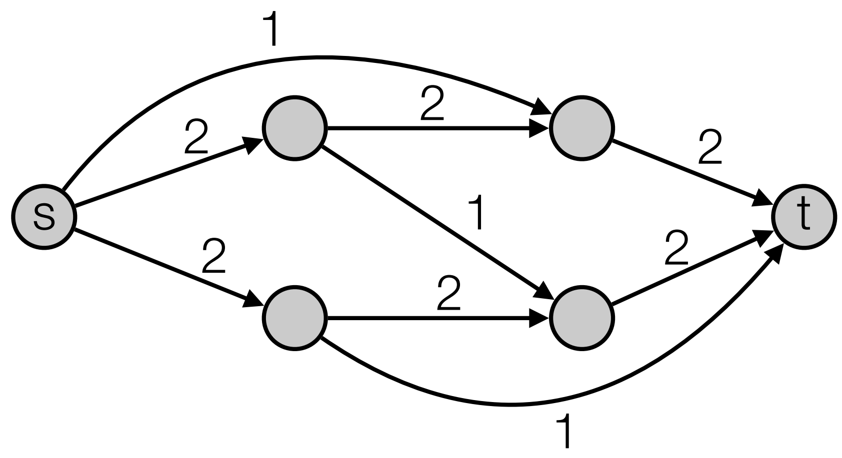

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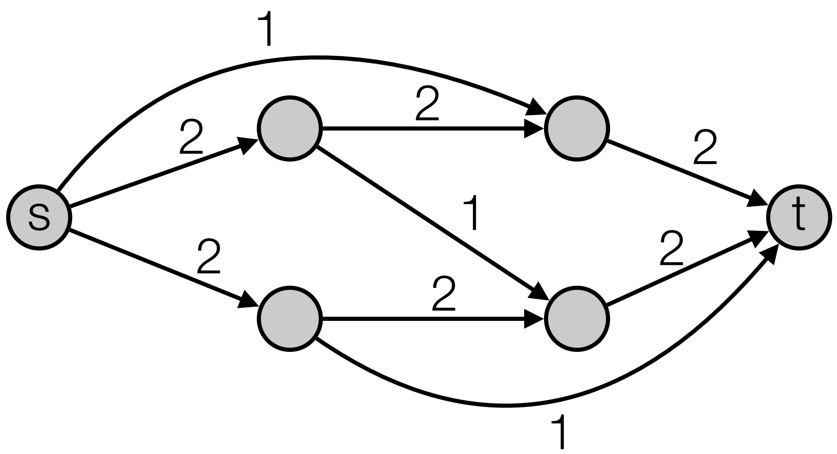

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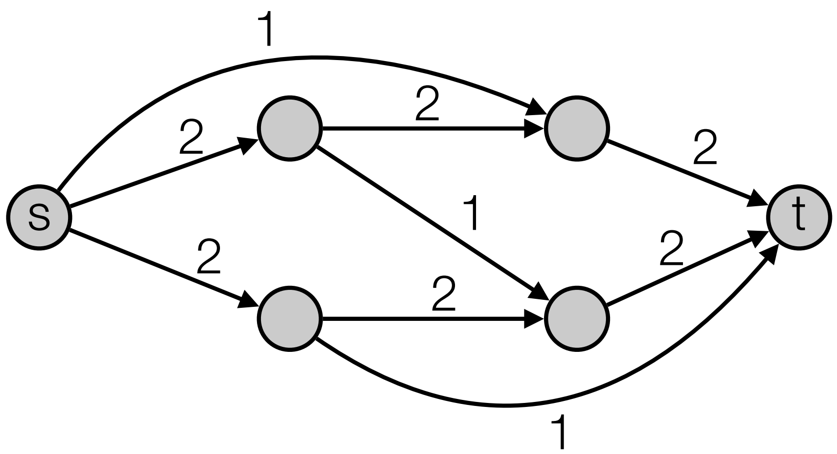

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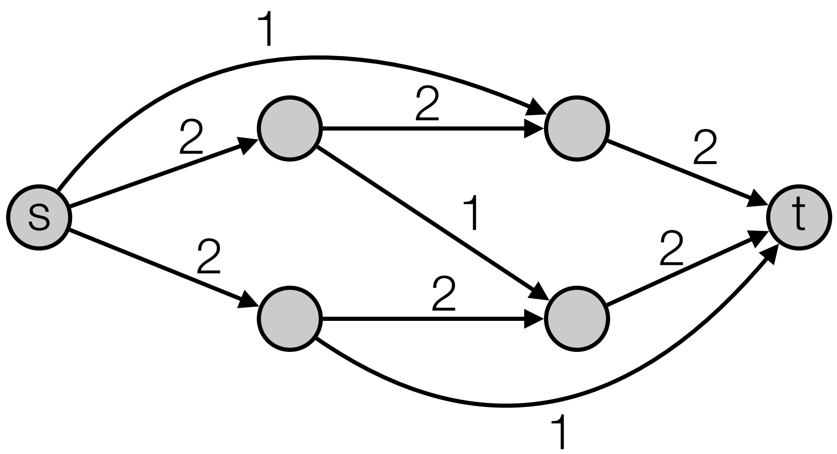

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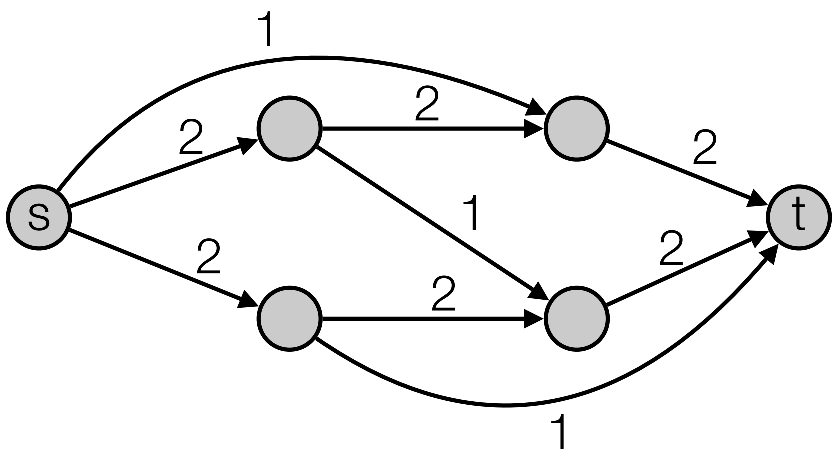

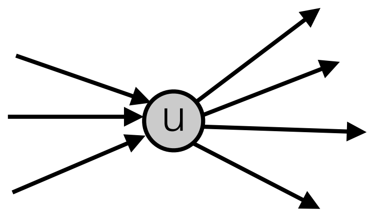

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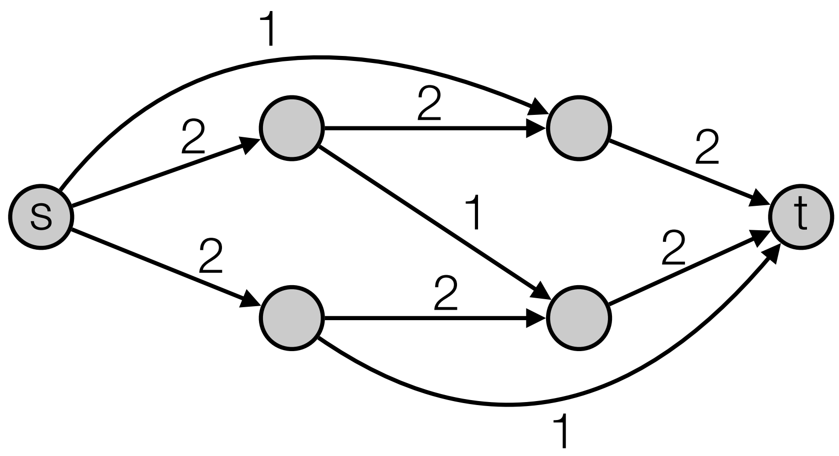

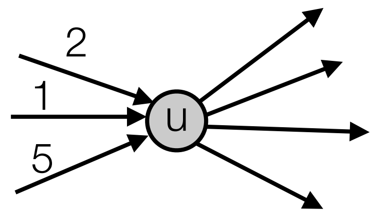

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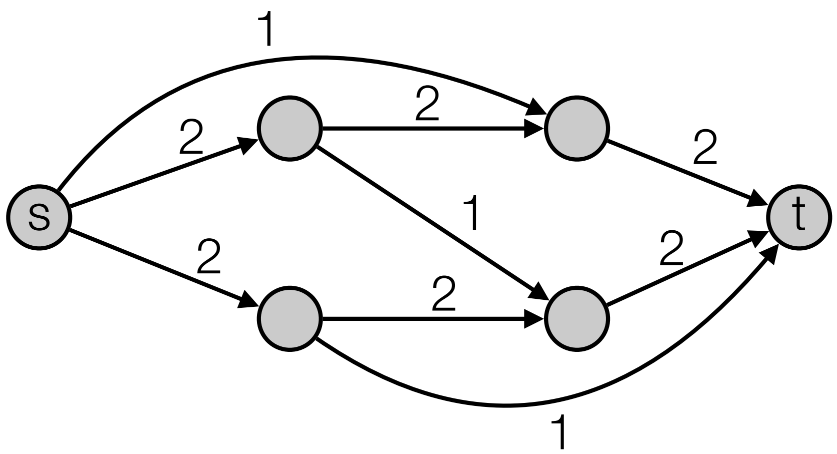

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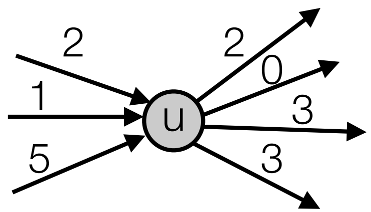

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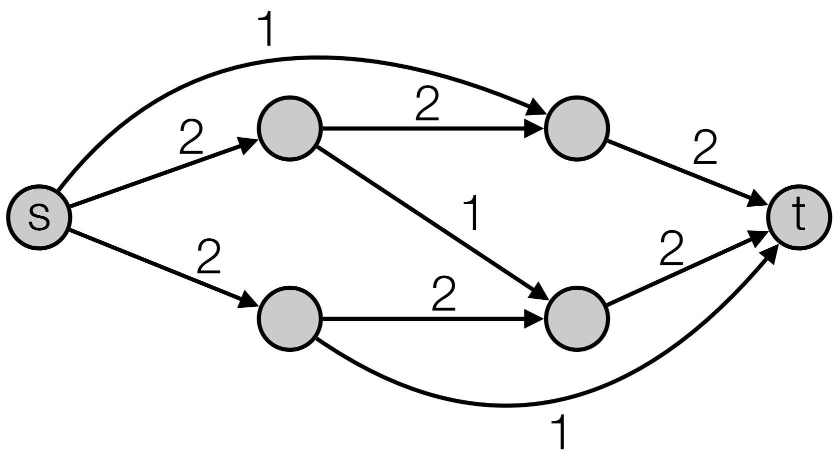

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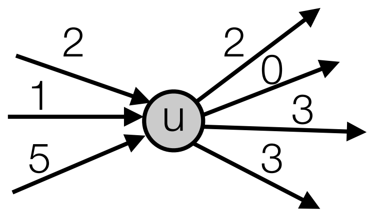

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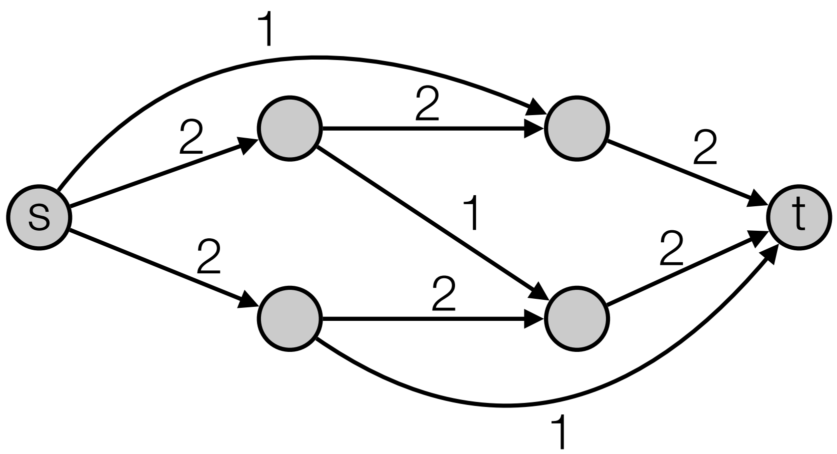

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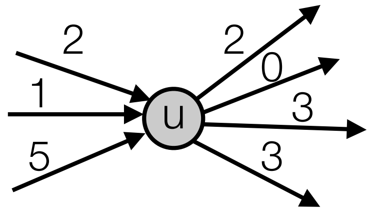

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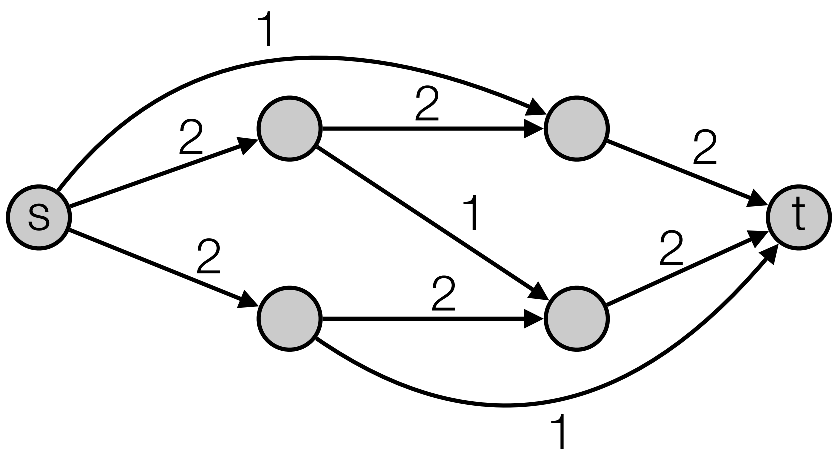

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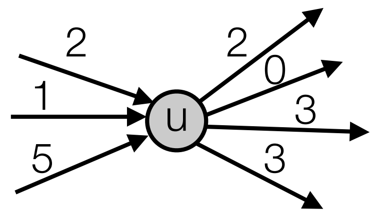

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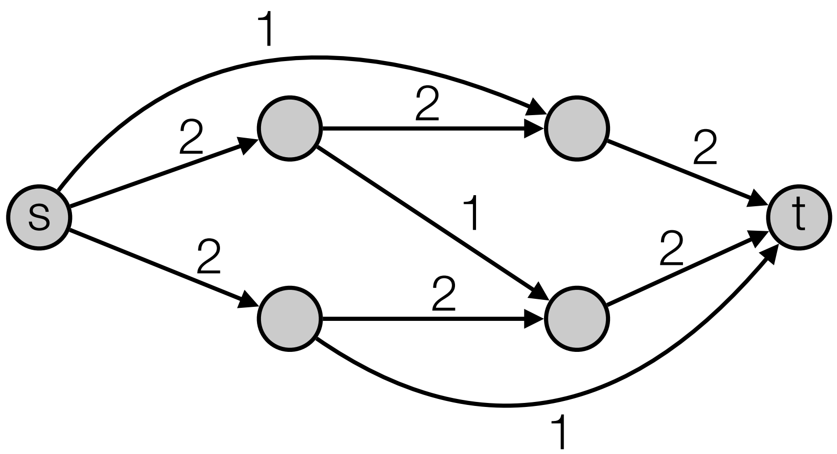

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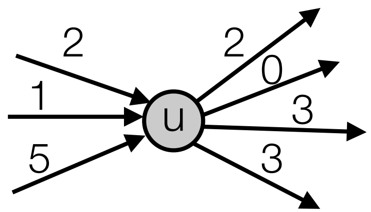

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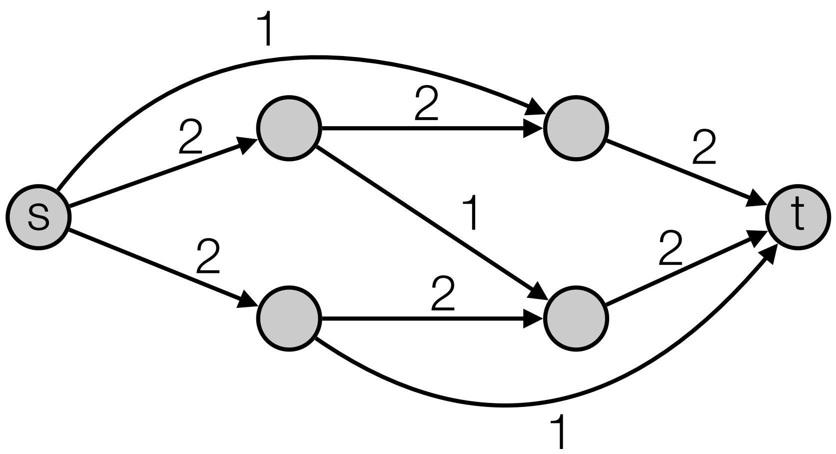
  - **Maximum flow problem:** find s-t flow of maximum value

Algorithm

- Find path where we can send more flow.



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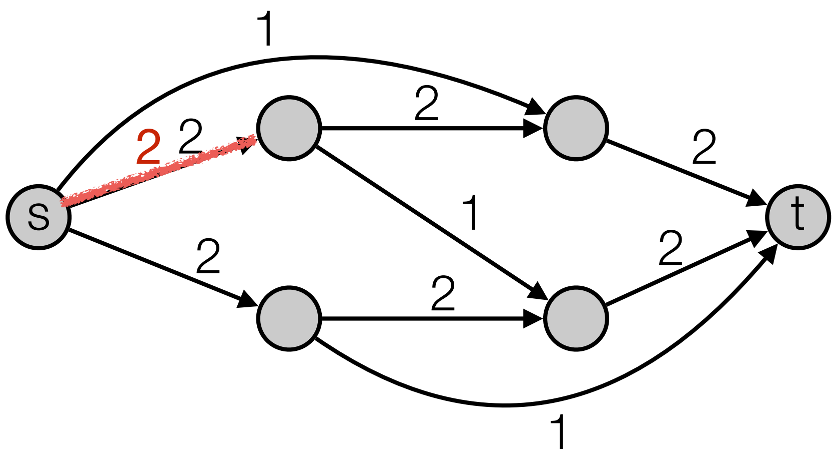

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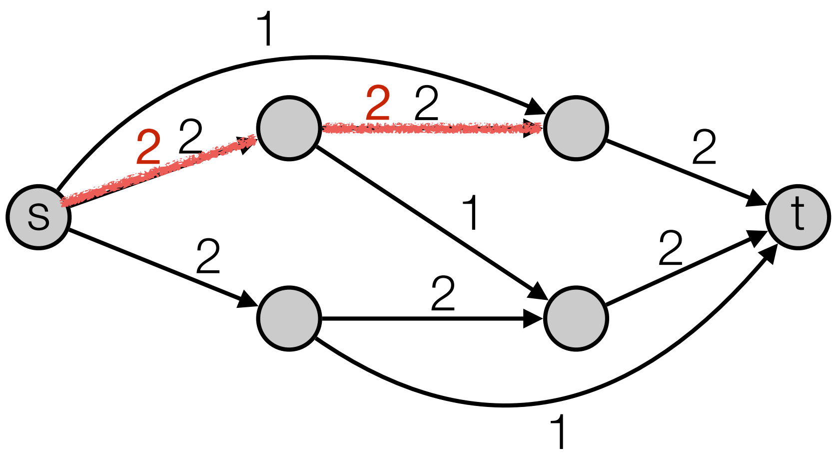

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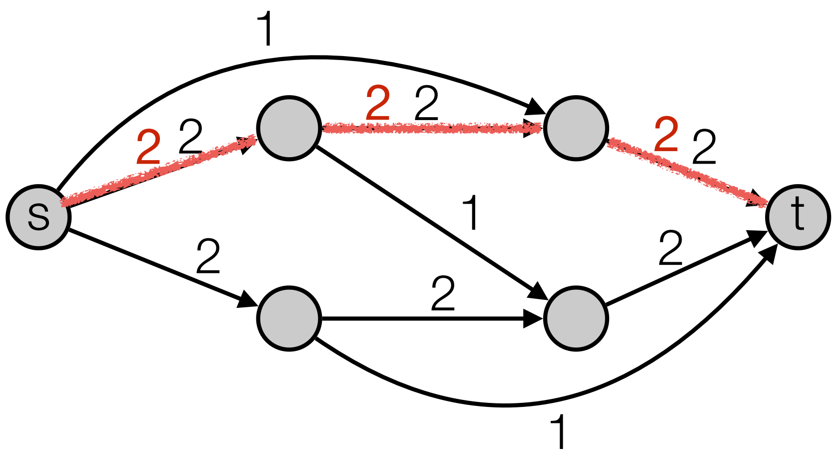

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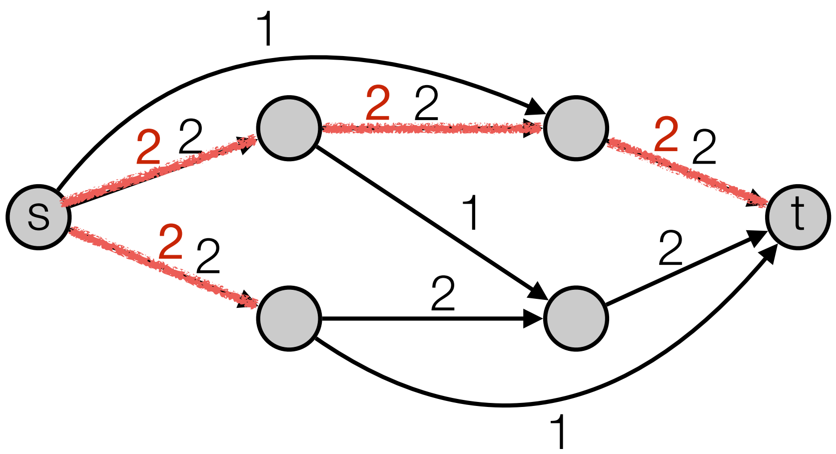

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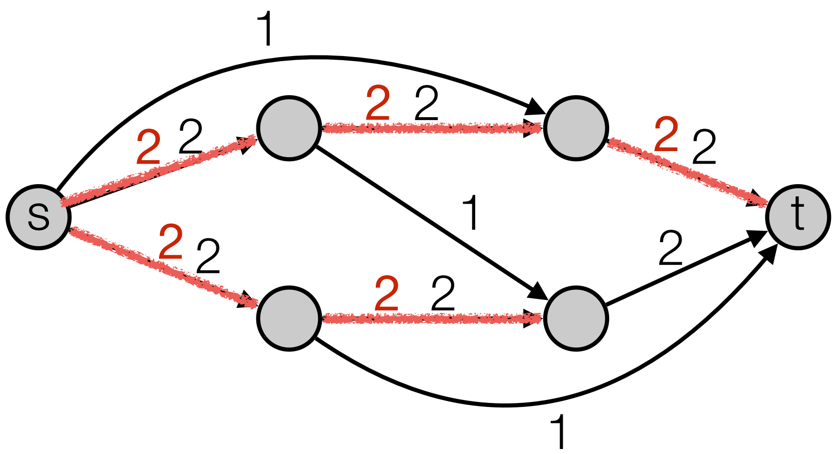

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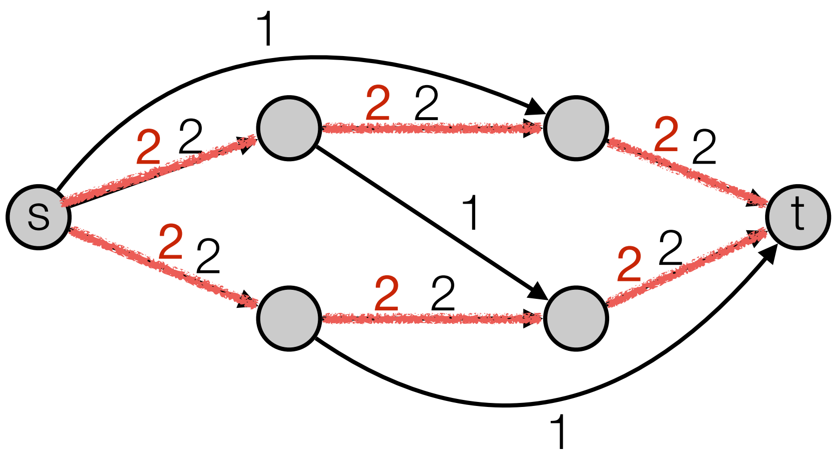

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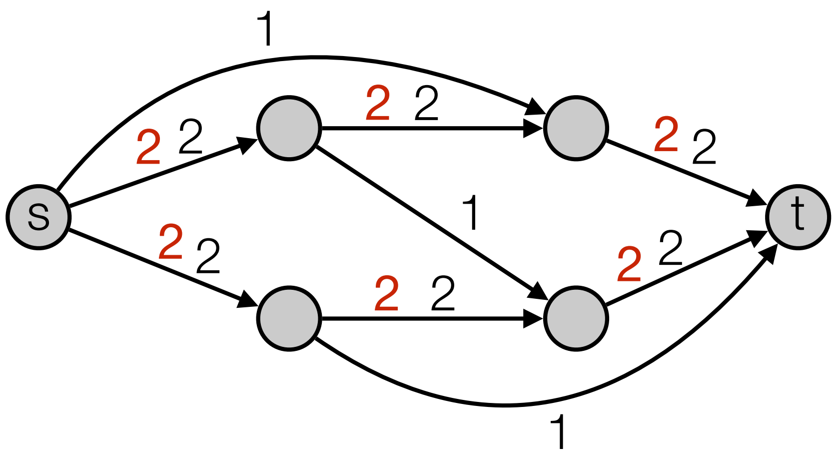

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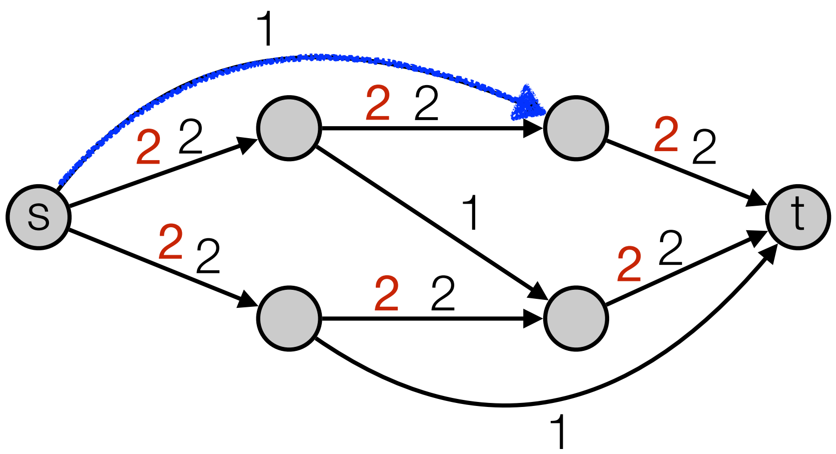

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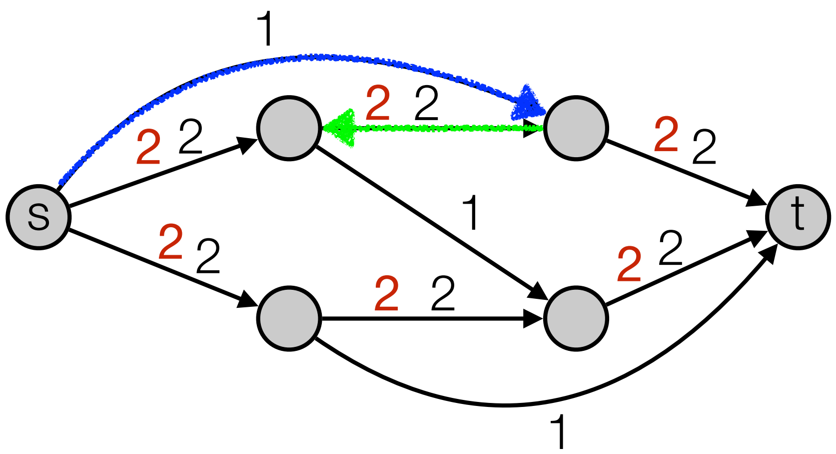

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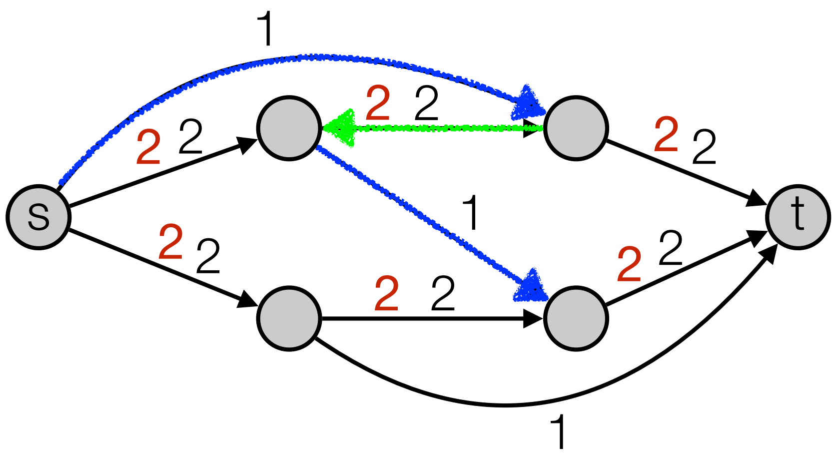

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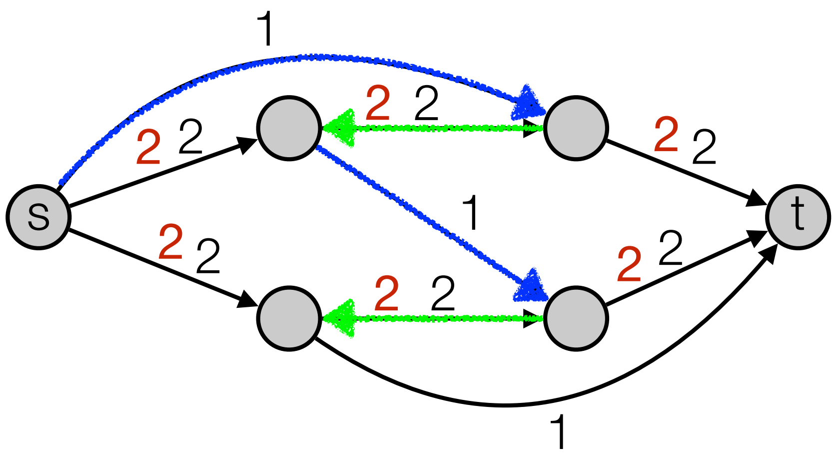

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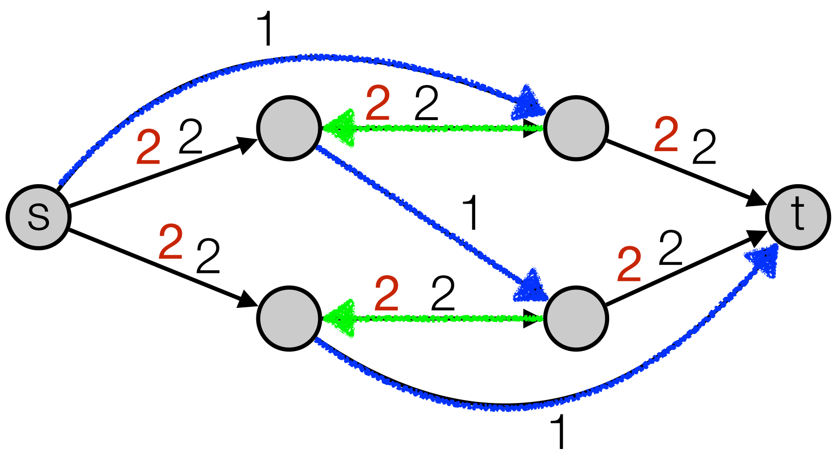

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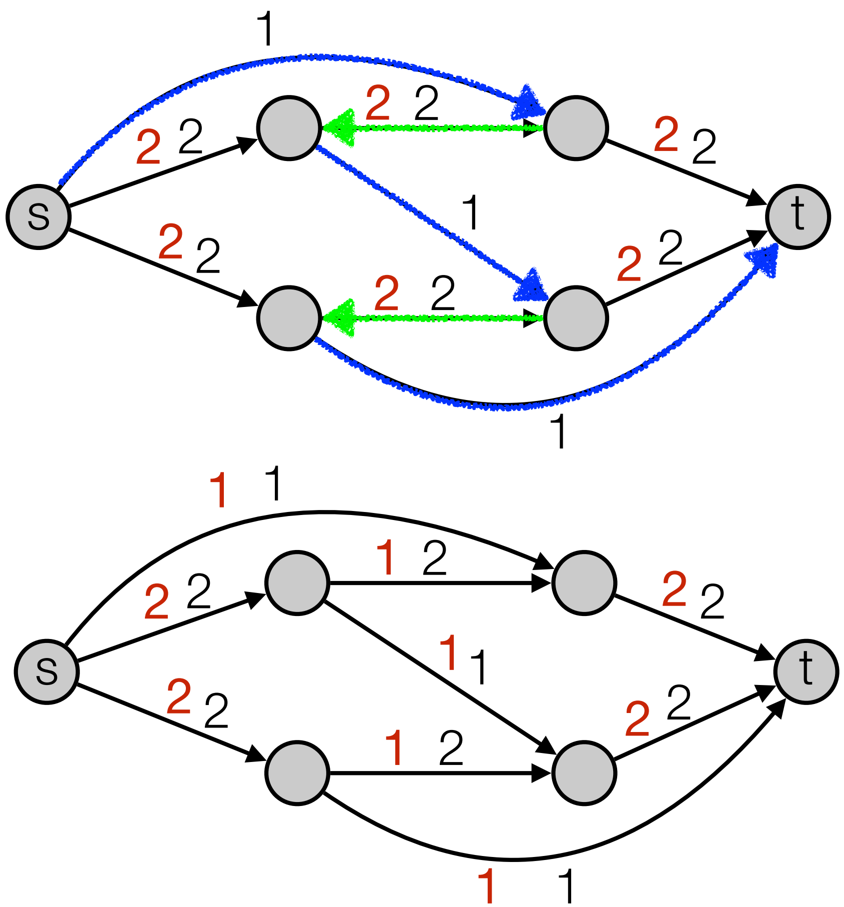

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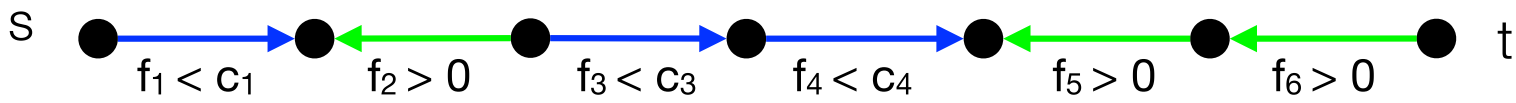

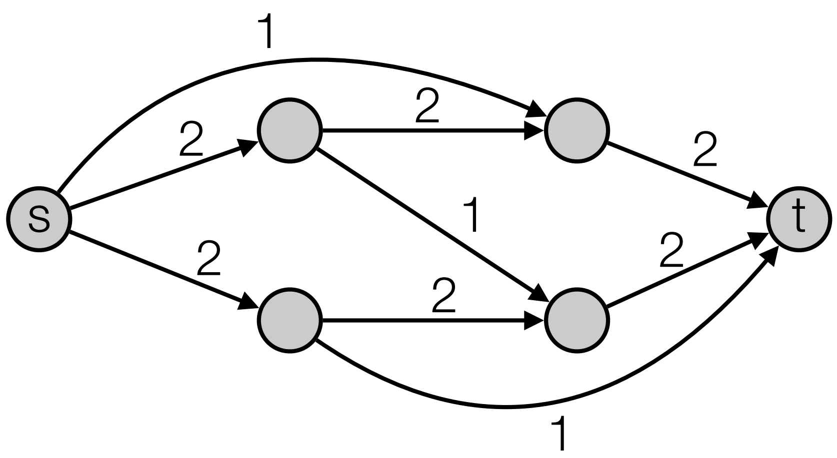

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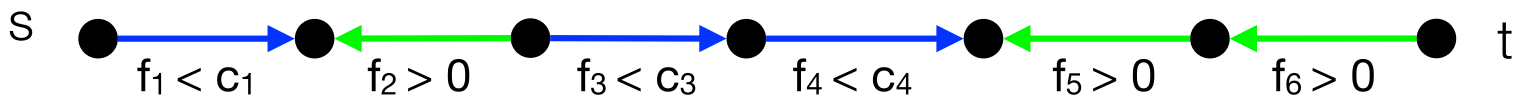

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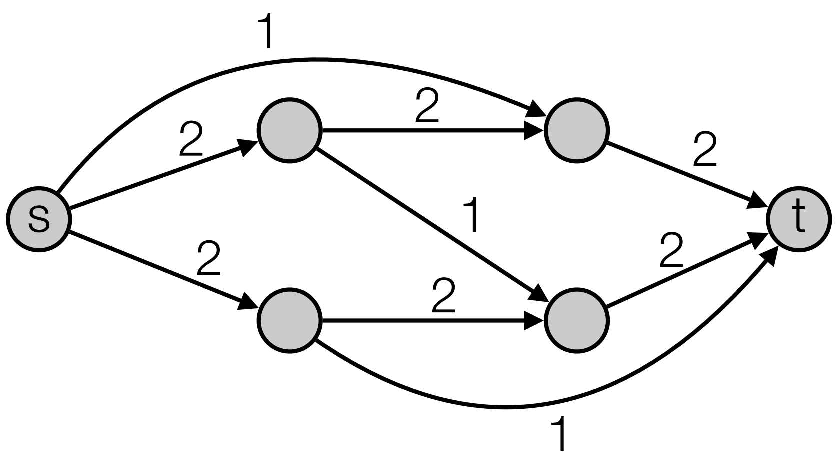

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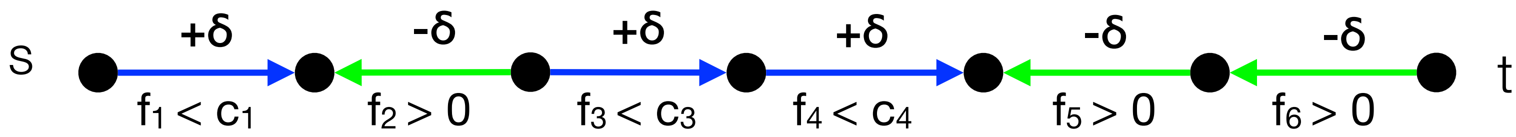

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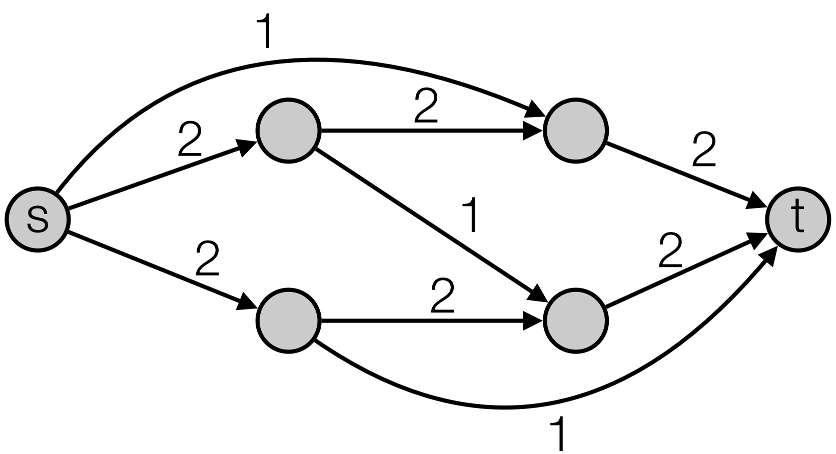

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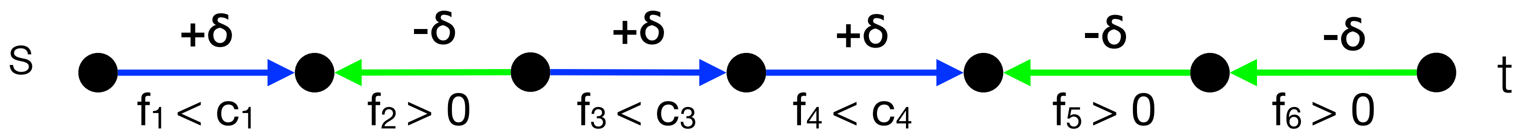

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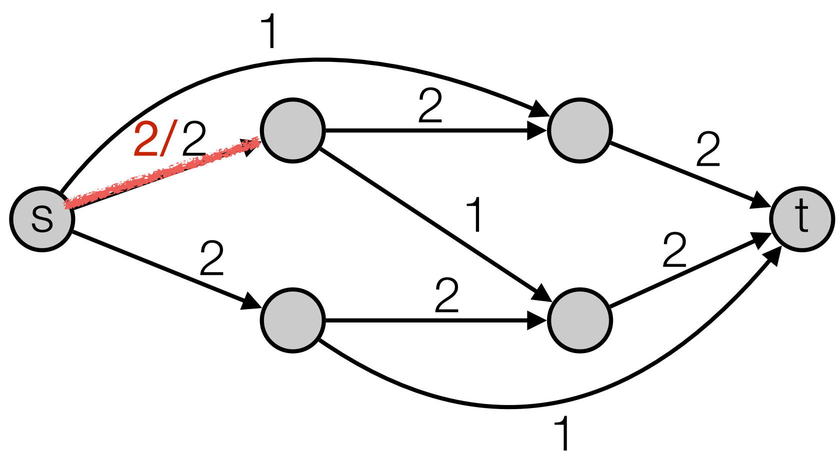

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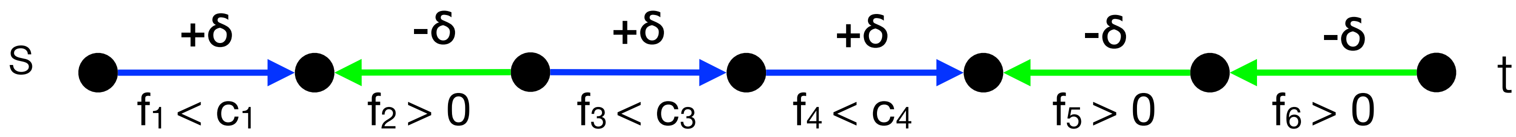

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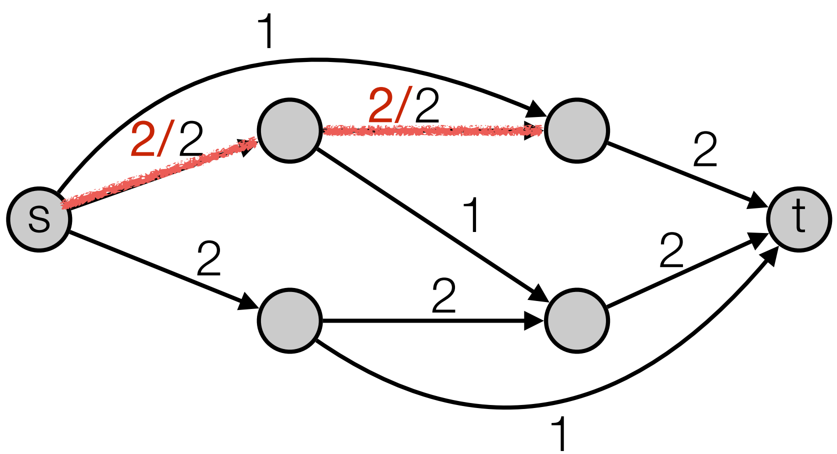

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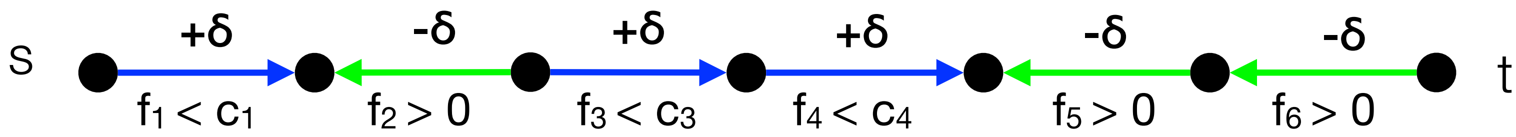

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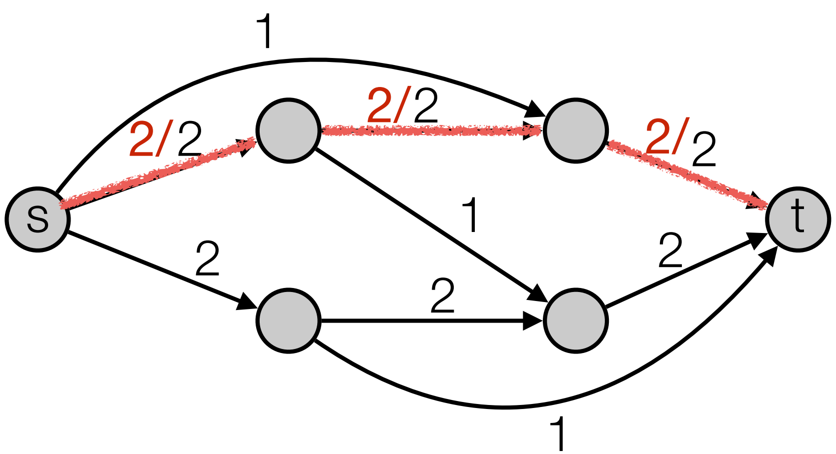

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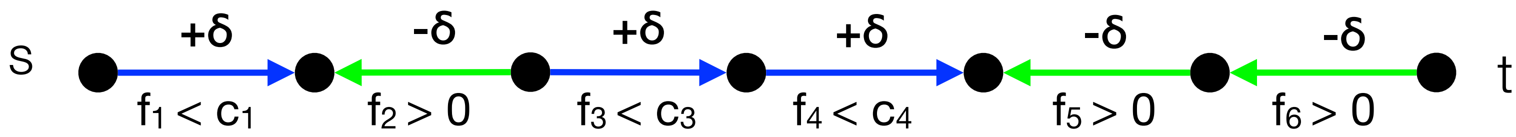

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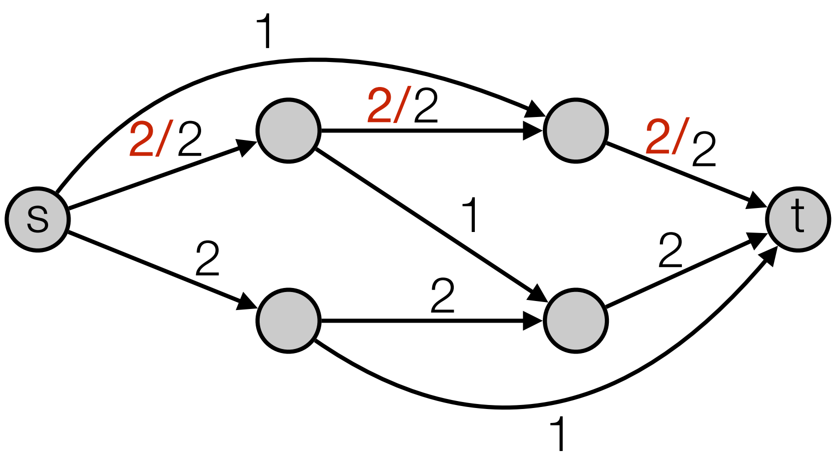

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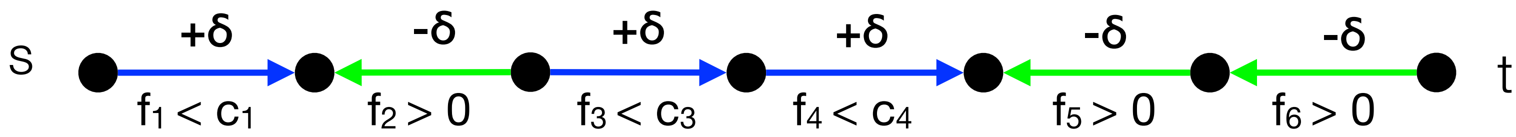

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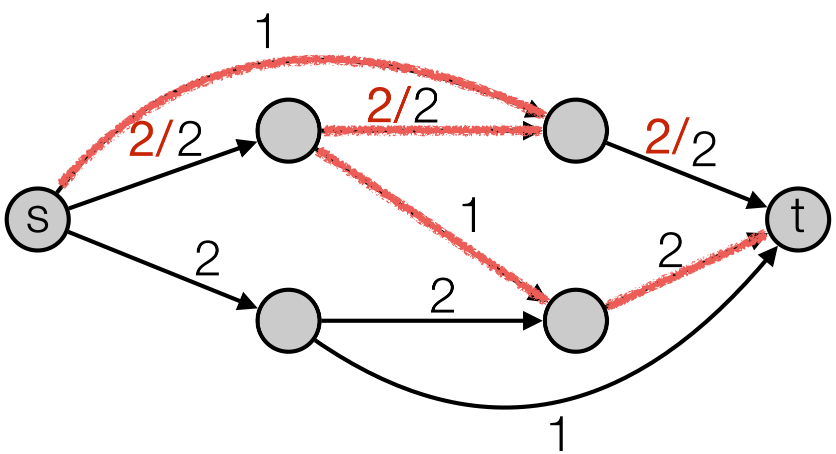

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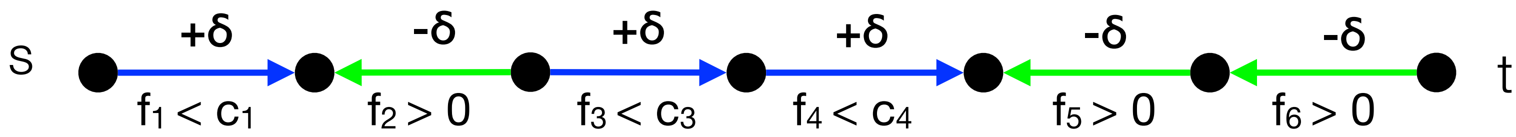

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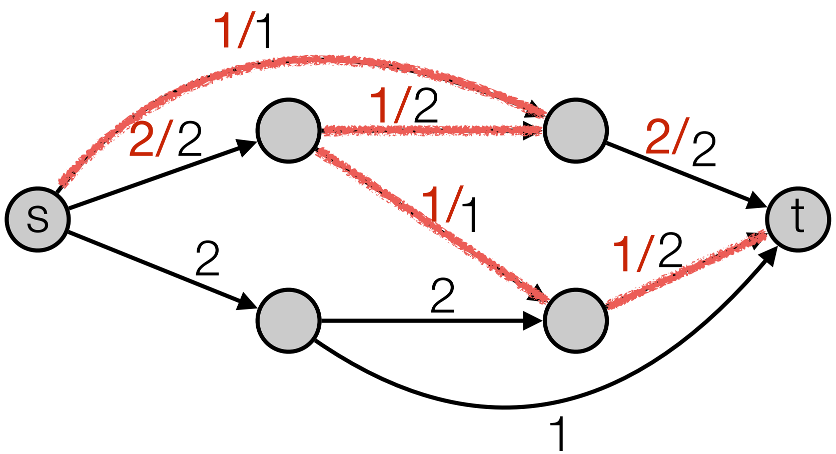

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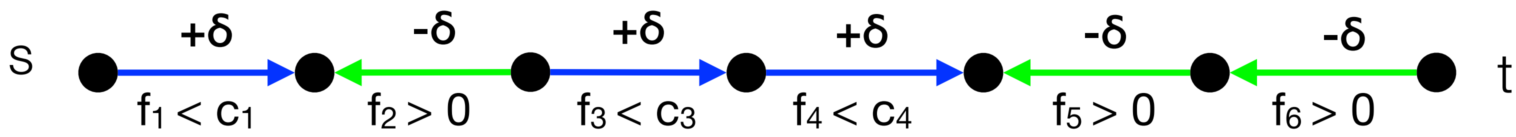

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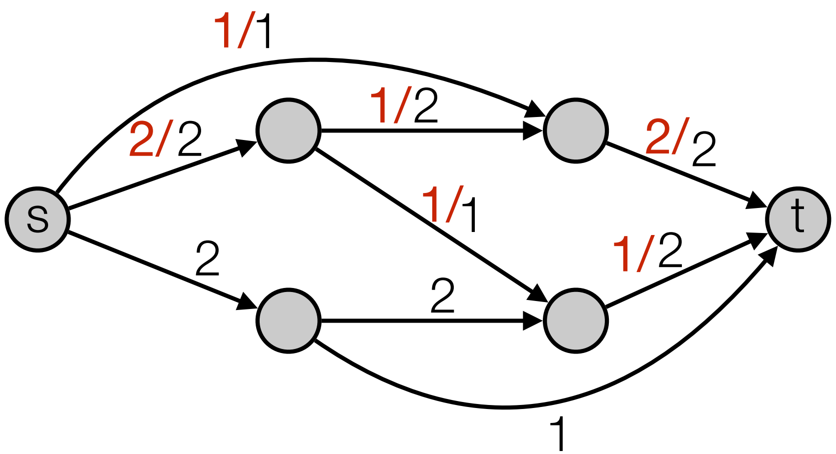

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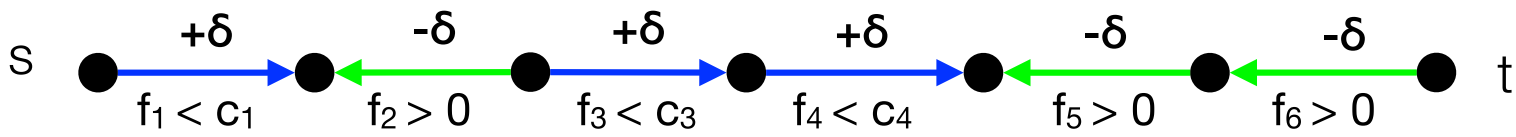

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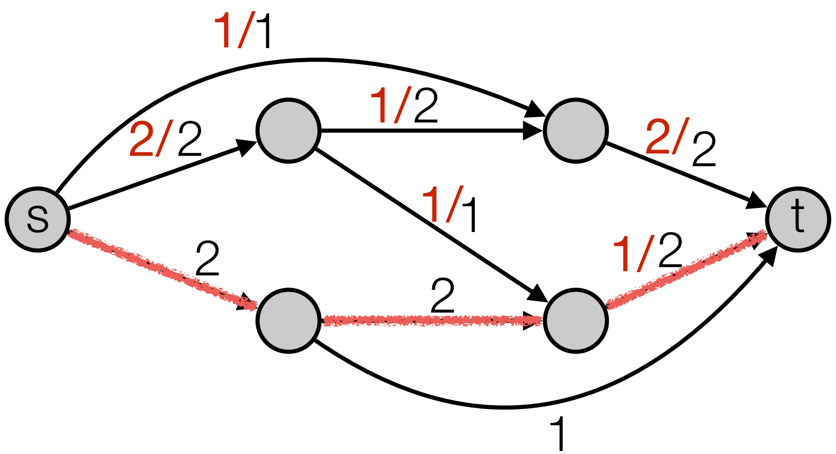

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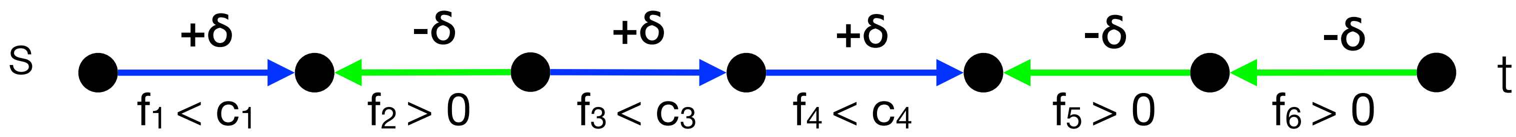

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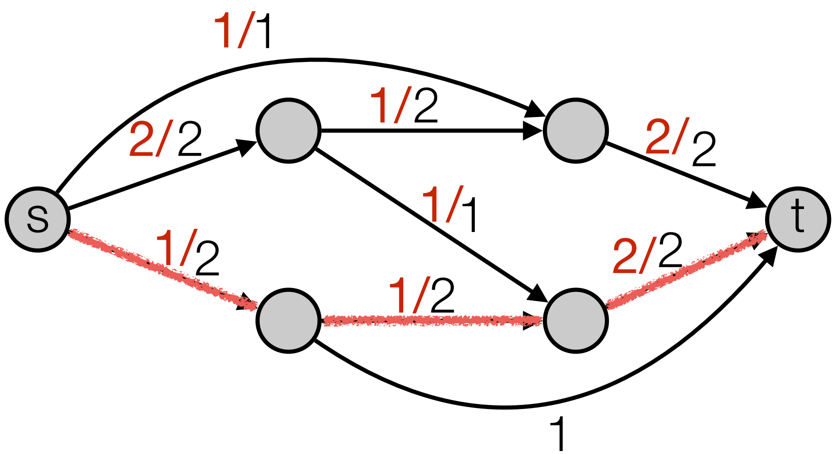

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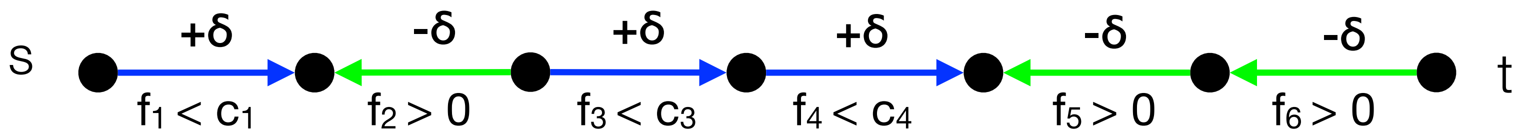

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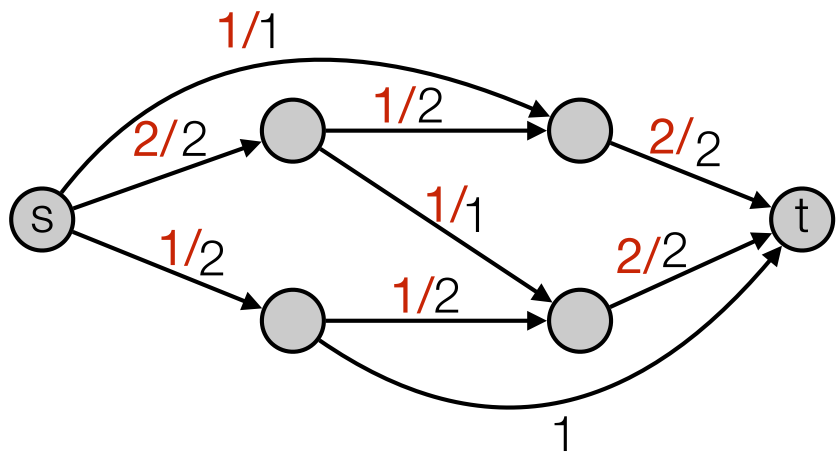

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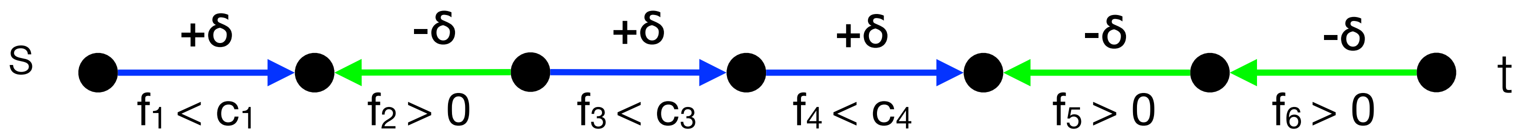

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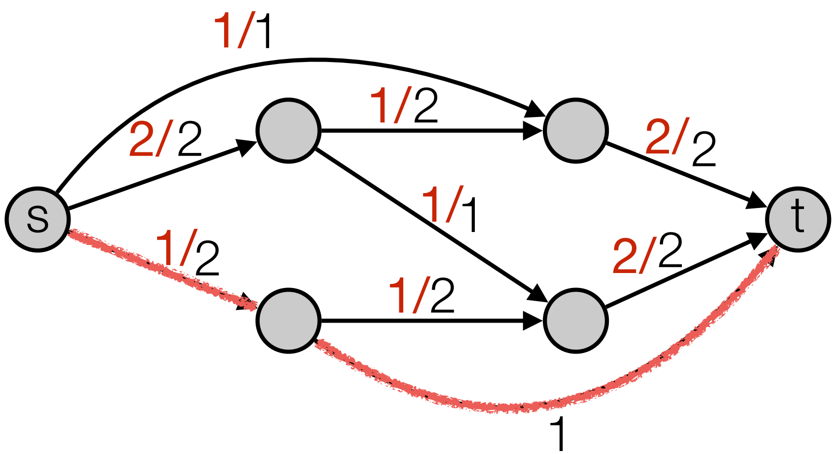

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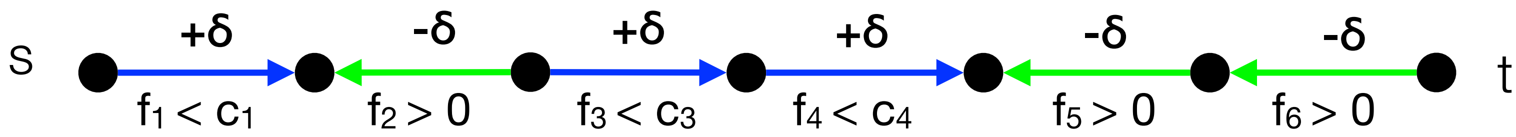

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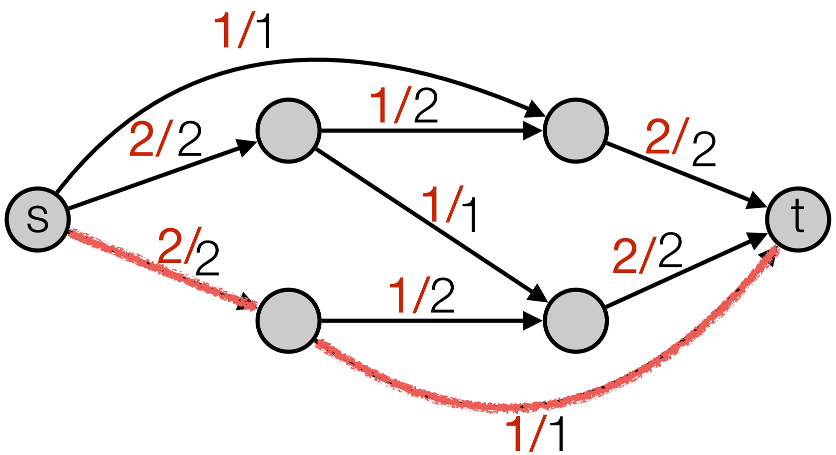

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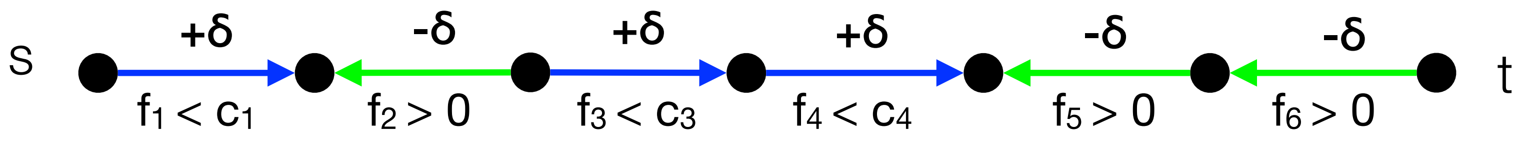

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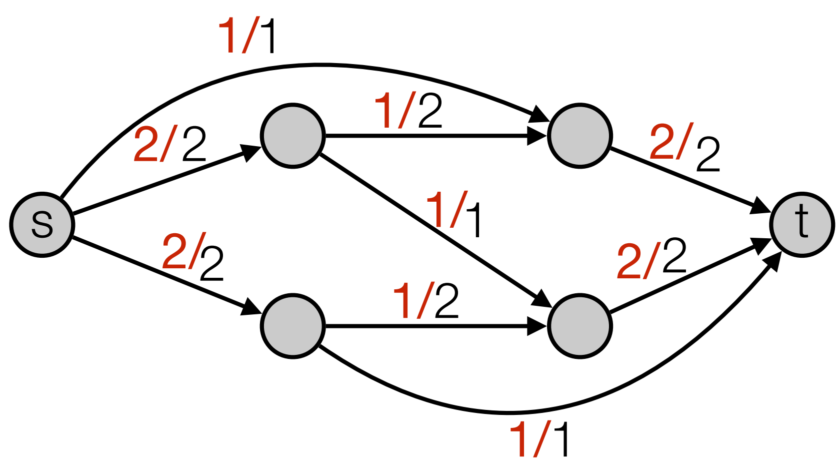

Augmenting Paths

- Augmenting path (definition different than in CLRS): s-t path where
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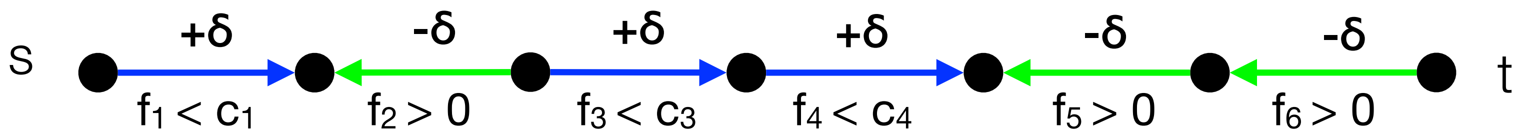

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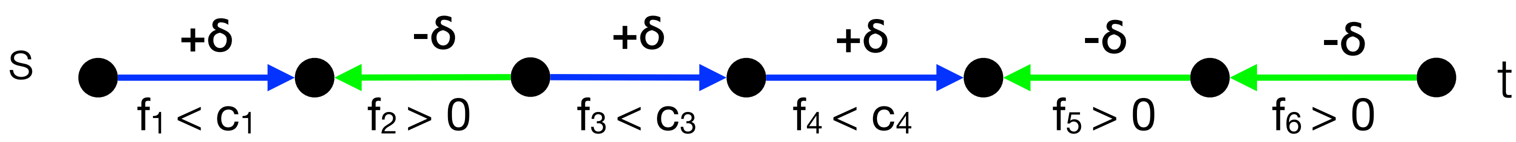

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- Ford-Fulkerson:
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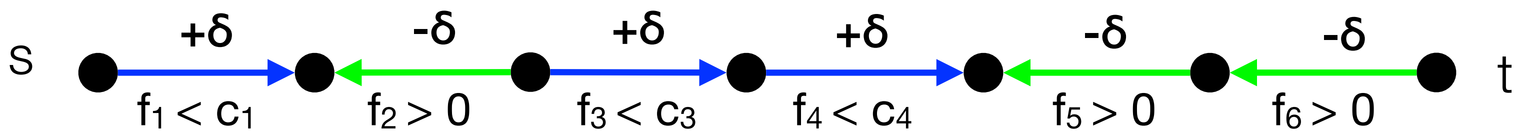

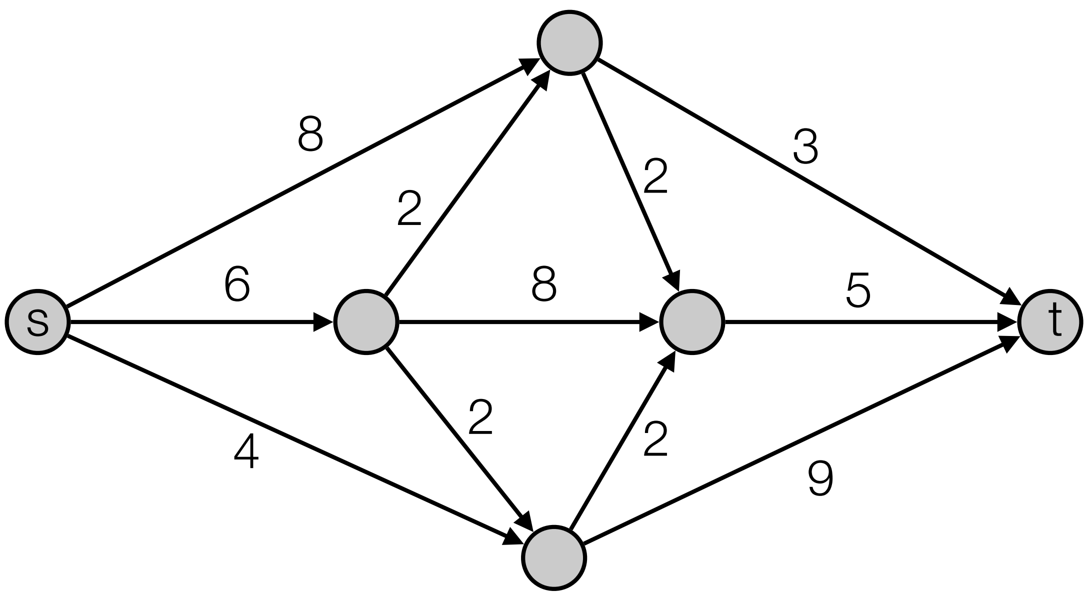

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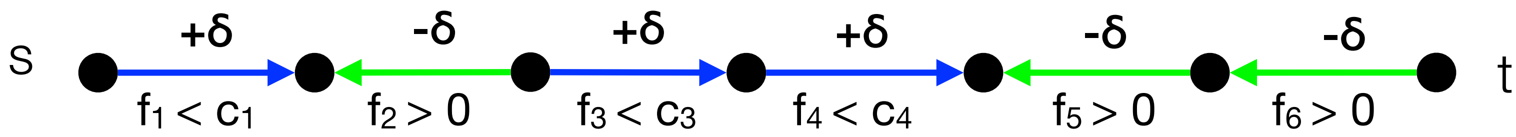

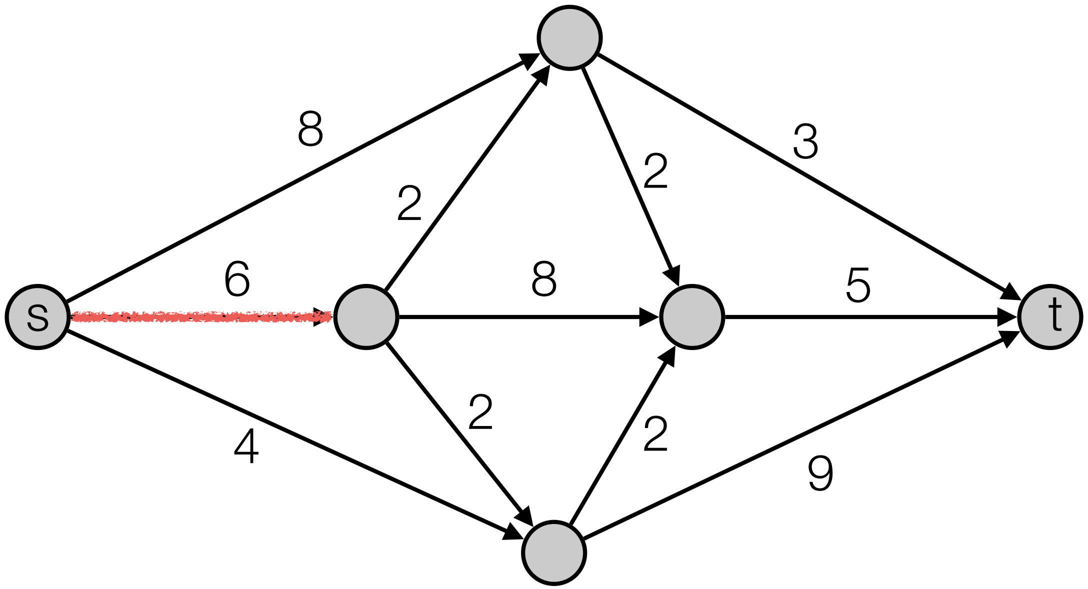

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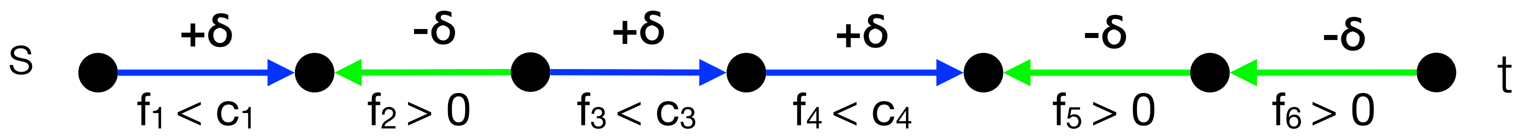

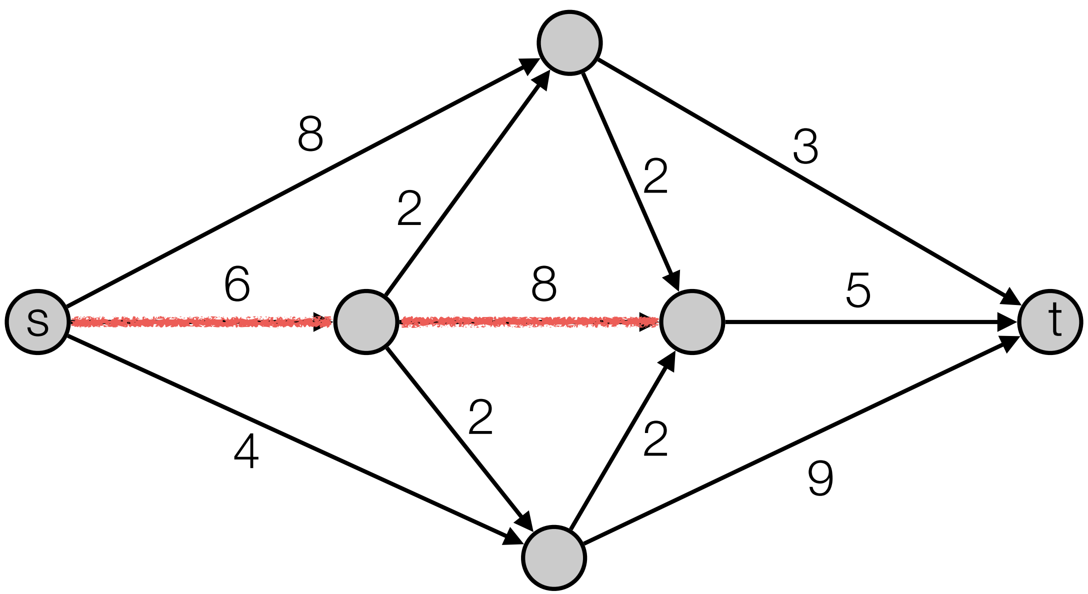

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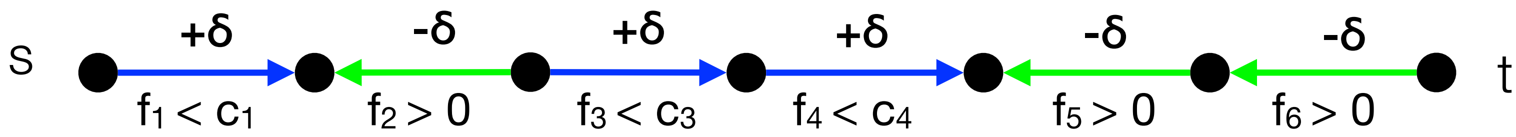

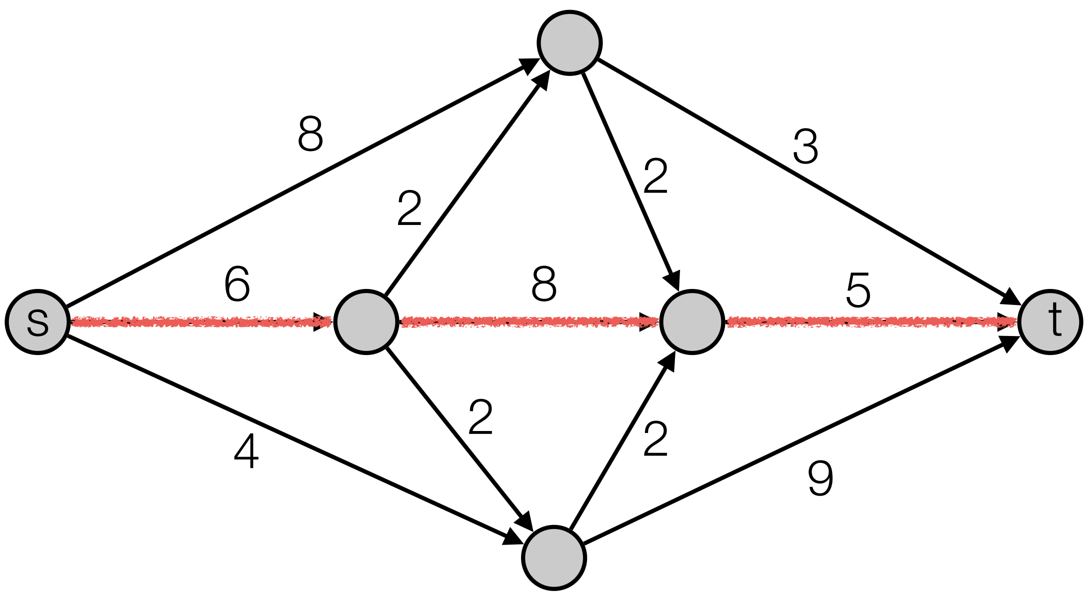

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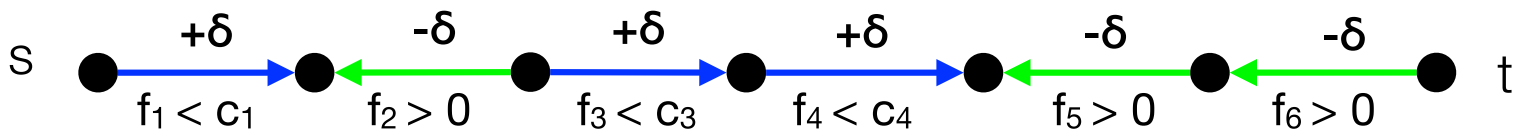

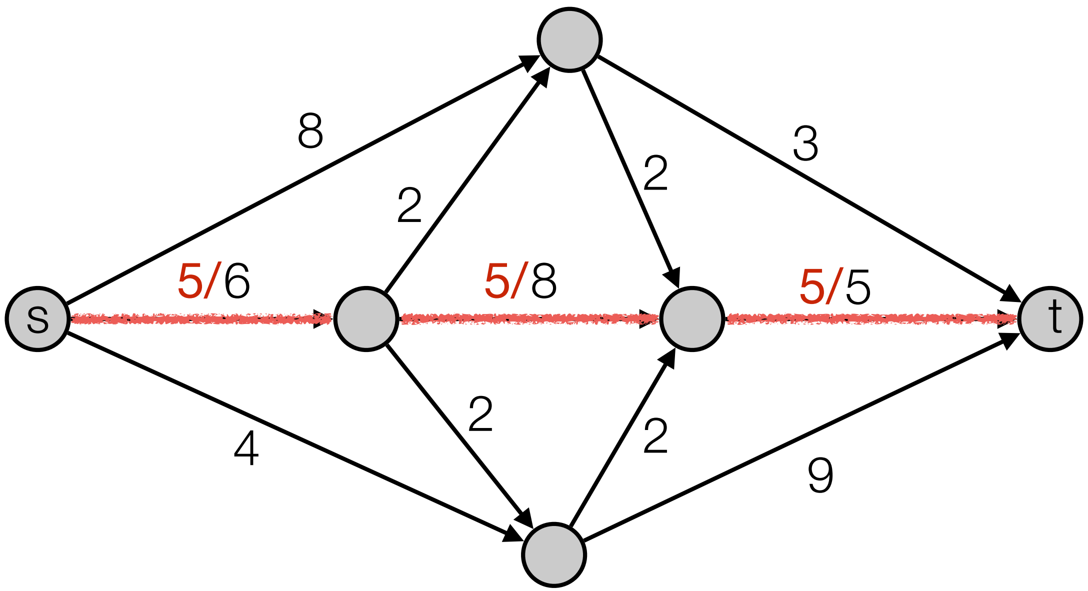

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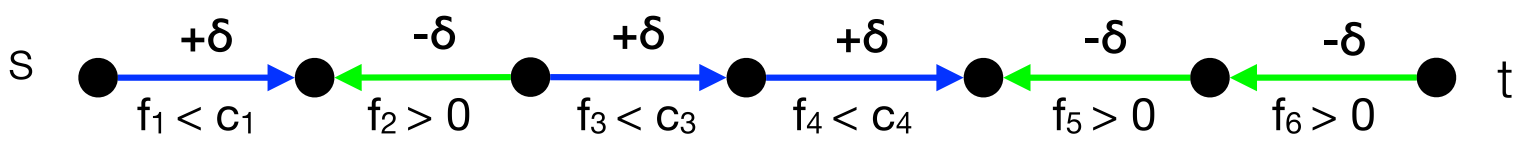

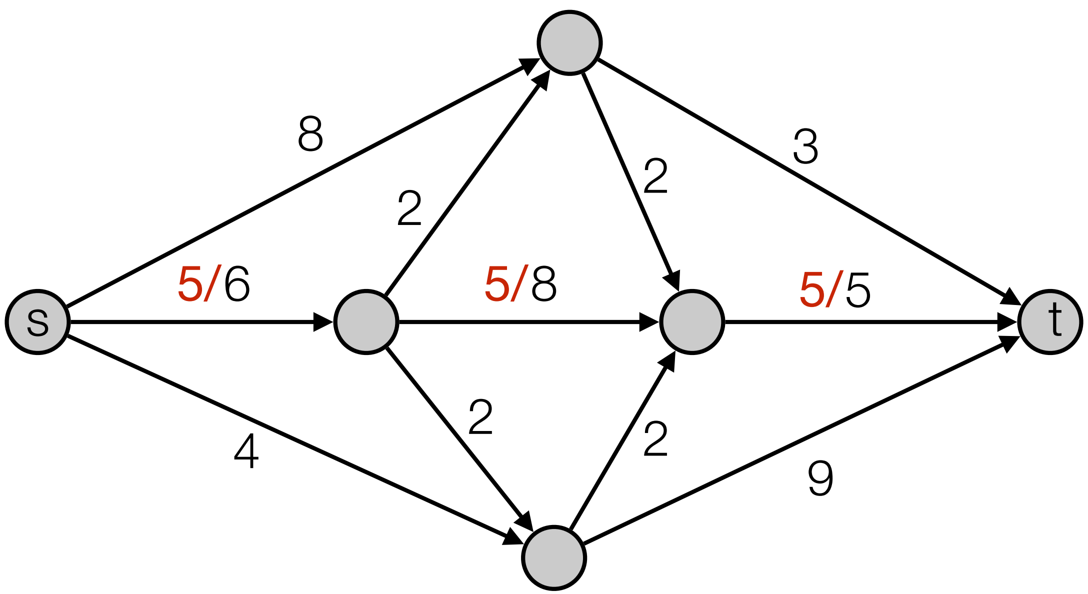

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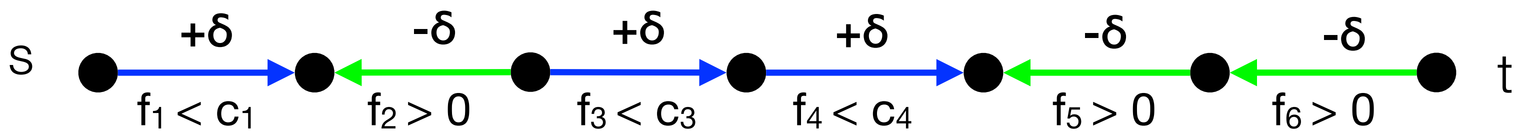

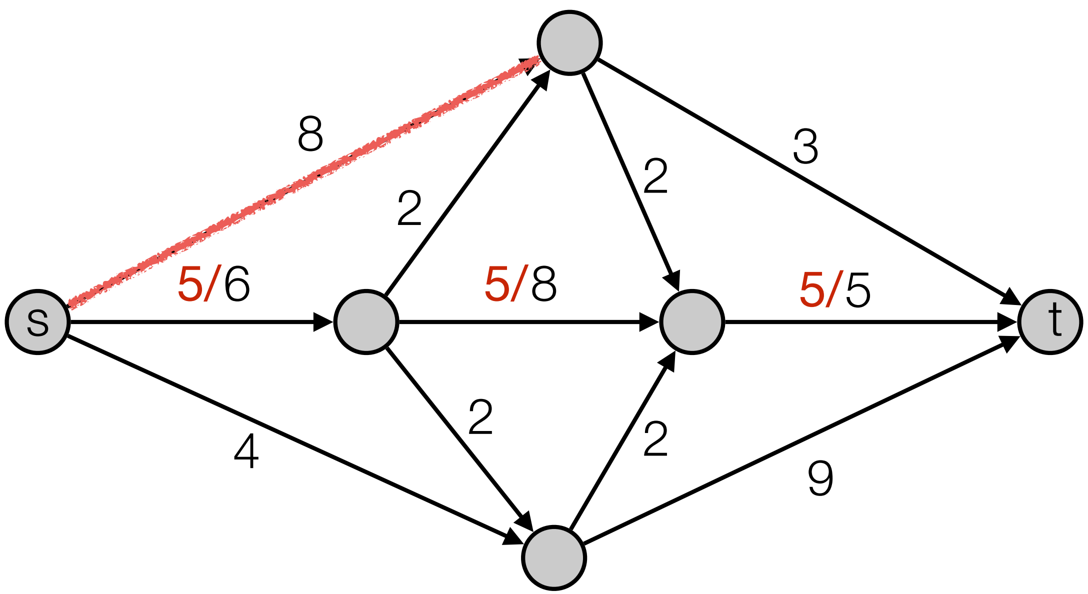

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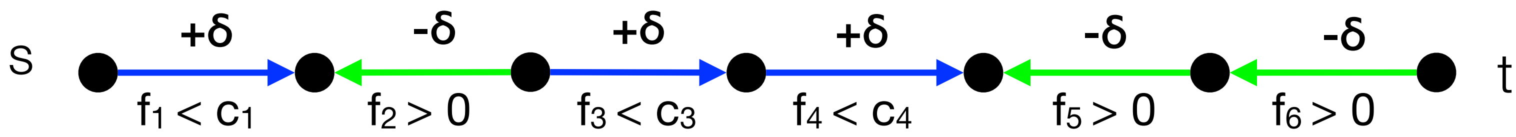

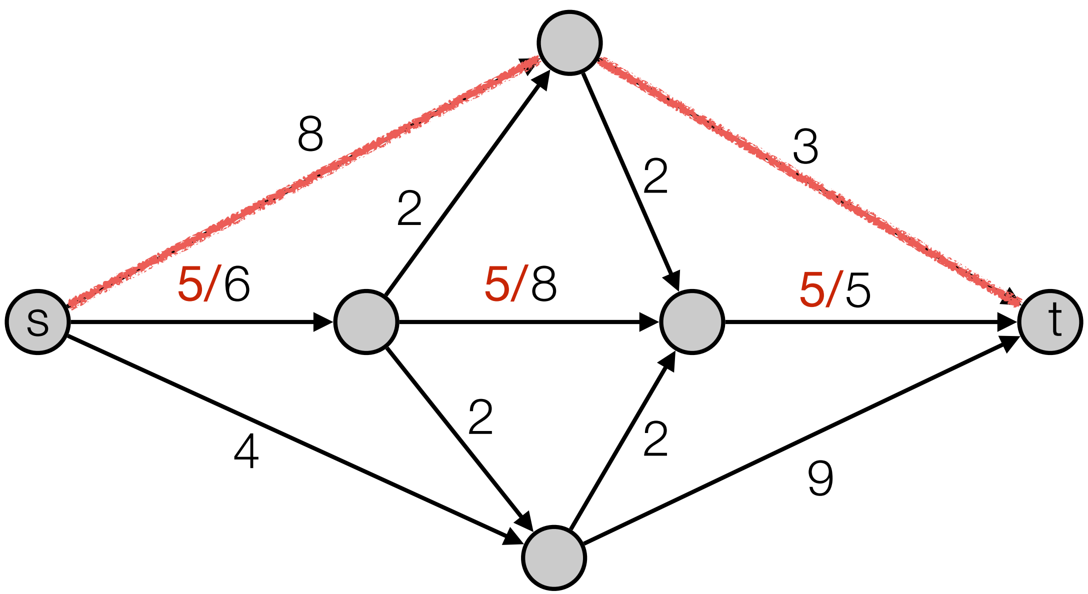

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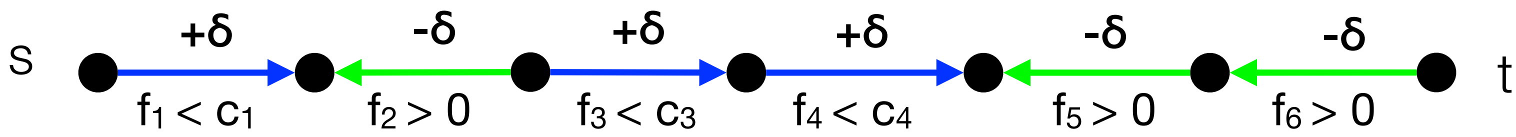

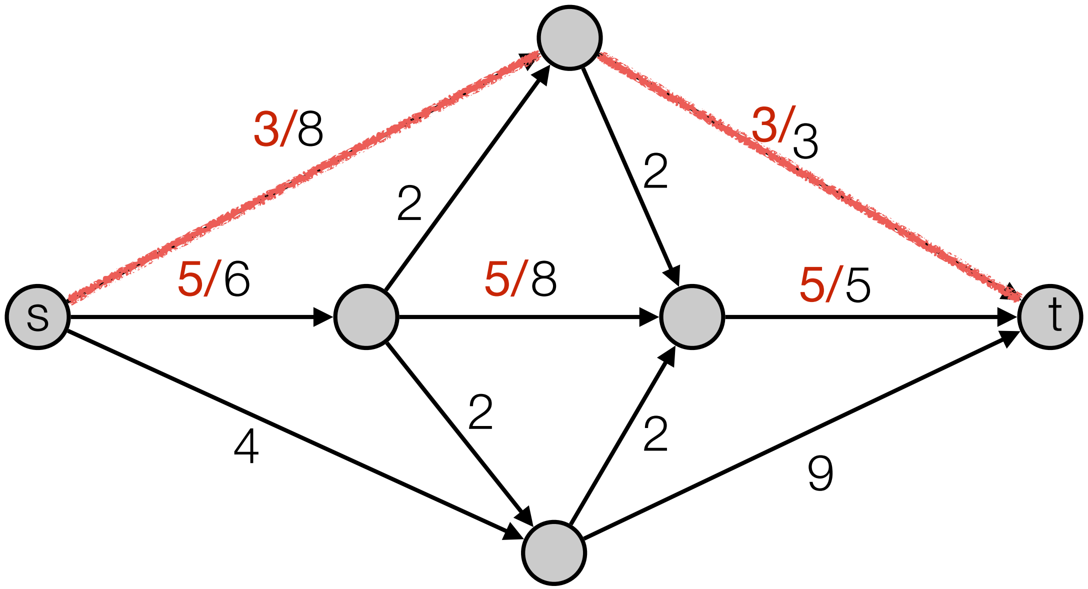

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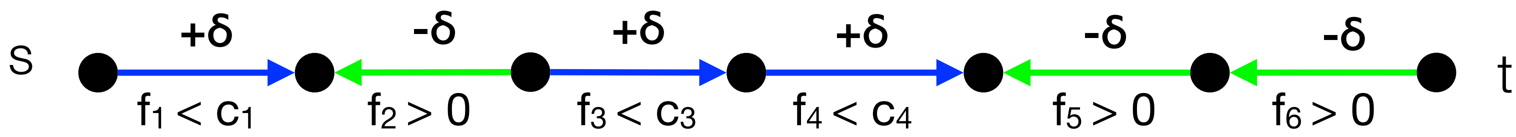

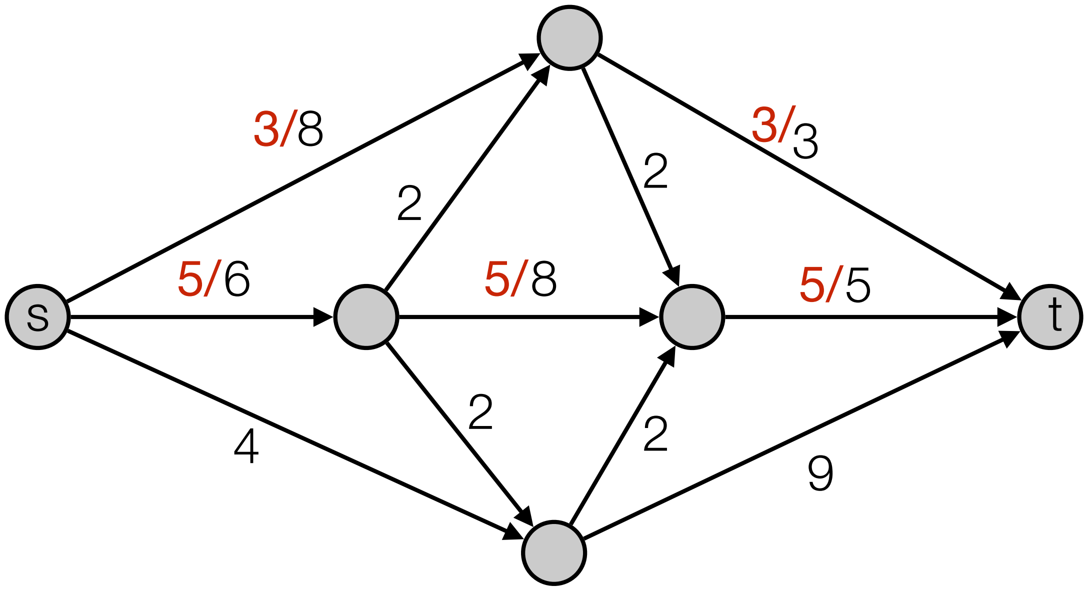

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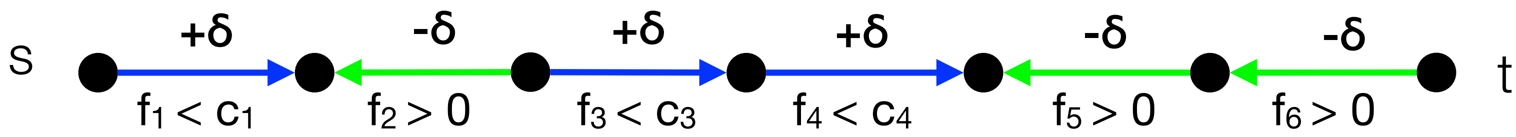

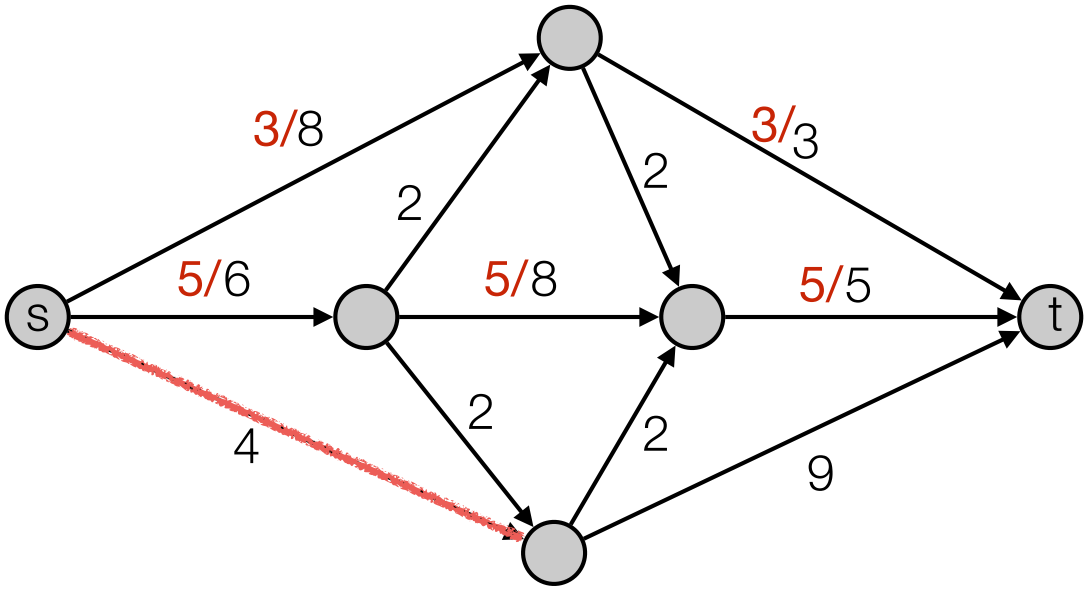

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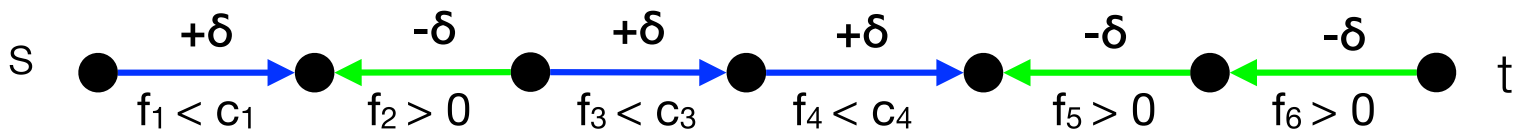

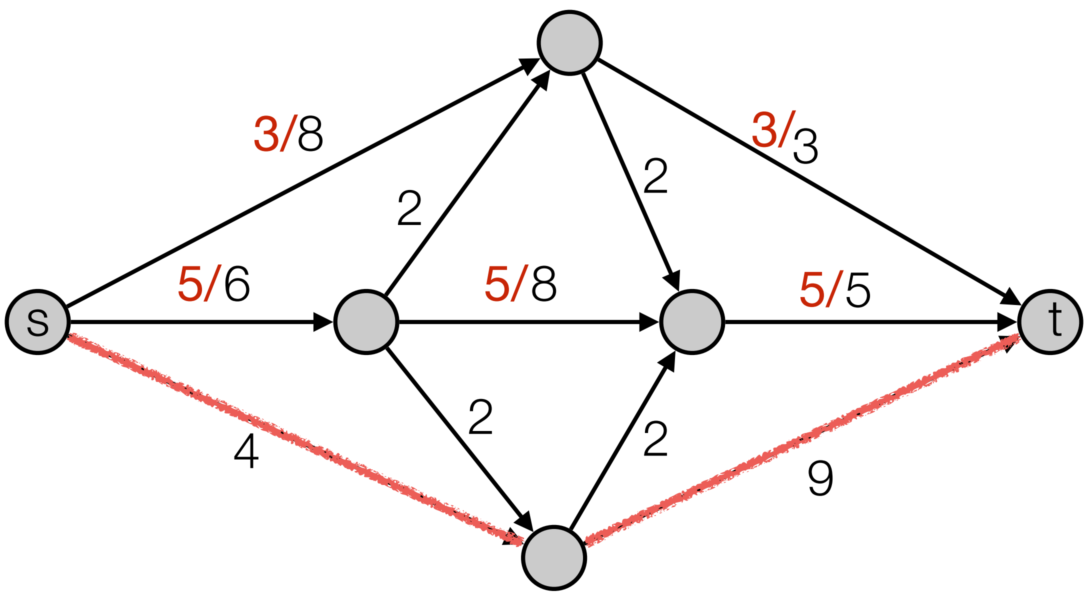

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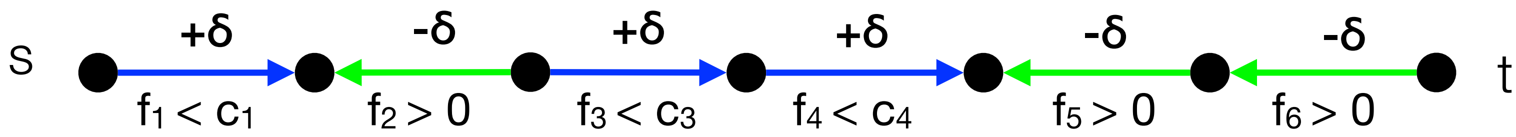

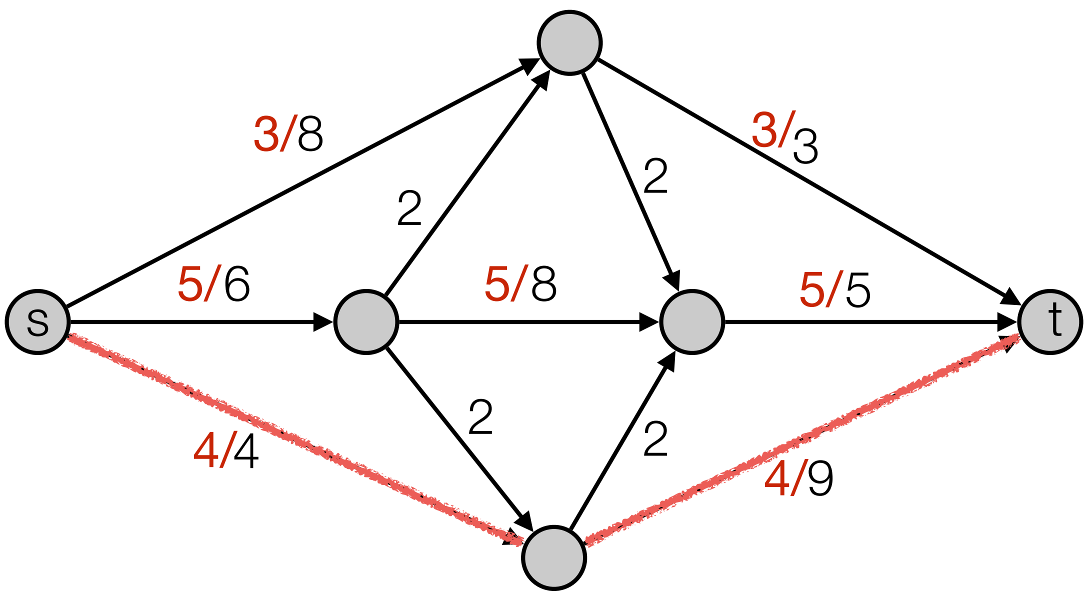

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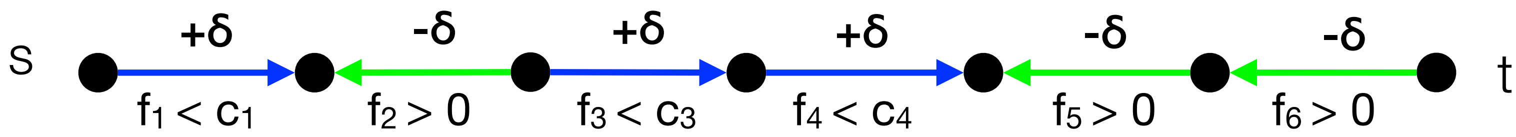

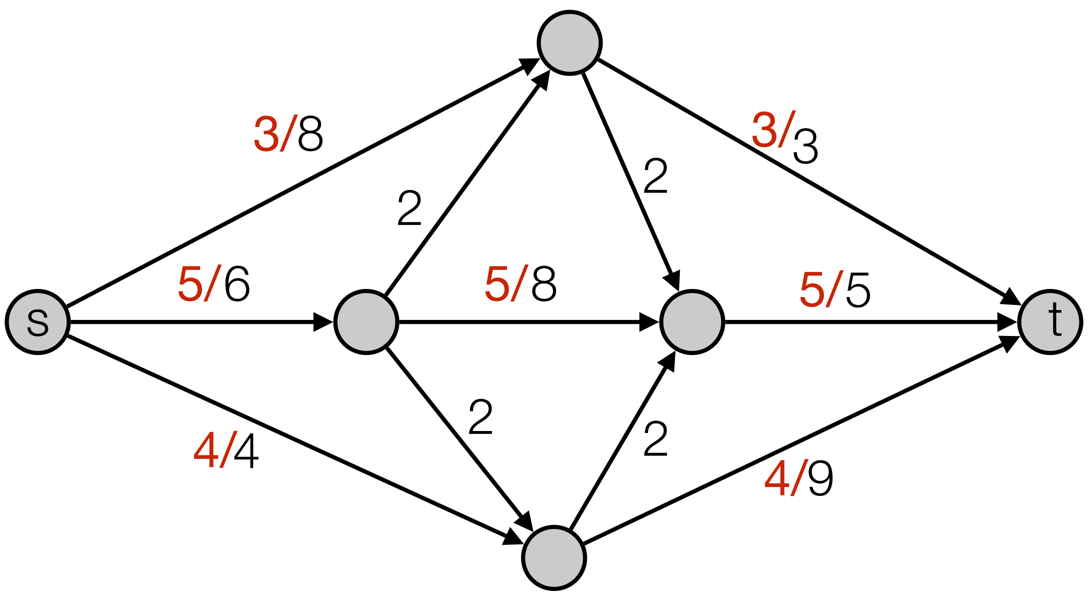

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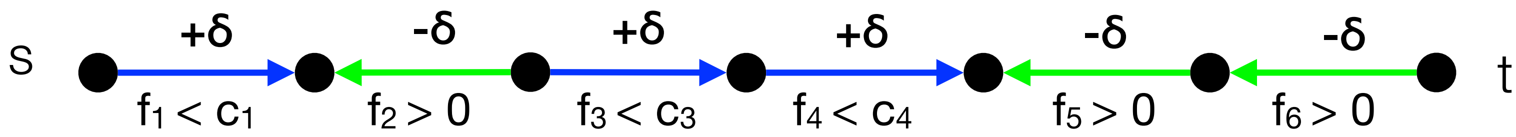

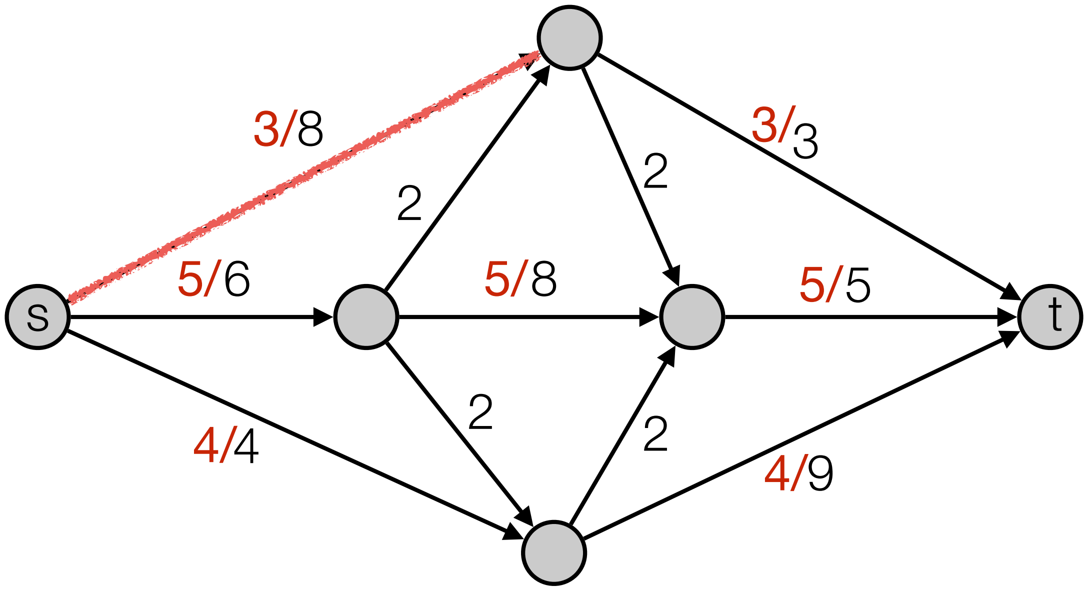

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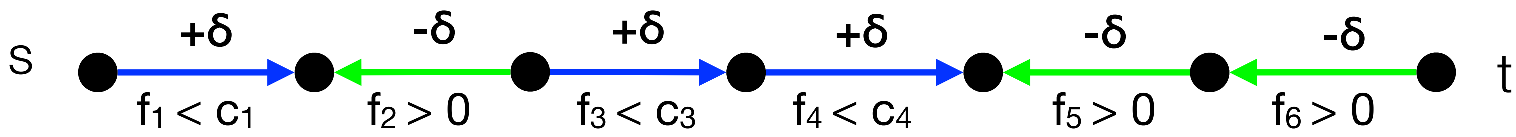

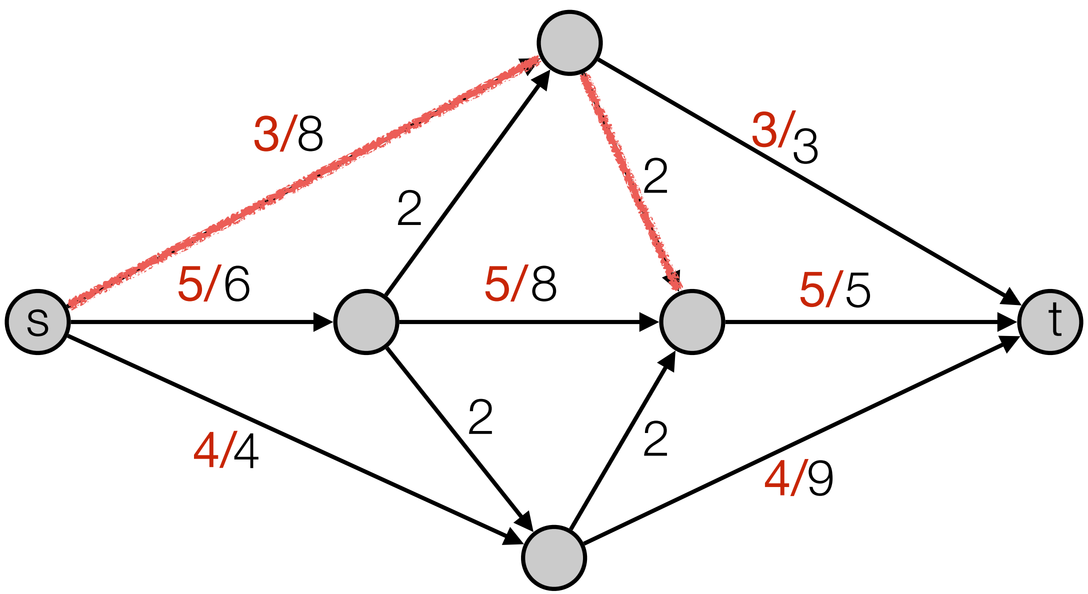

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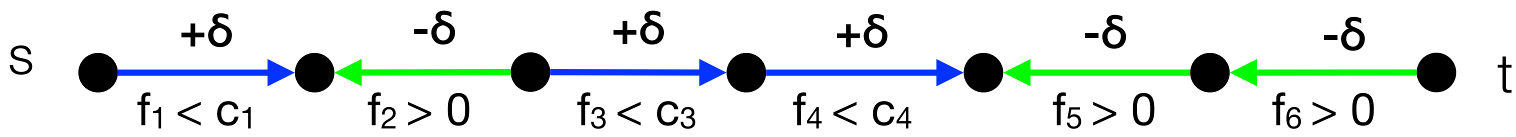

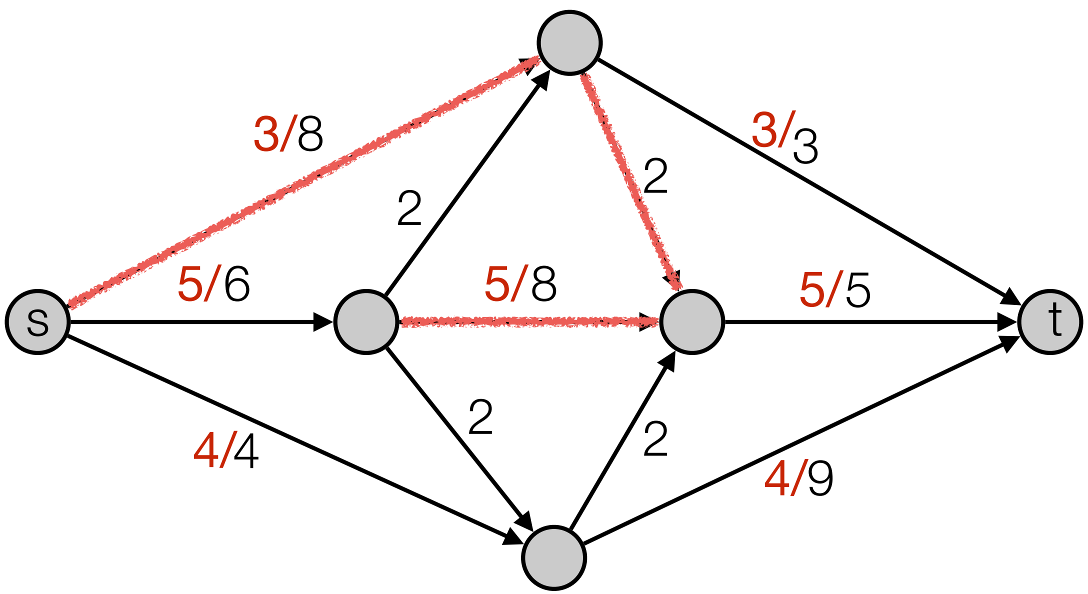

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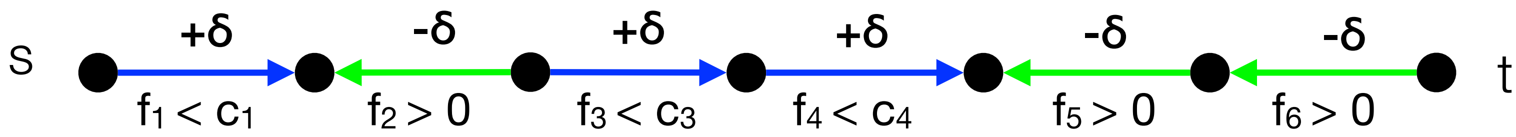

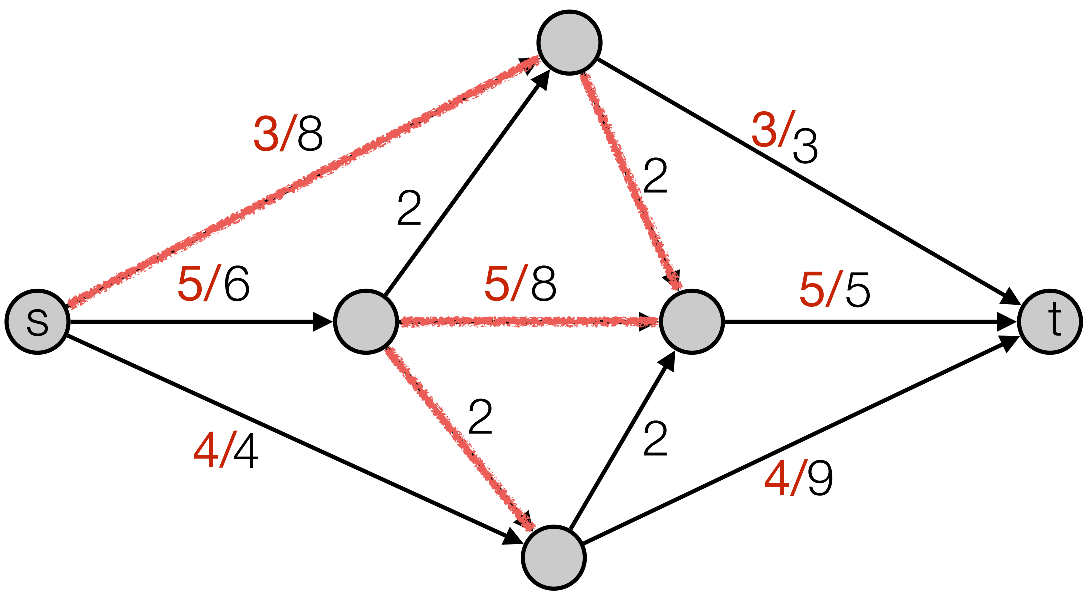

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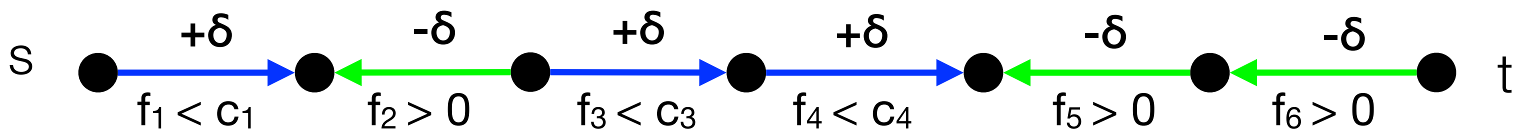

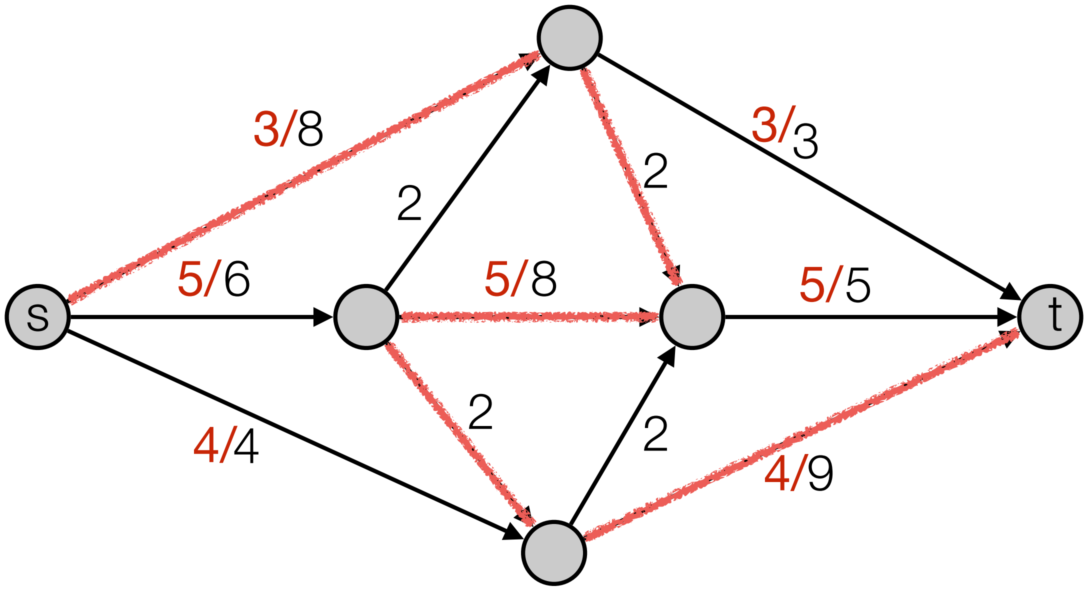

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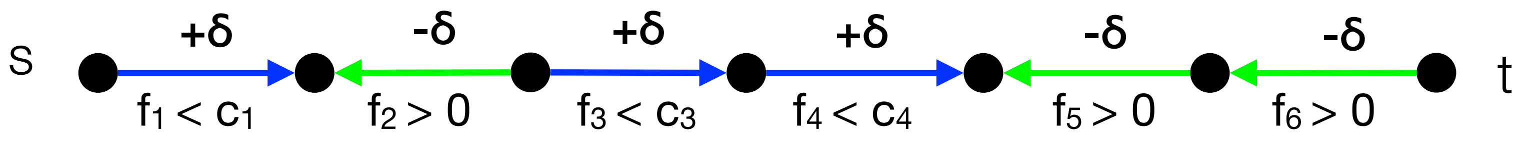

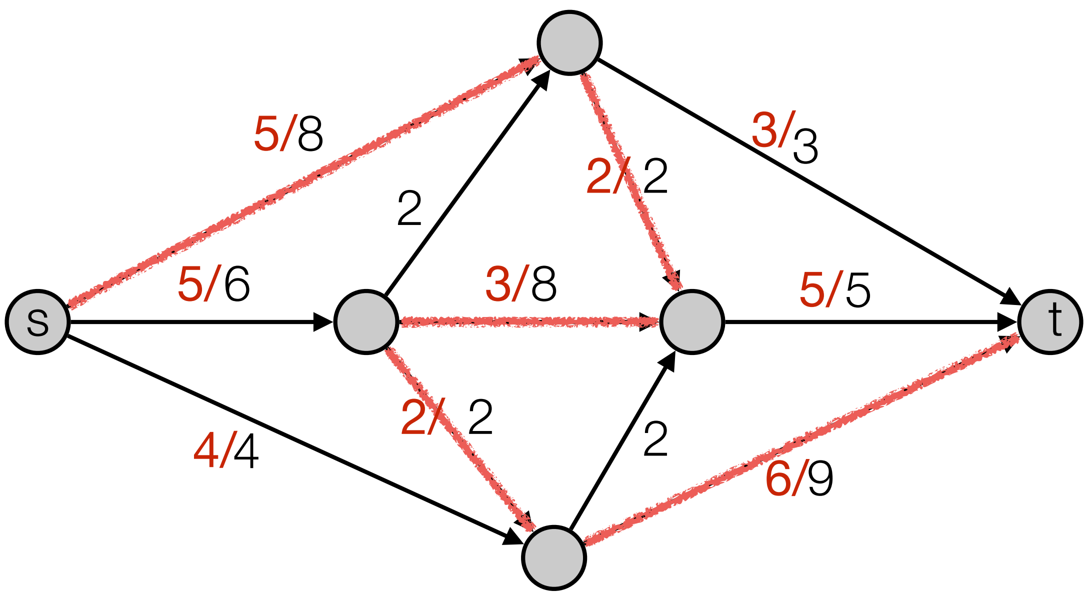

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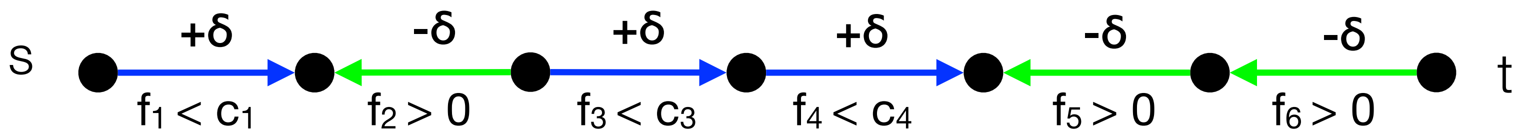

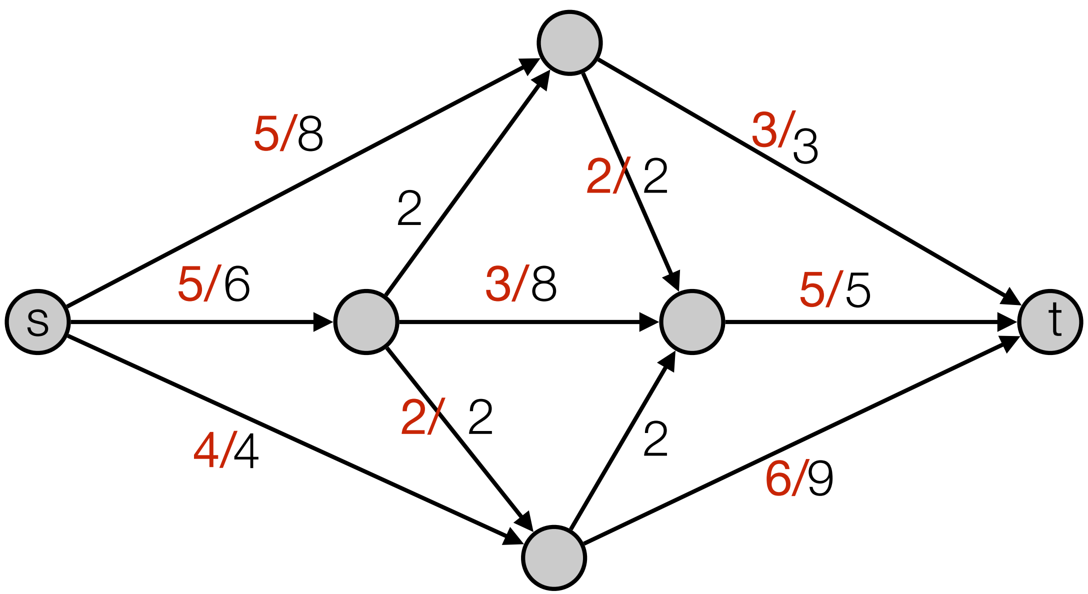

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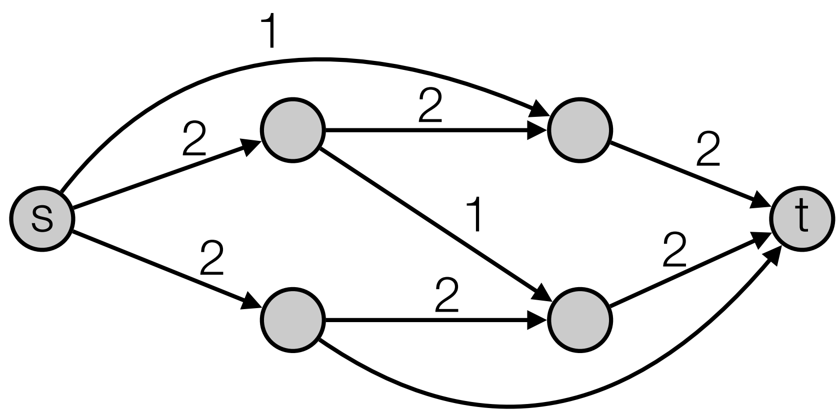
- Total running time = O(|f*| m).

Residual networks



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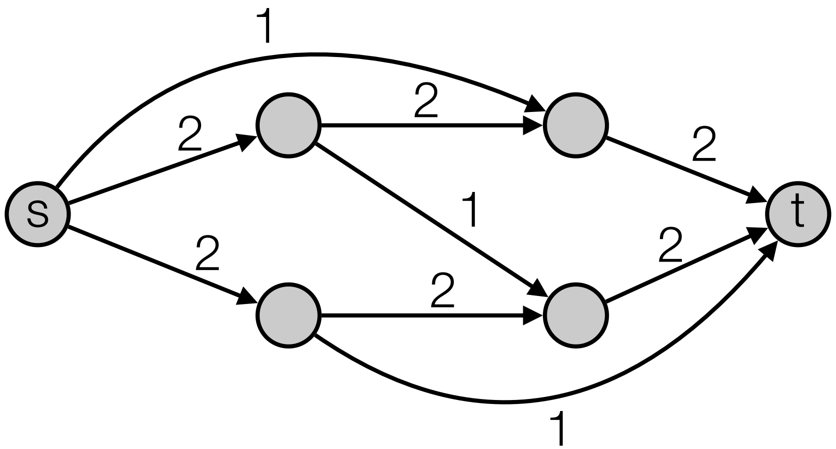

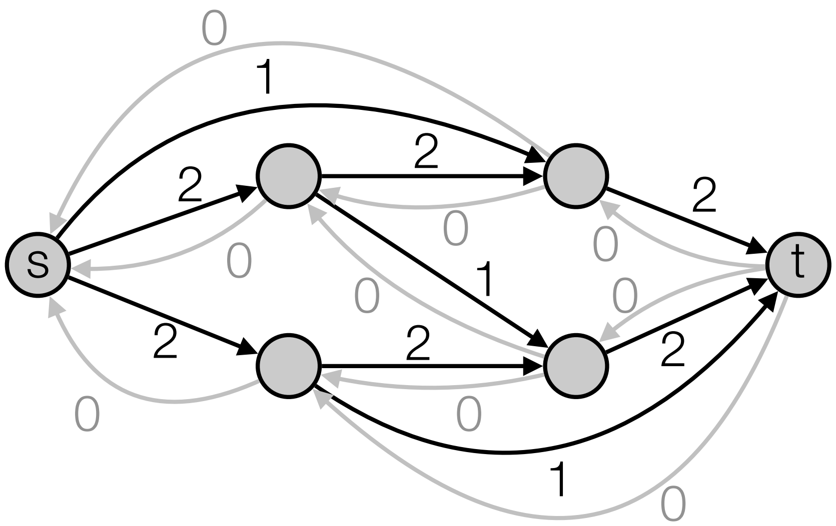

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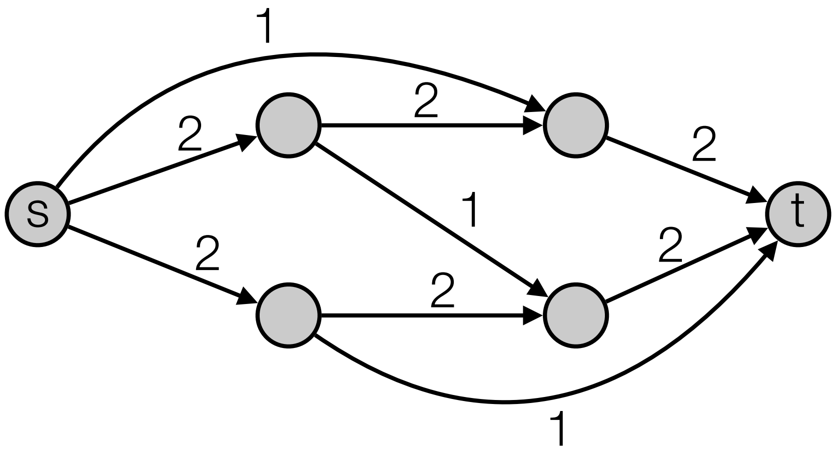

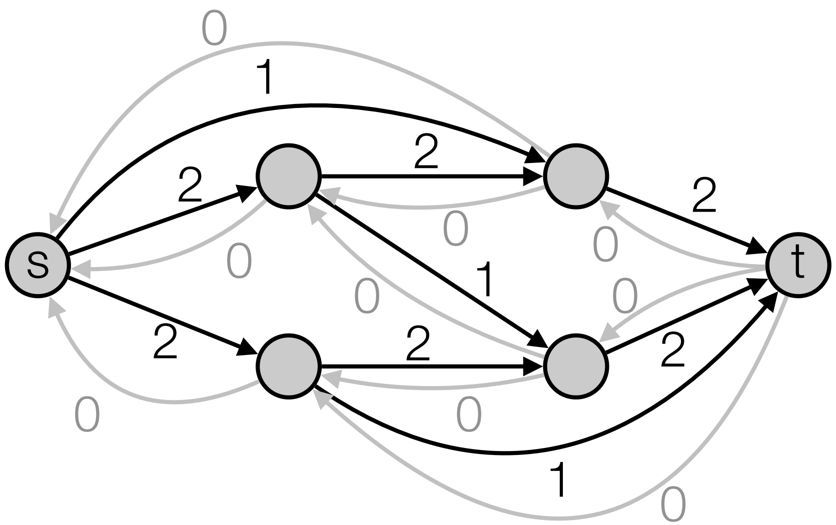

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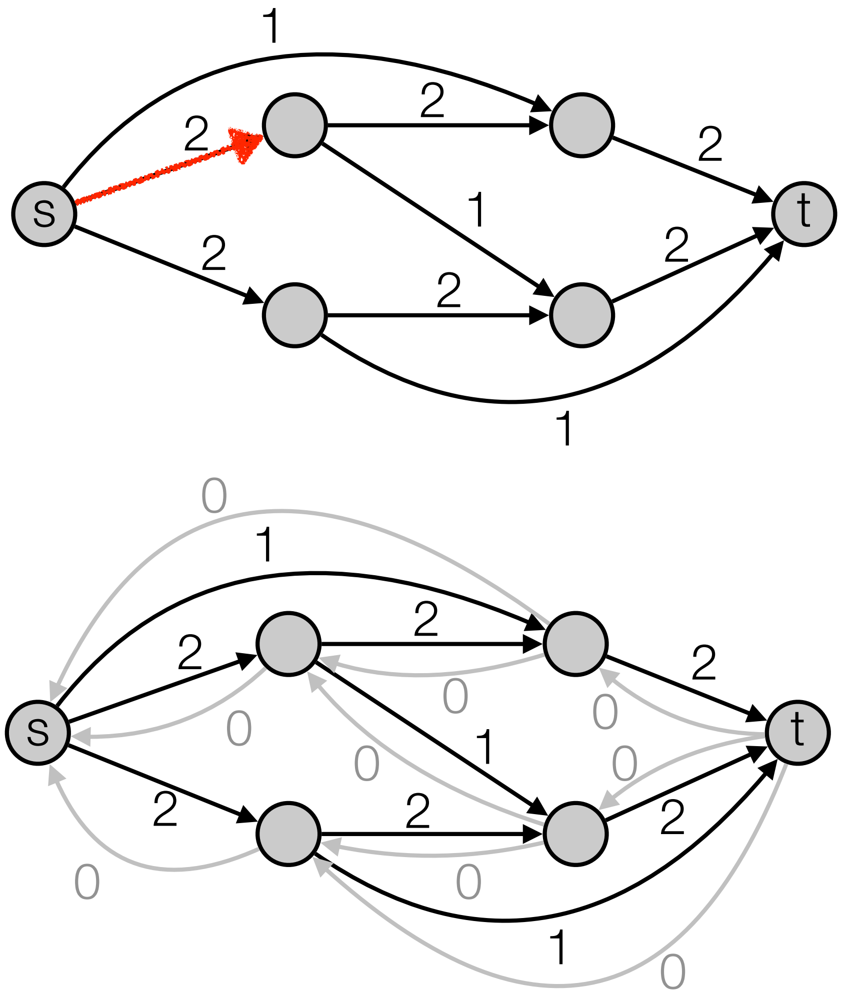

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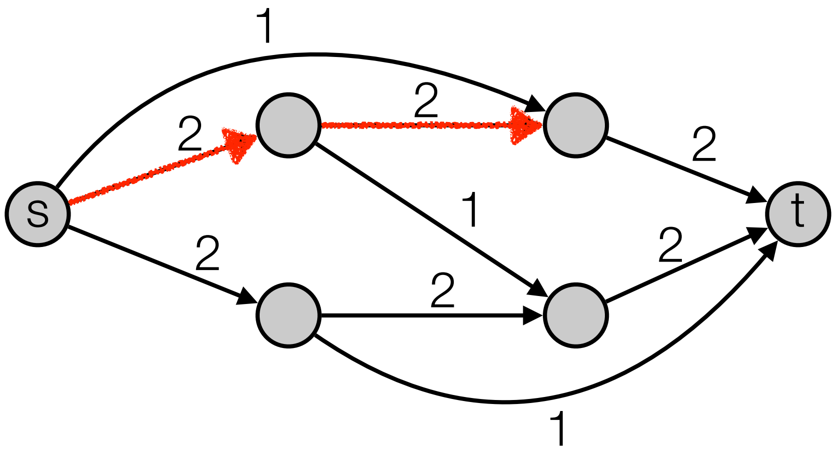

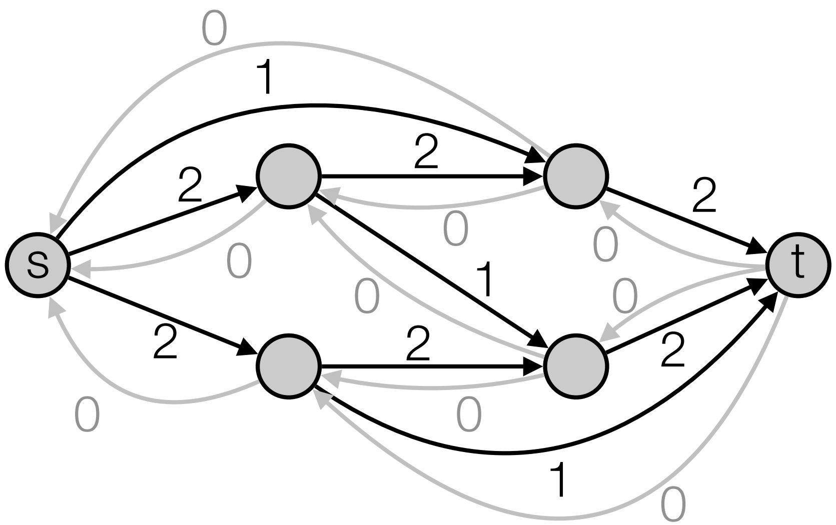

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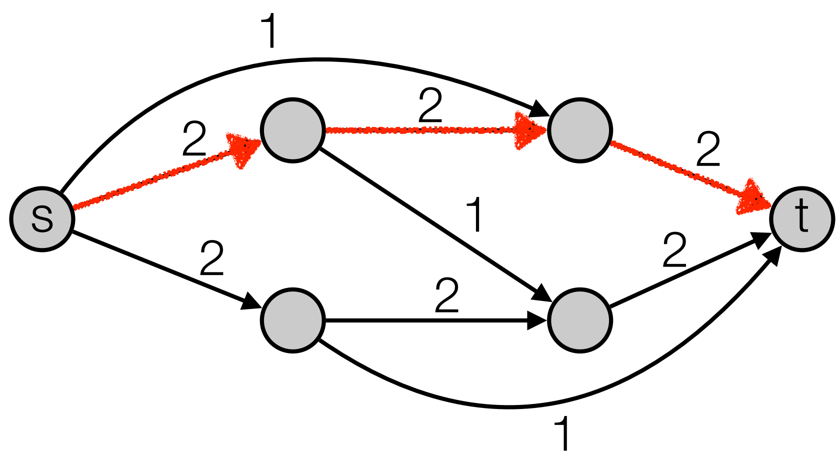

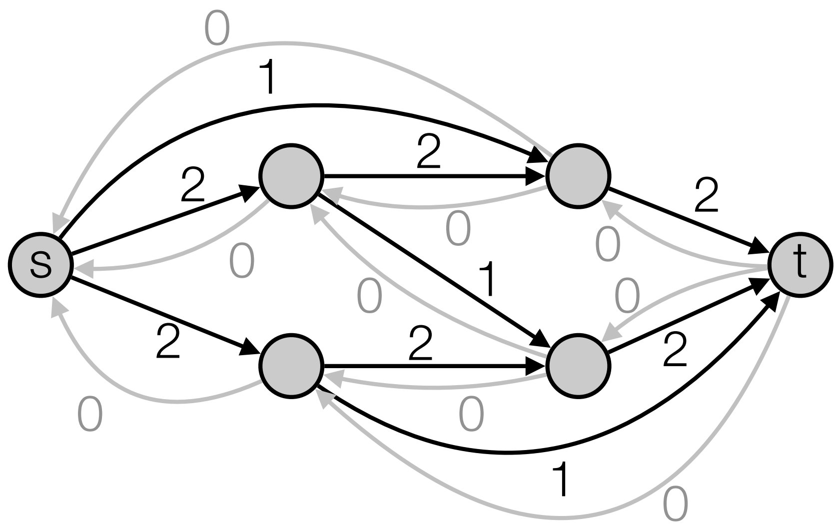

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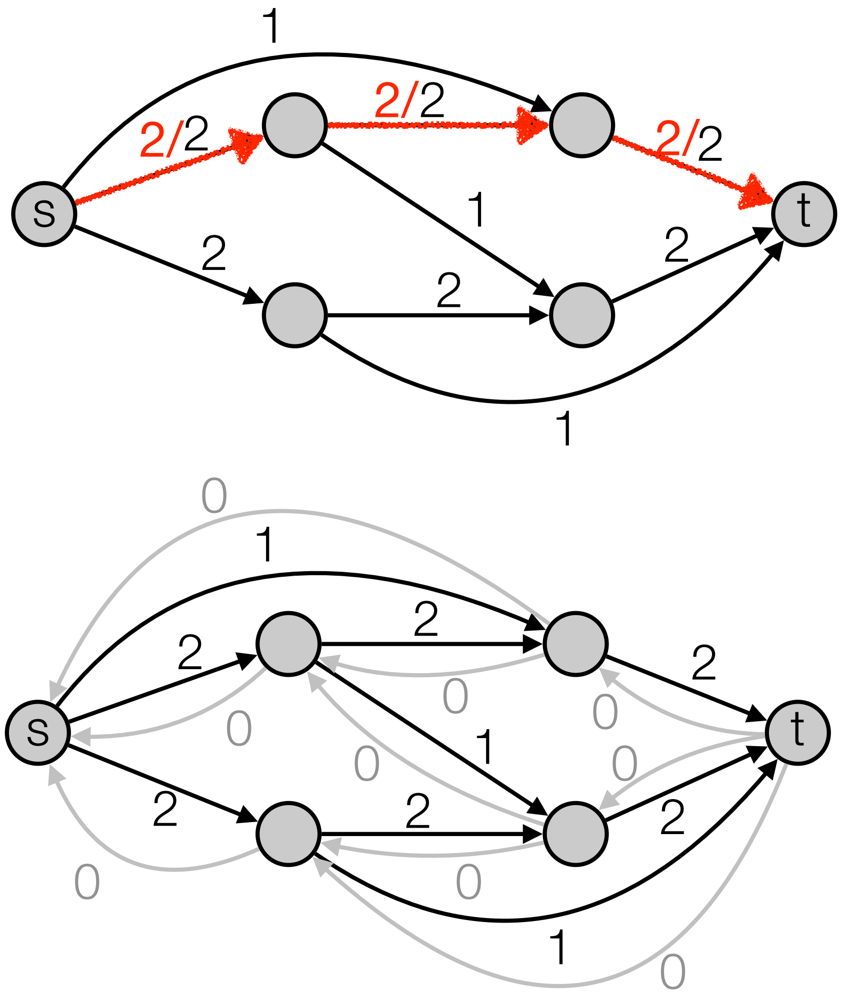

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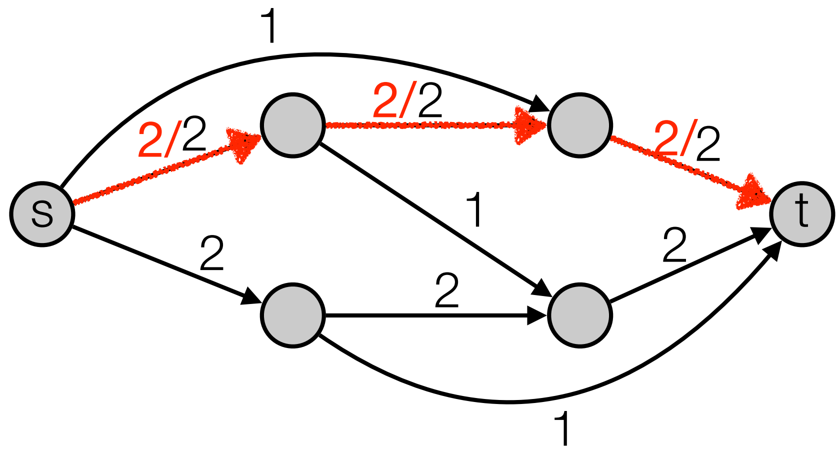

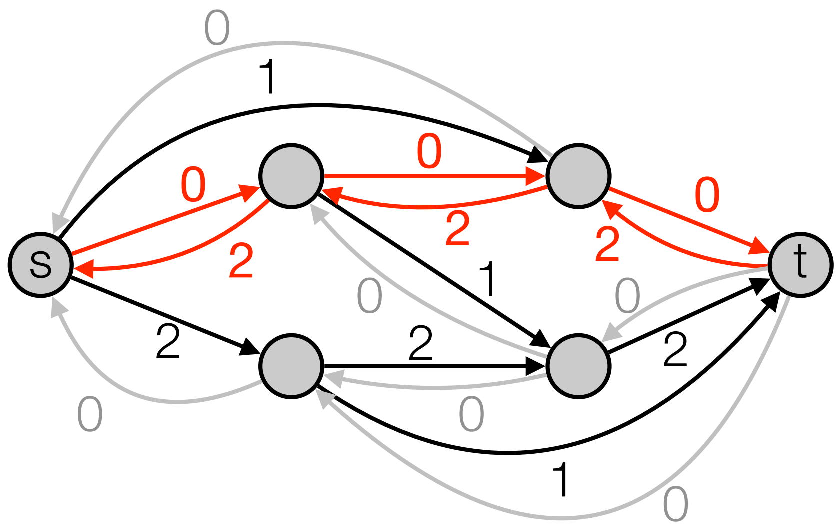

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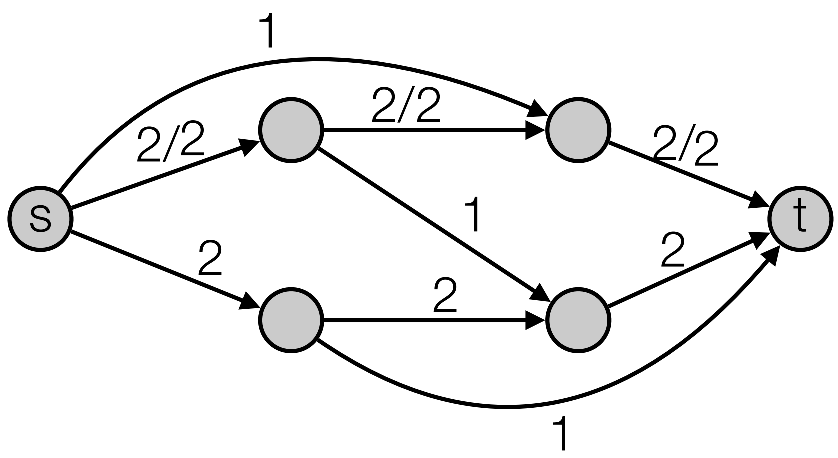

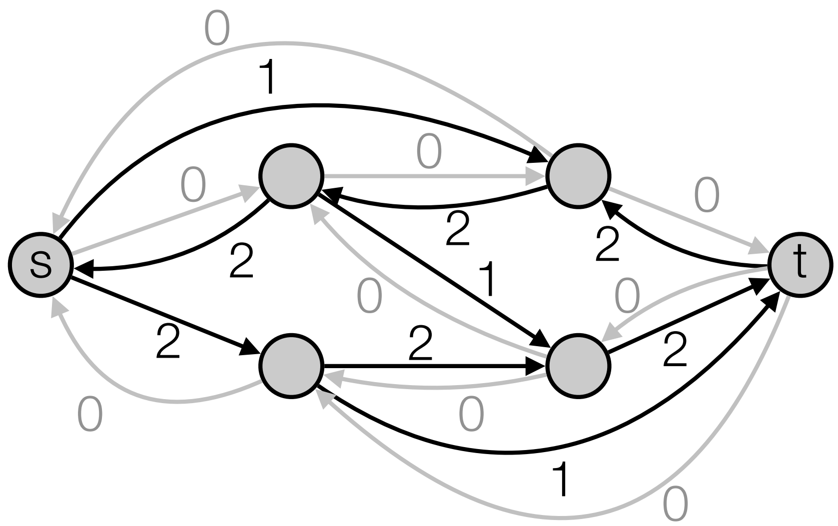

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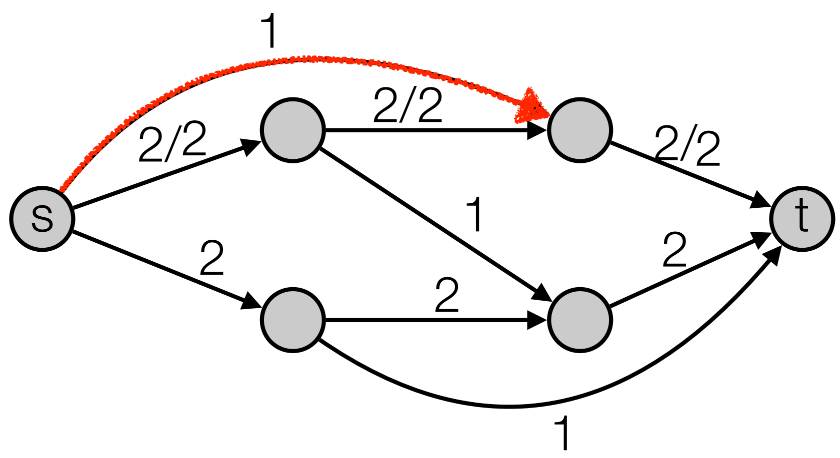

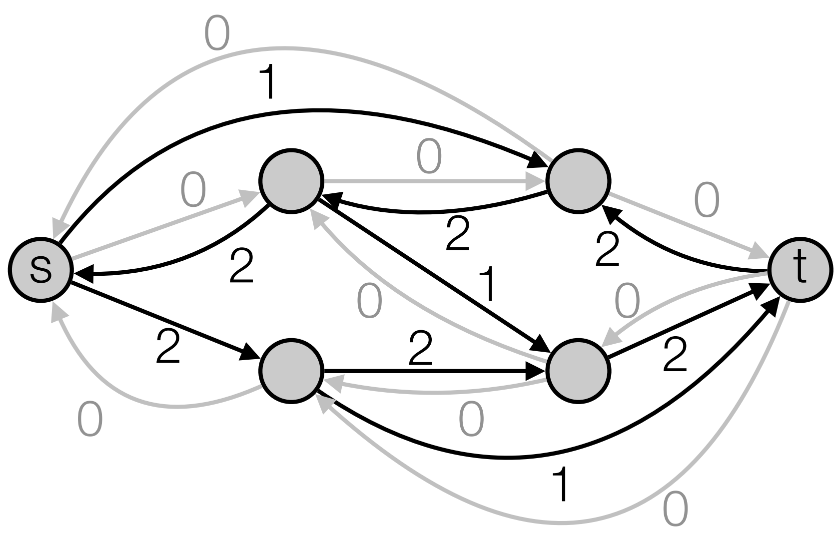

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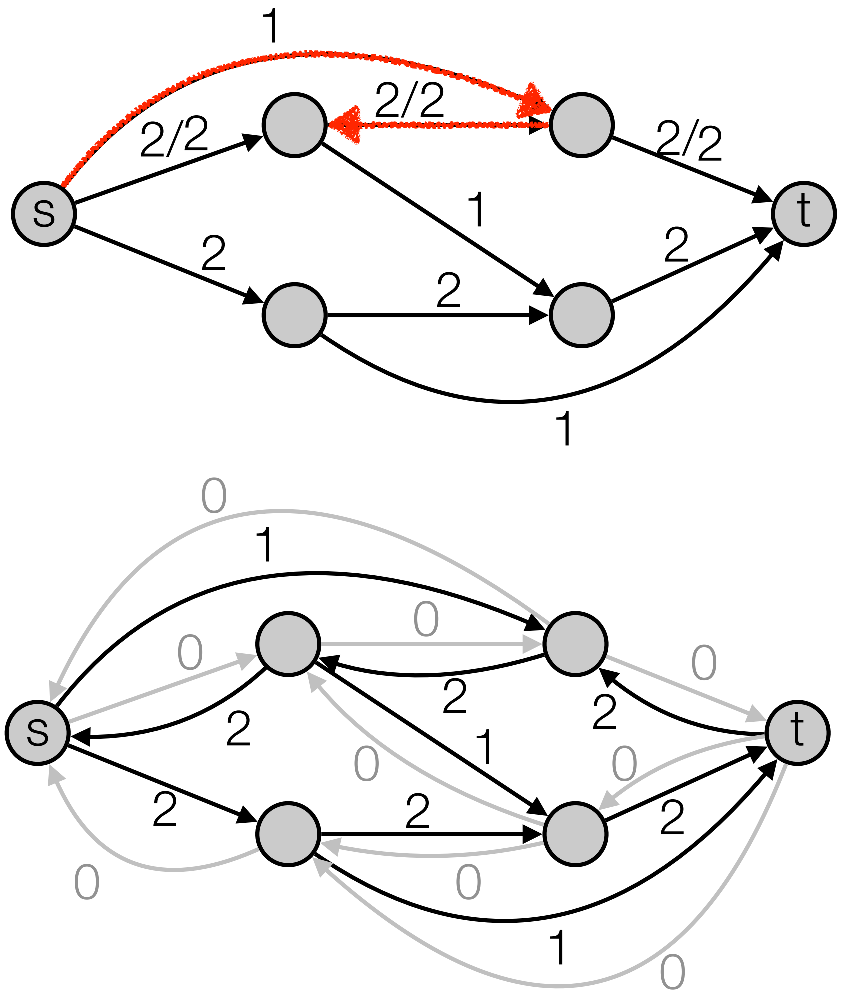

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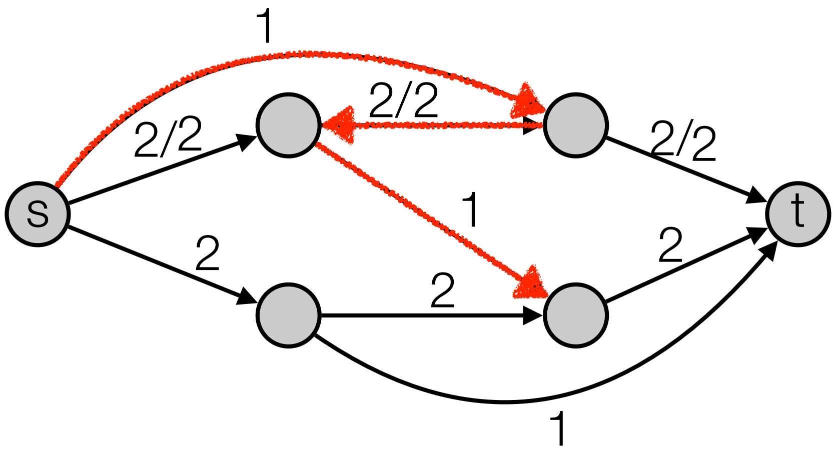

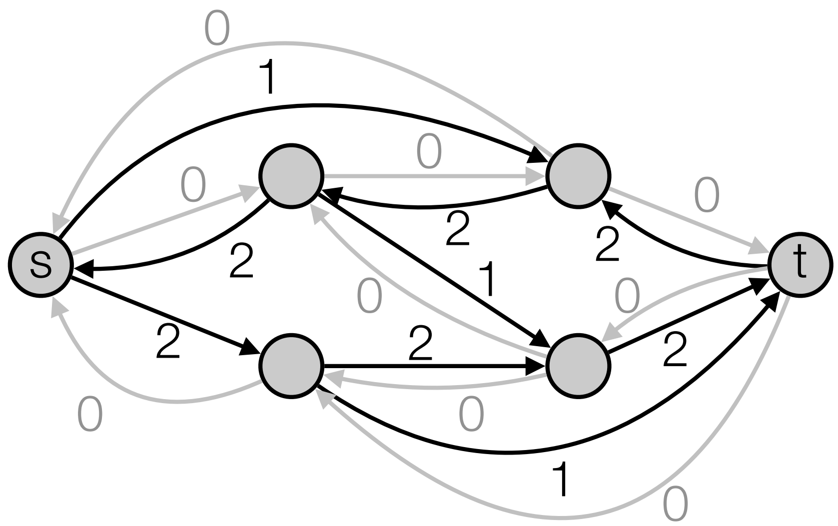

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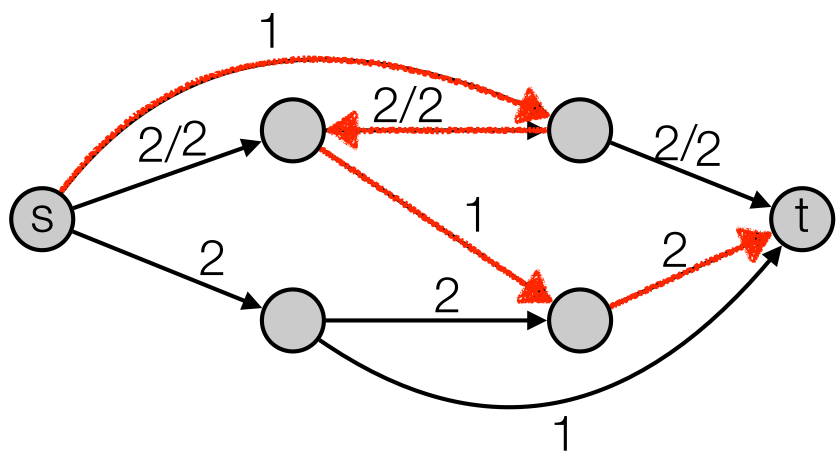

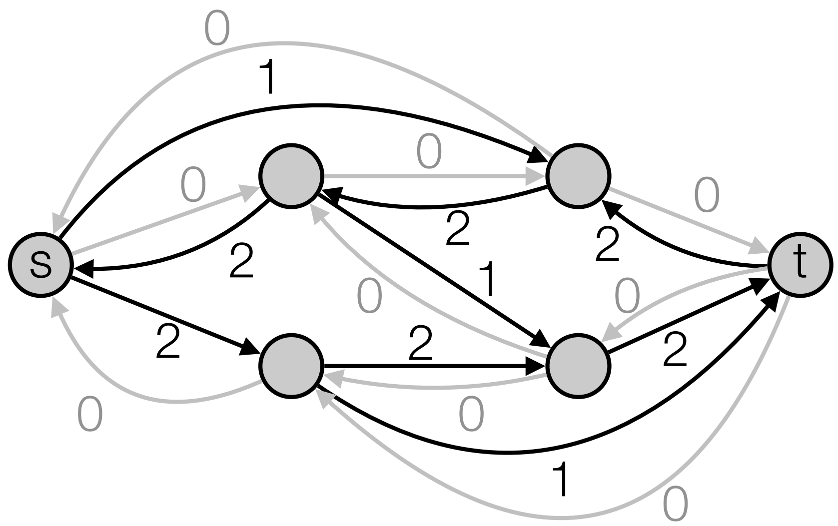

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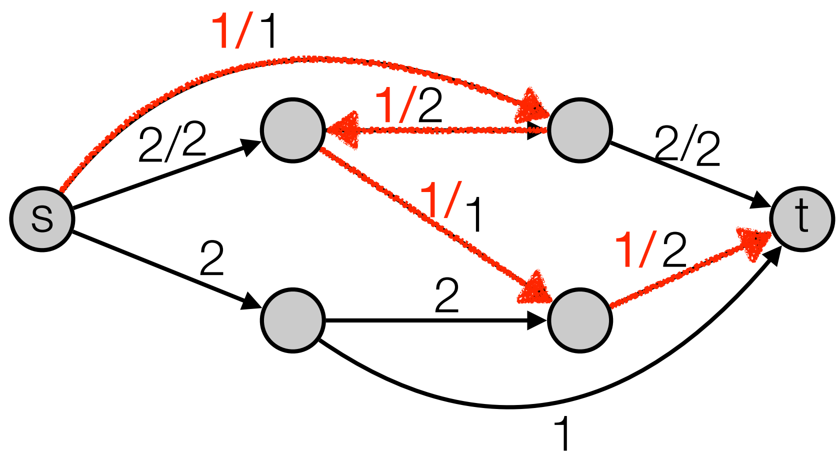

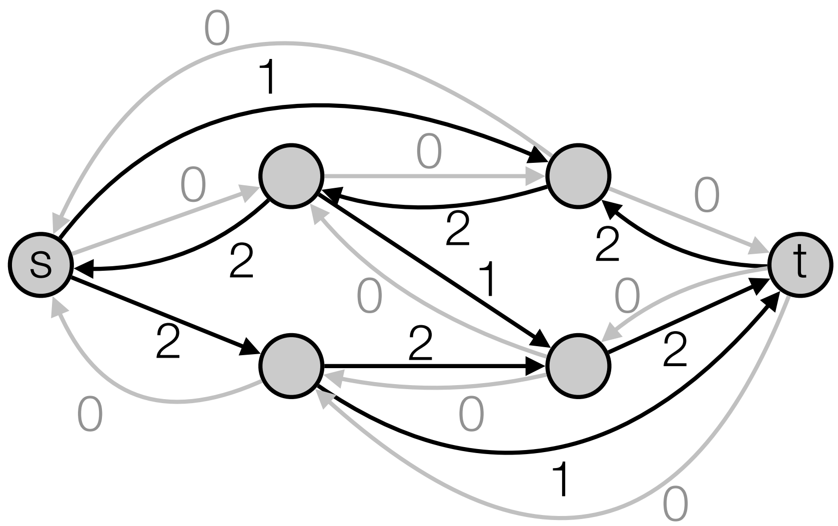

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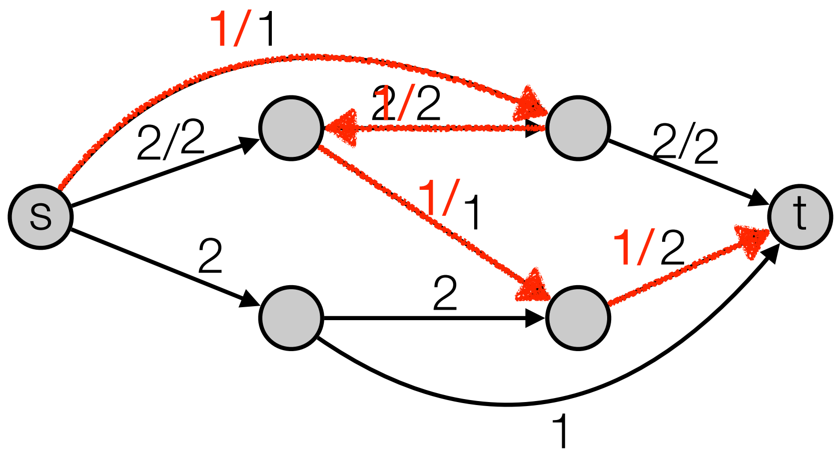

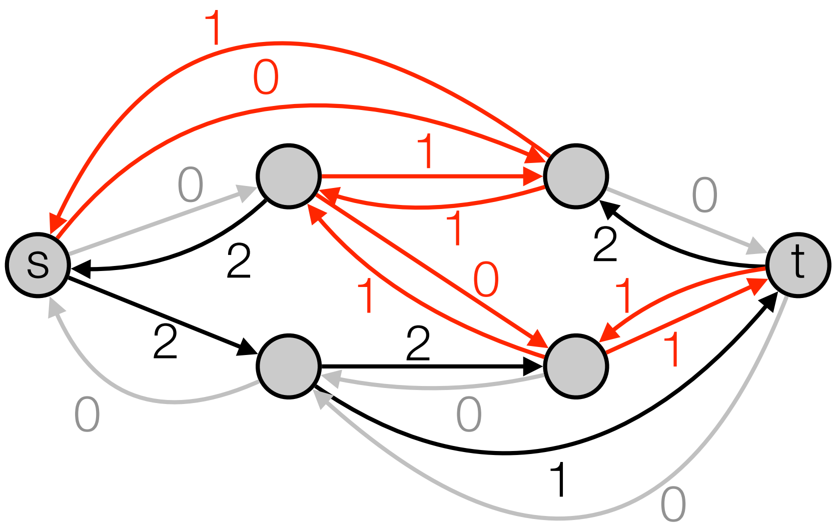

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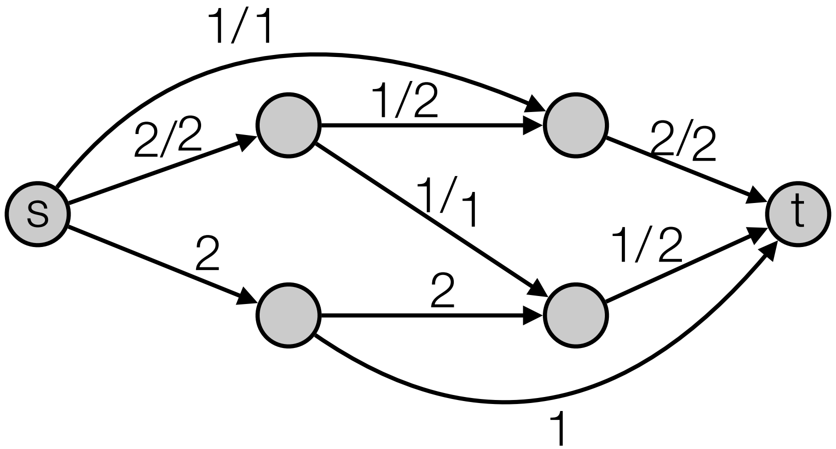
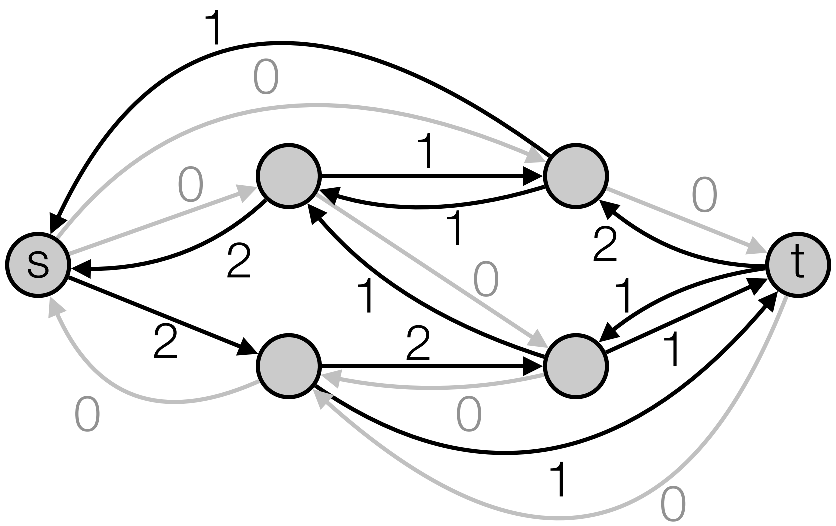
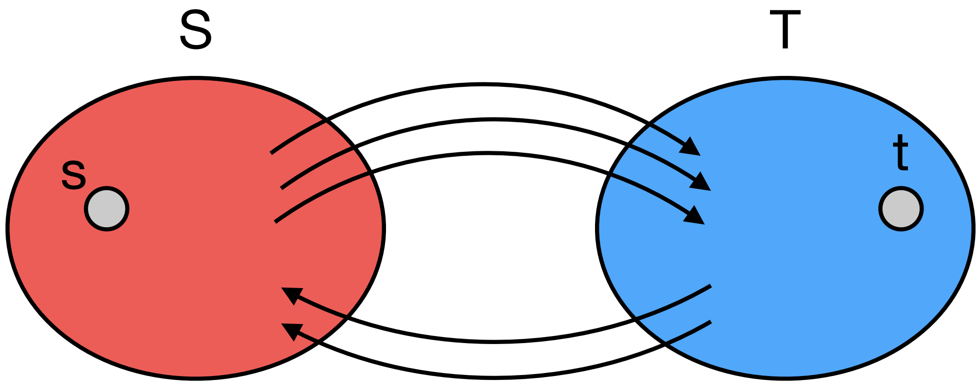




s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



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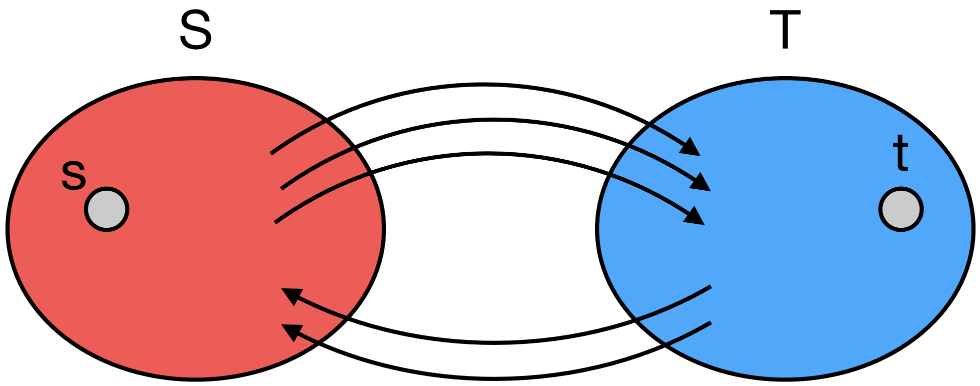

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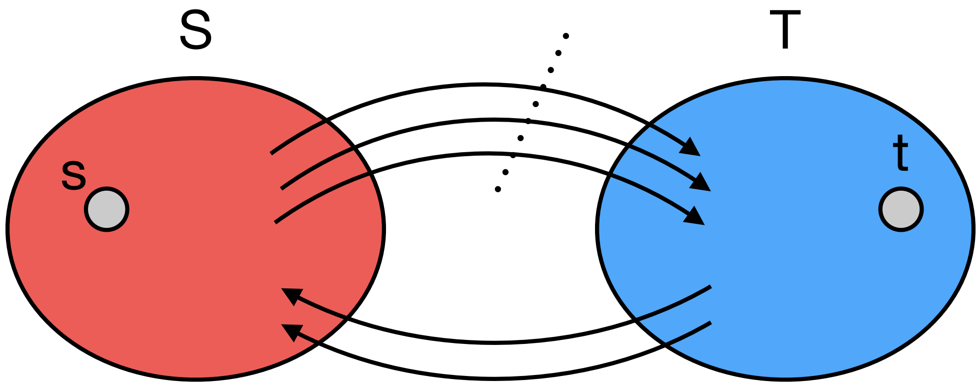

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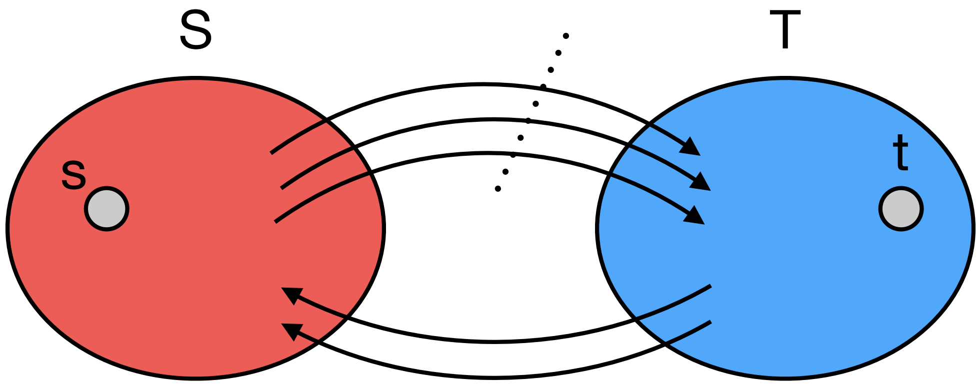

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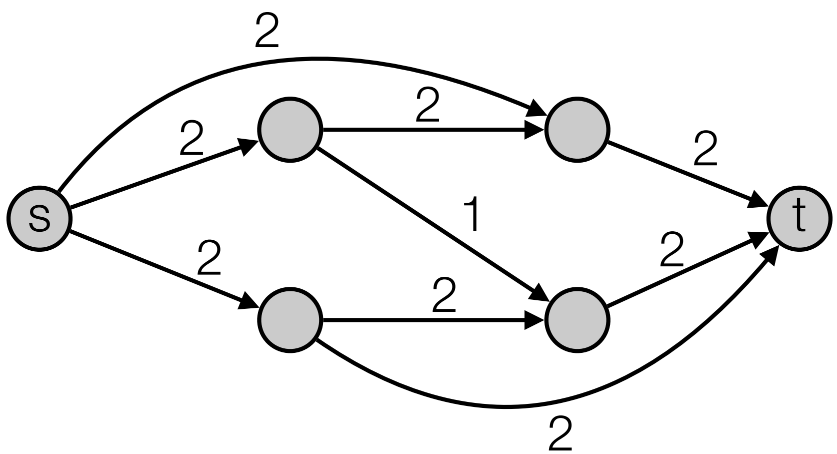

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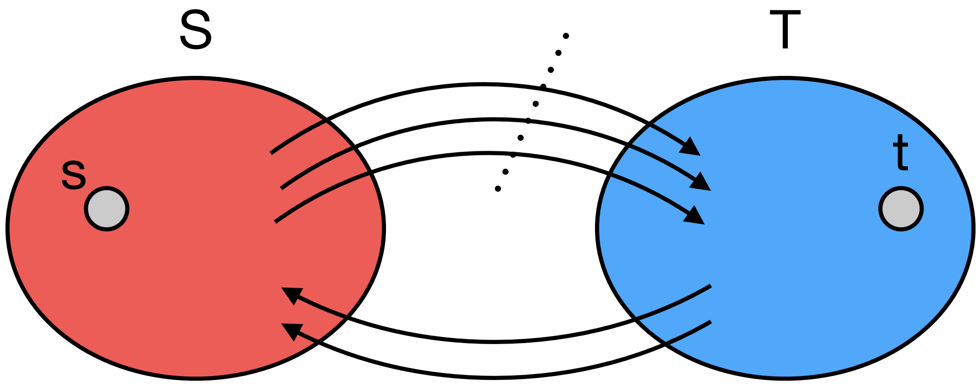

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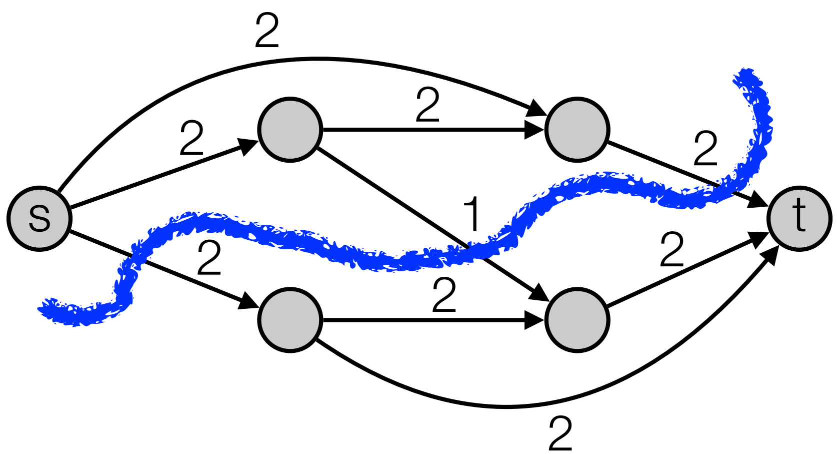

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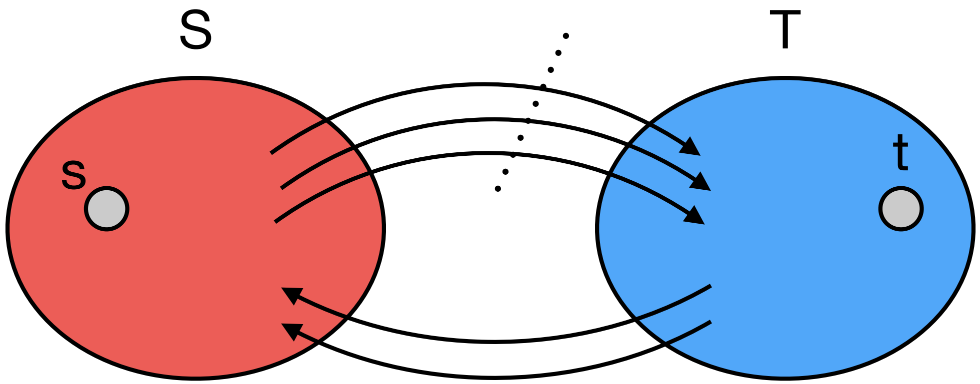

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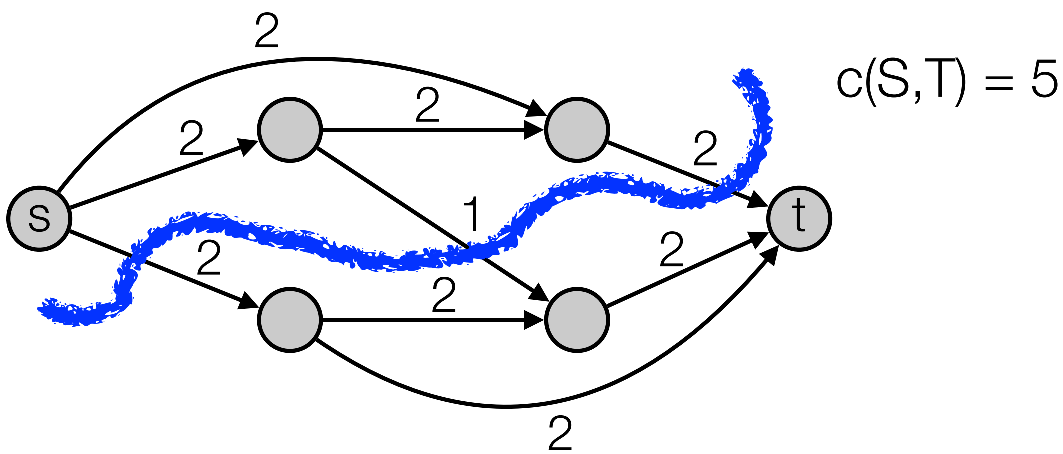

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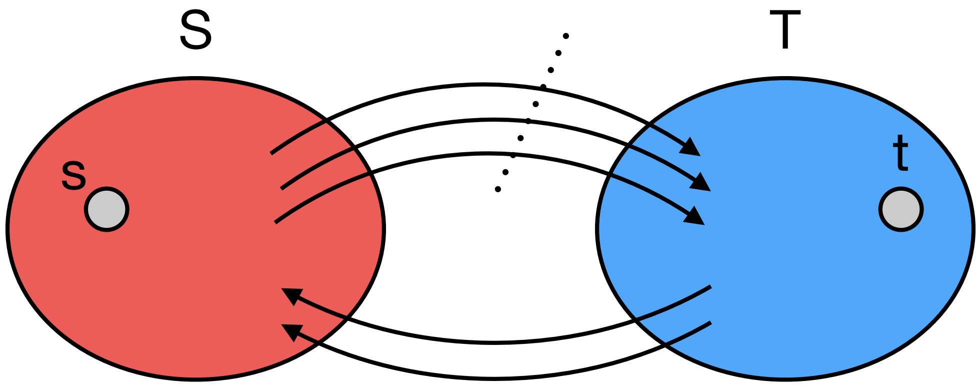

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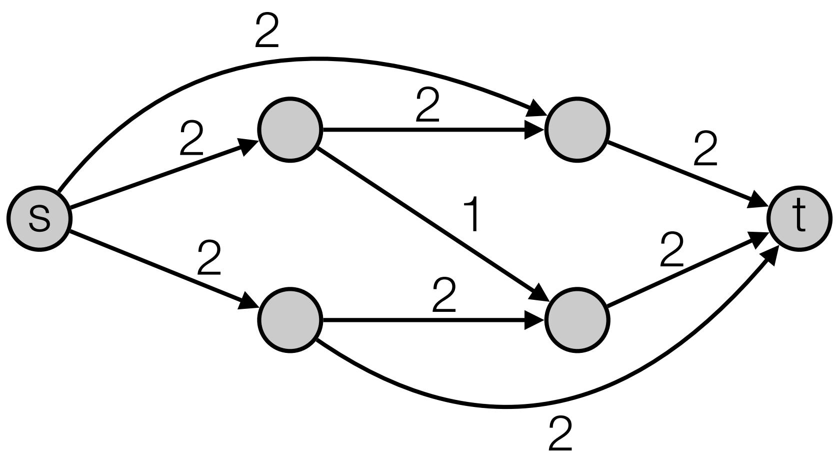

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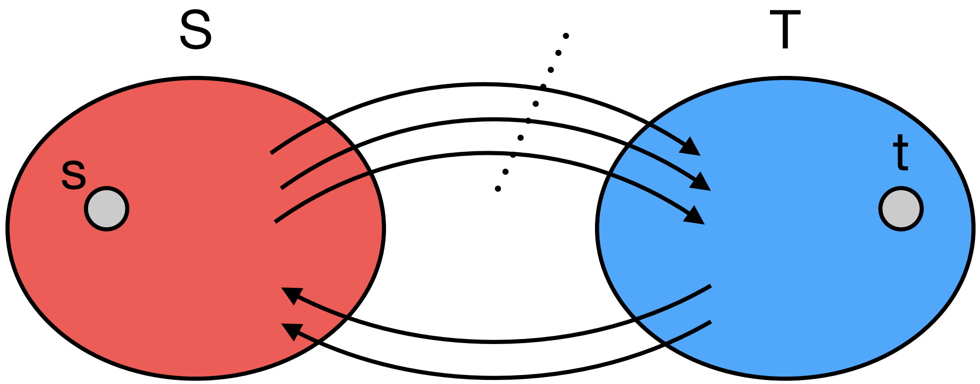

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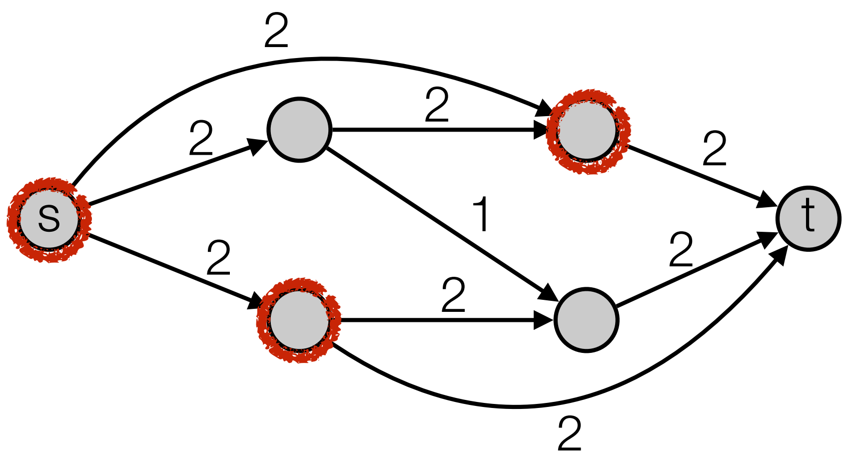

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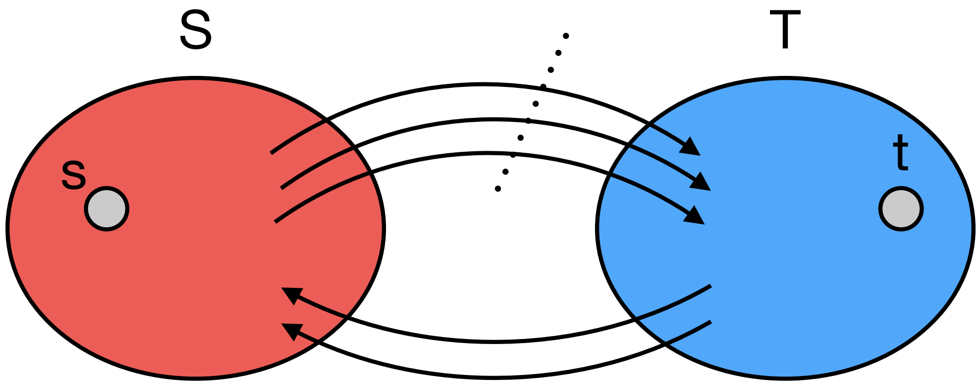
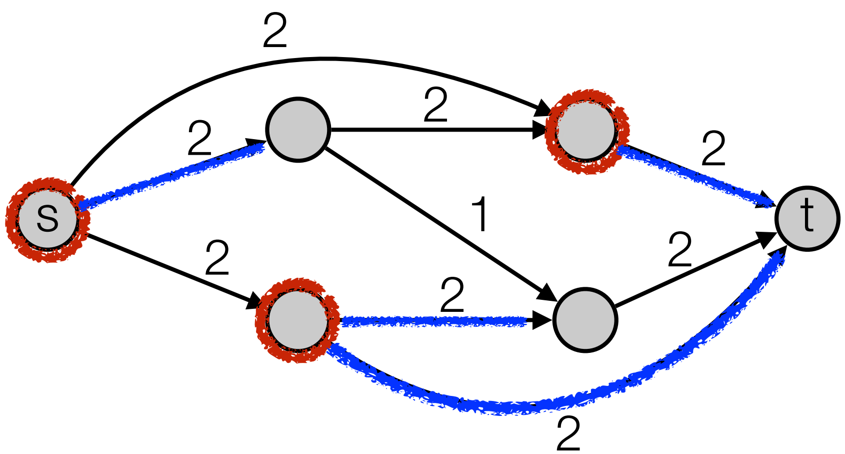


- Capacity of cut: total capacity of edges going *from* S to T.



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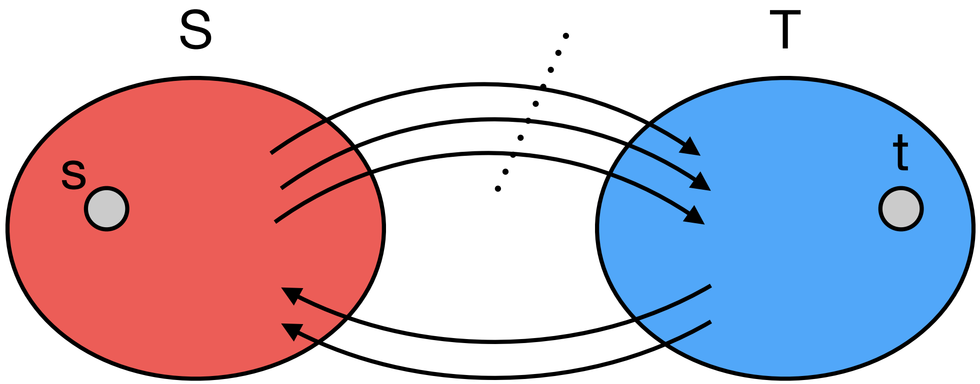
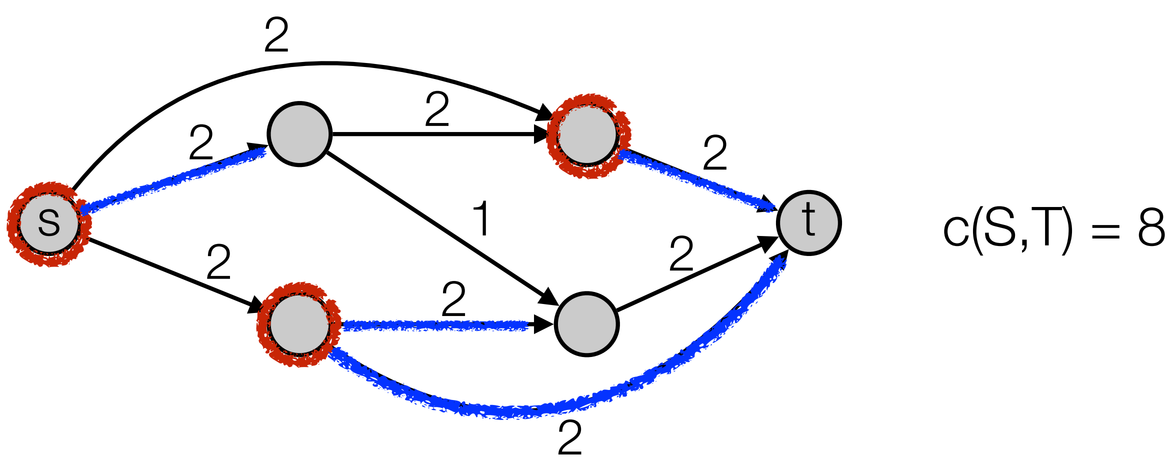


- Capacity of cut: total capacity of edges going *from* S to T.



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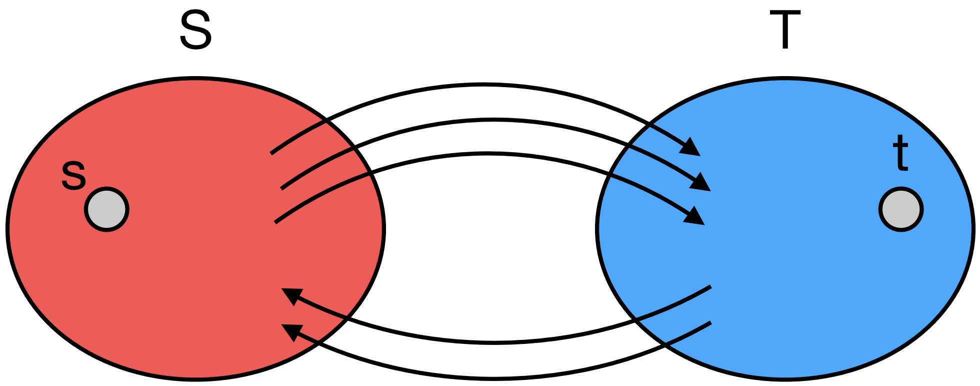
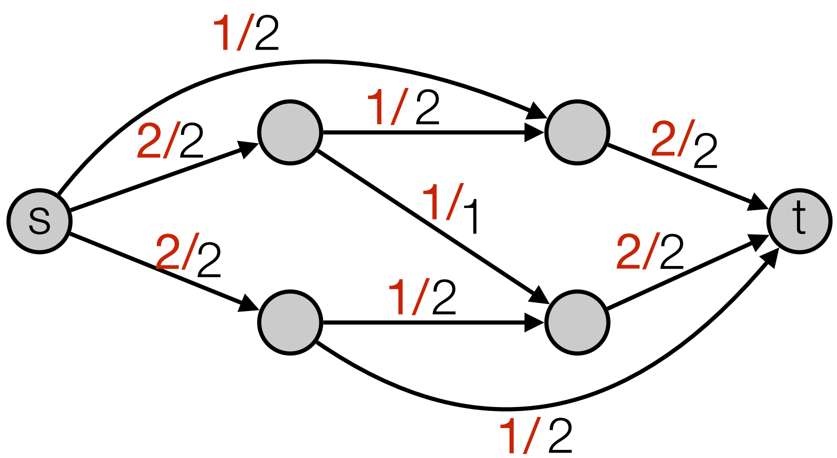


- Flow across cut: = flow *from* S to T *minus flow from* T to S.



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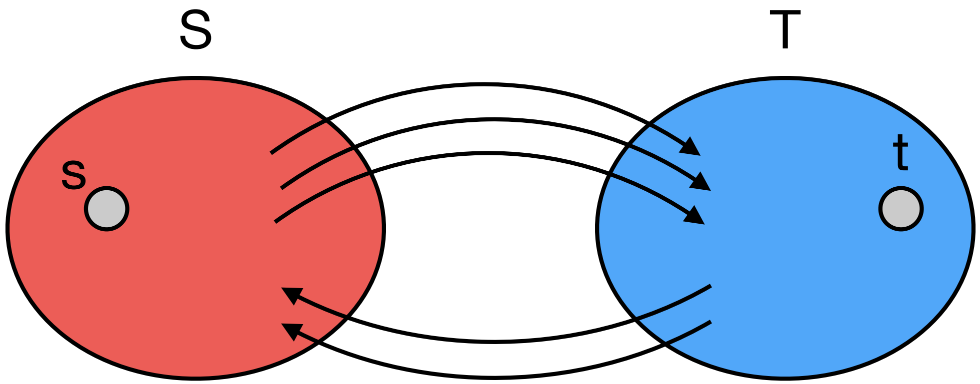

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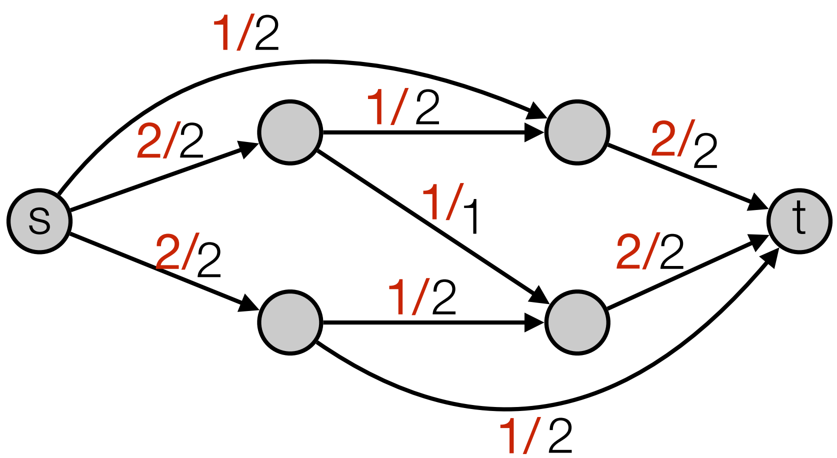

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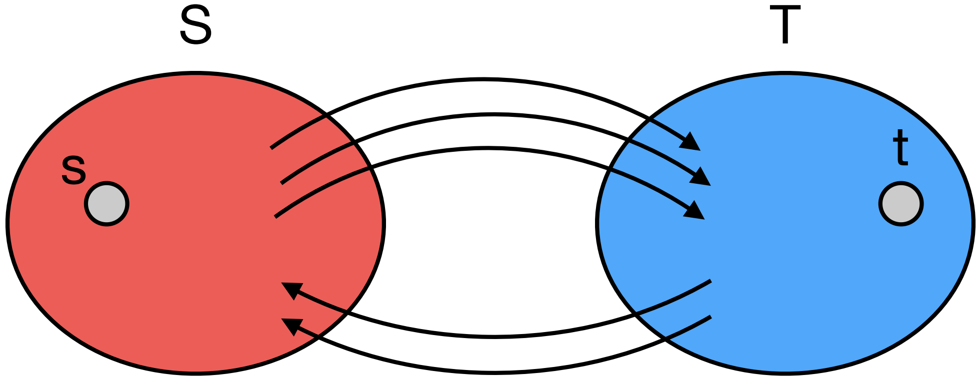

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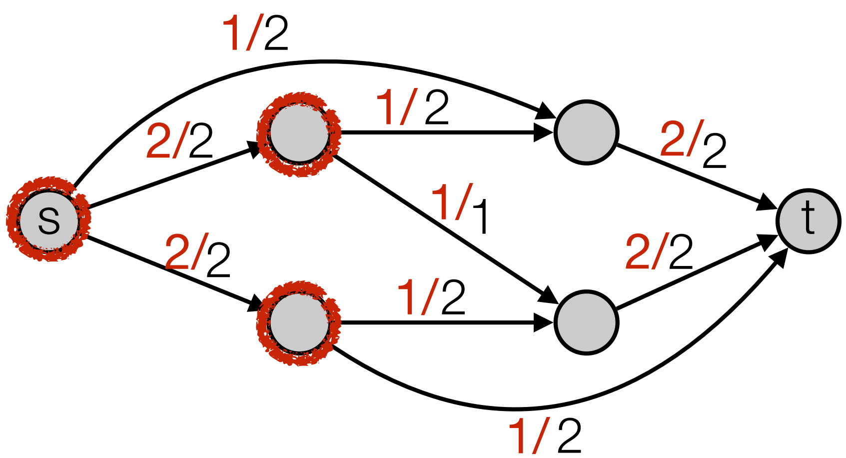

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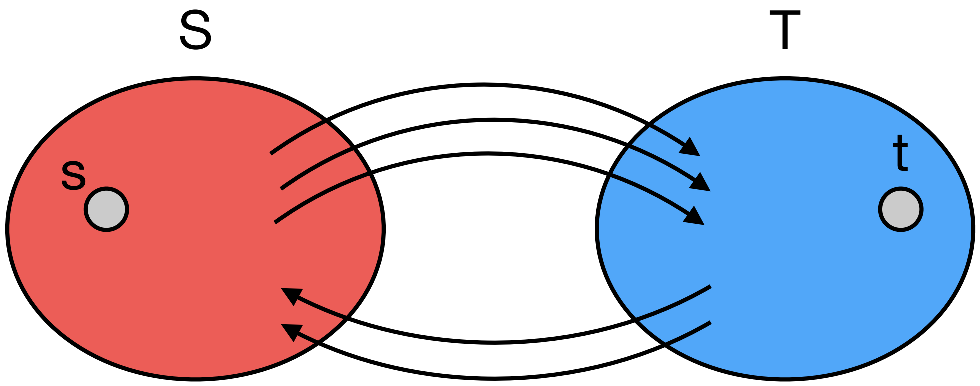

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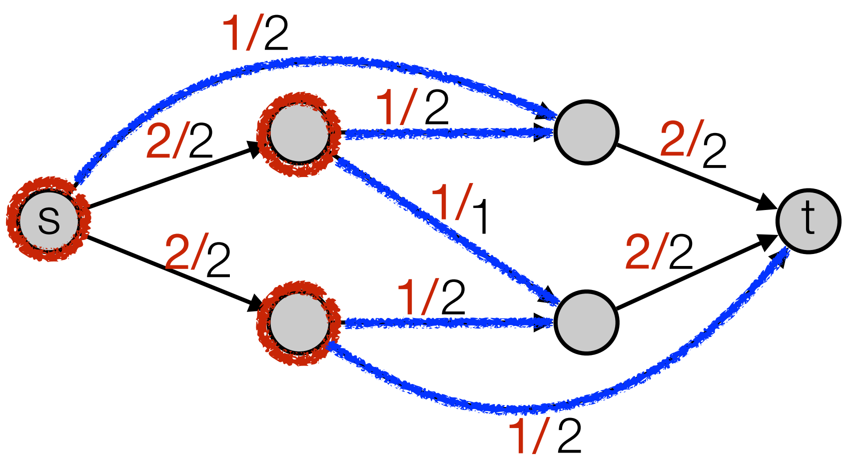

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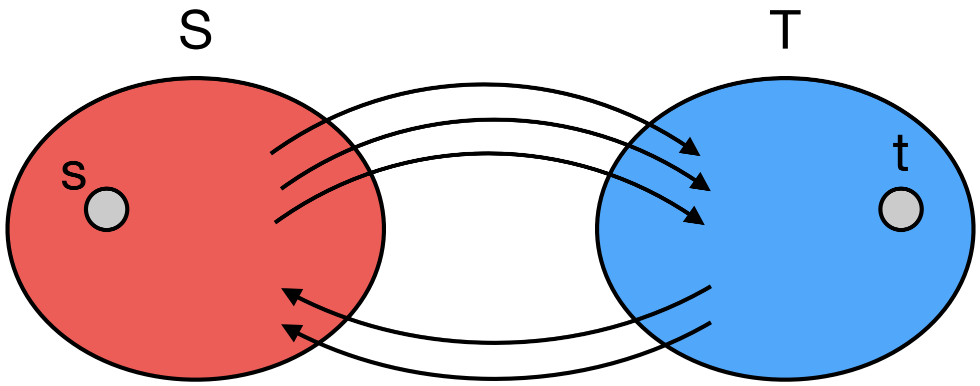

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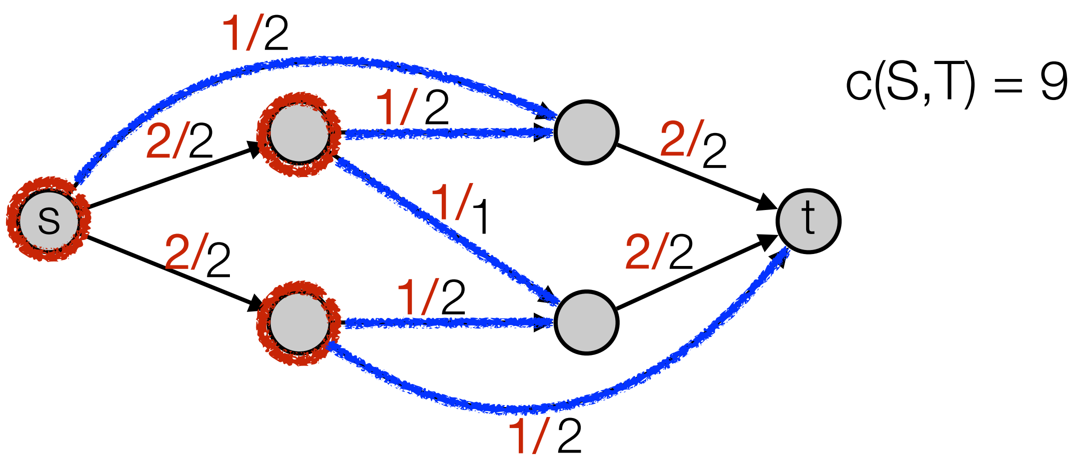

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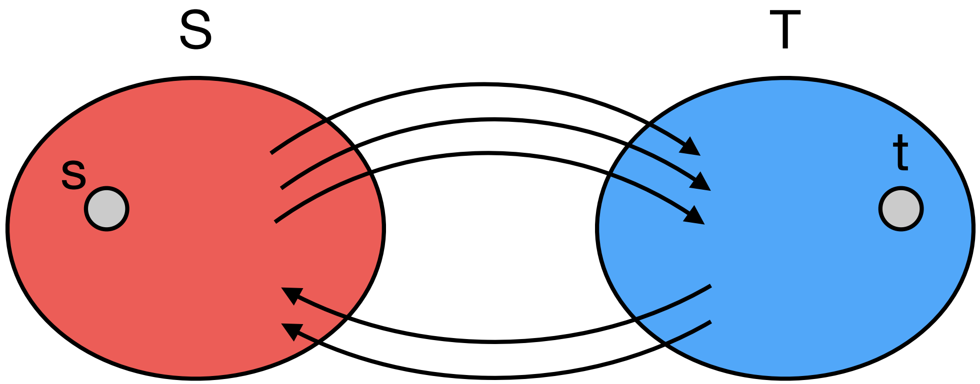

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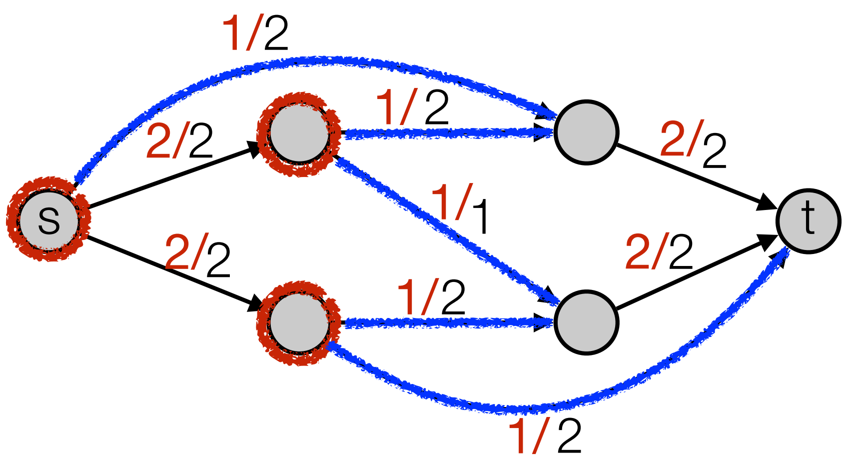
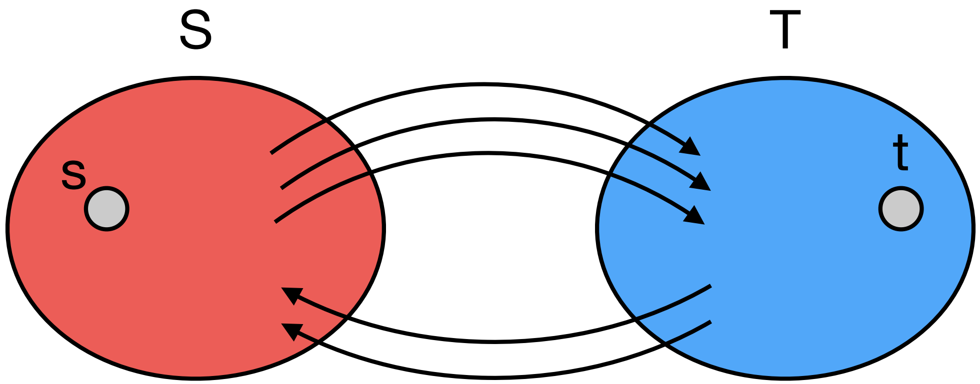


c(S,T) = 9   f(S,T) = 5

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



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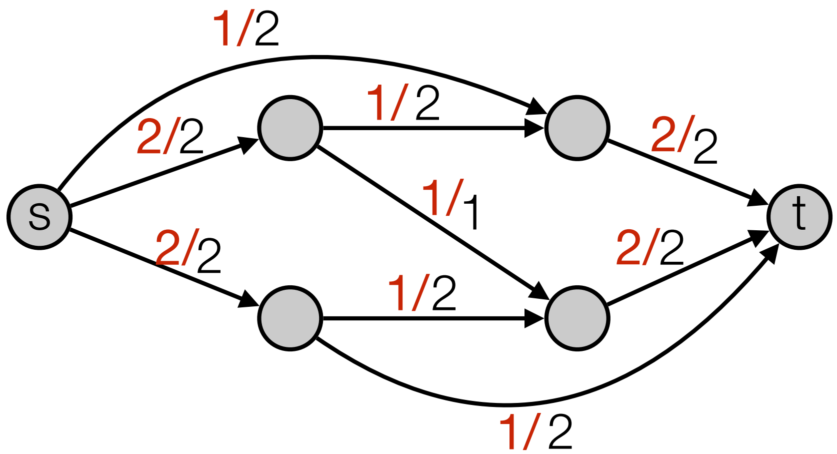

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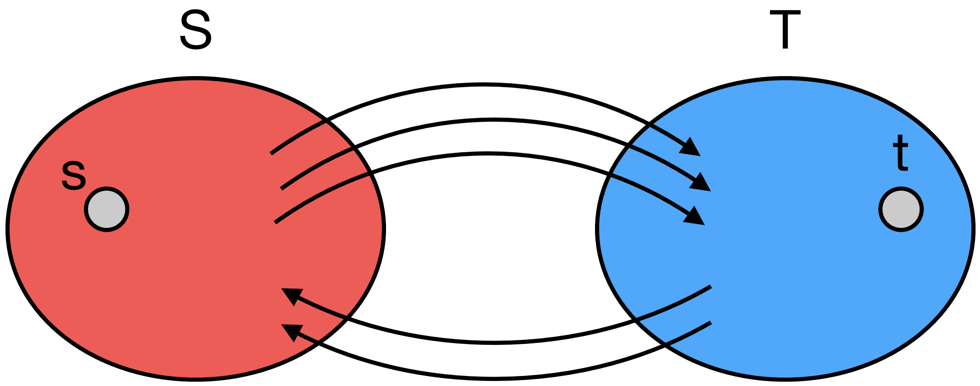

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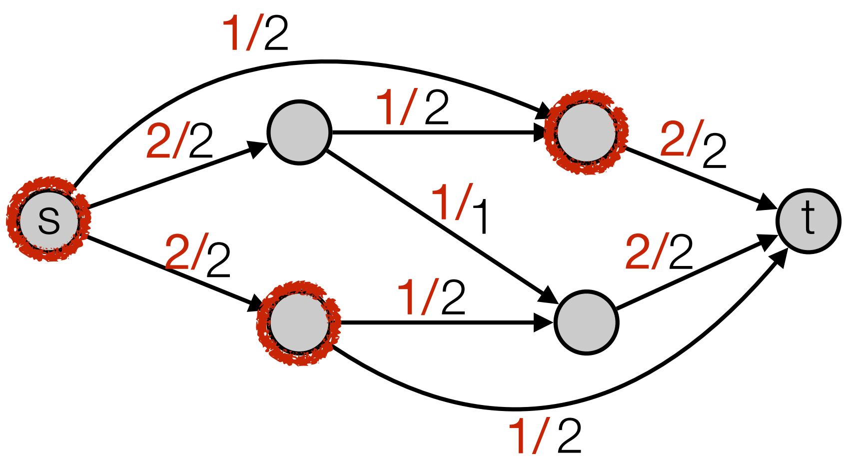

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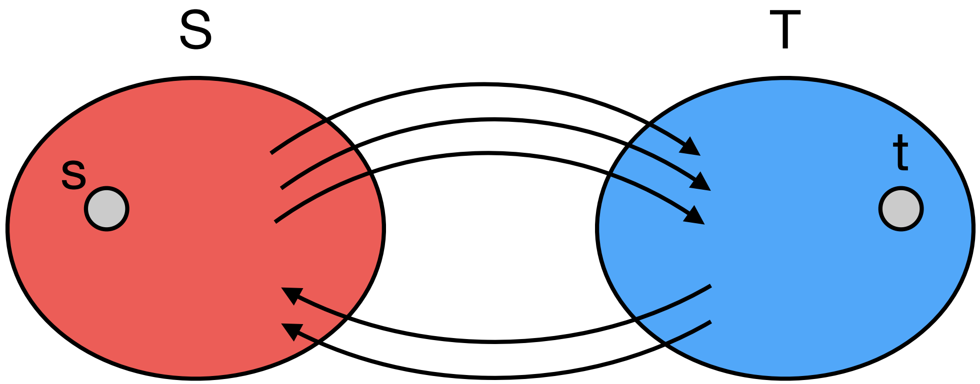

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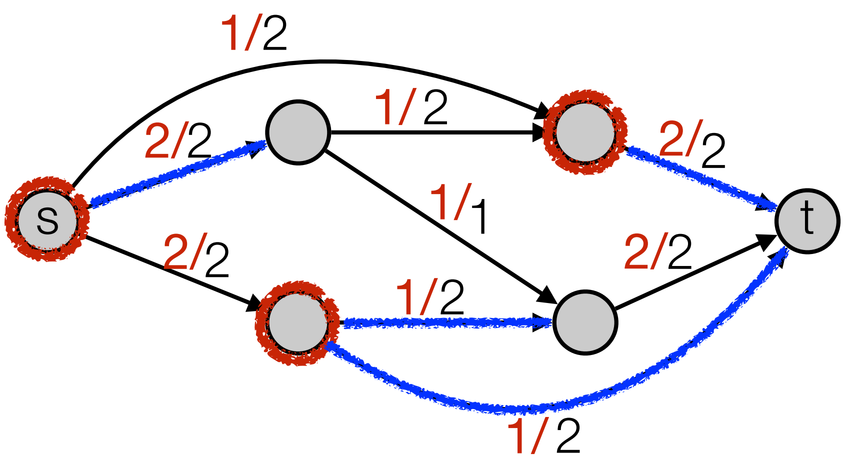

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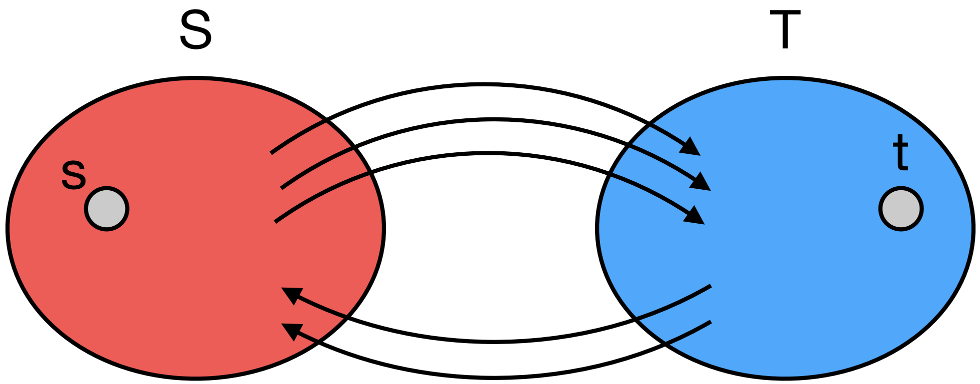

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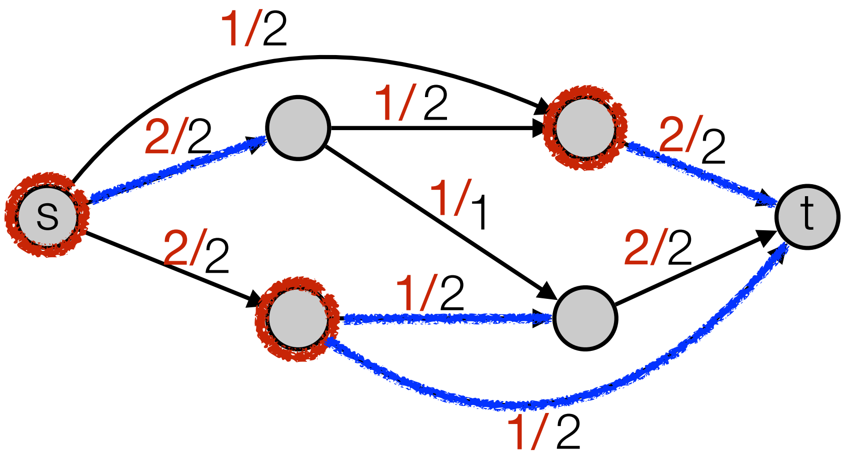
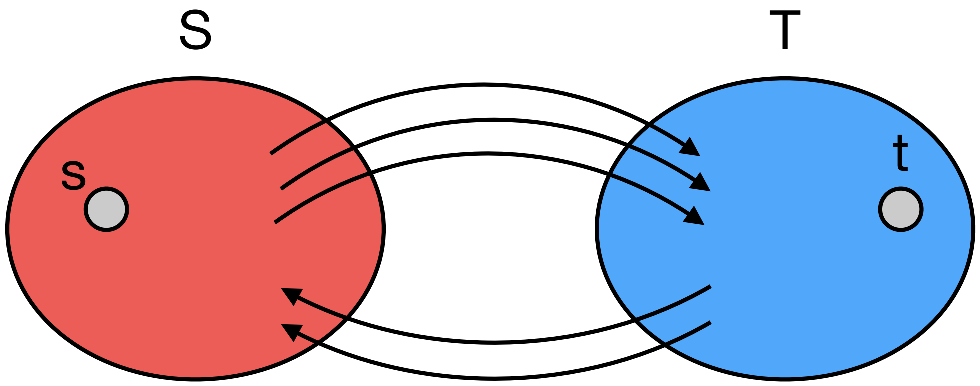


c(S,T) = 8

s-t Cuts

- Cut: Partition of vertices into S and T, such that s ∈ S and t ∈ T.



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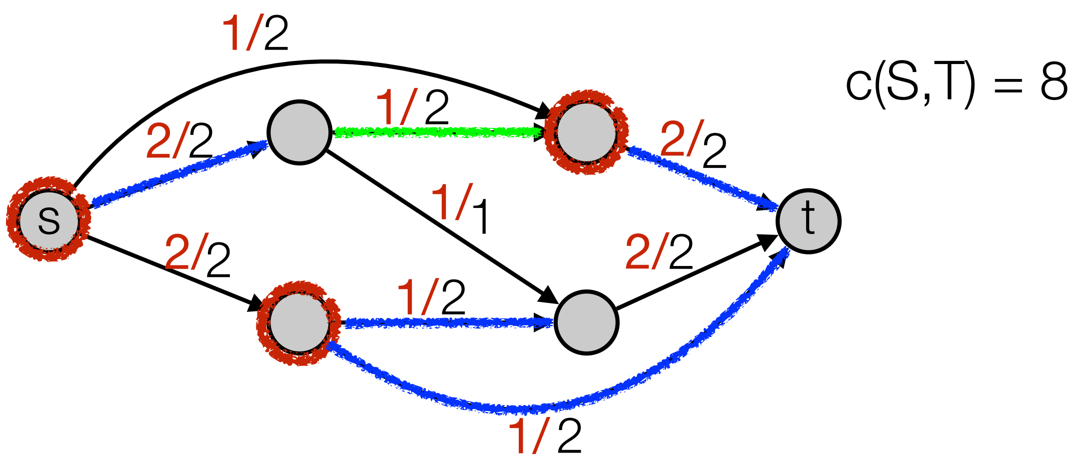

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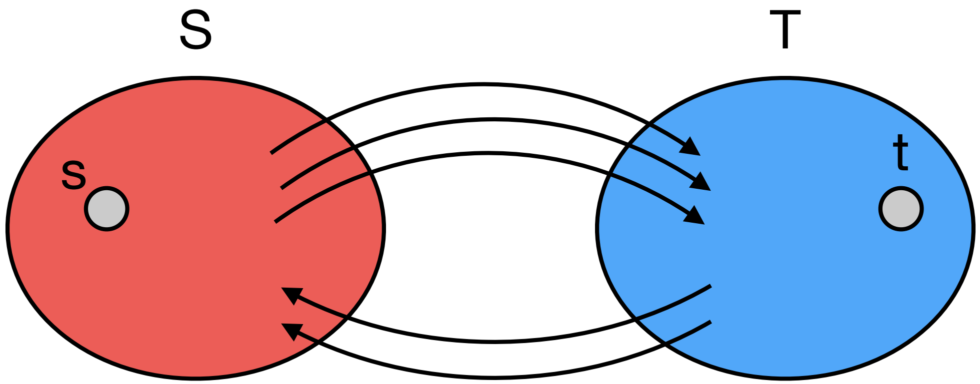

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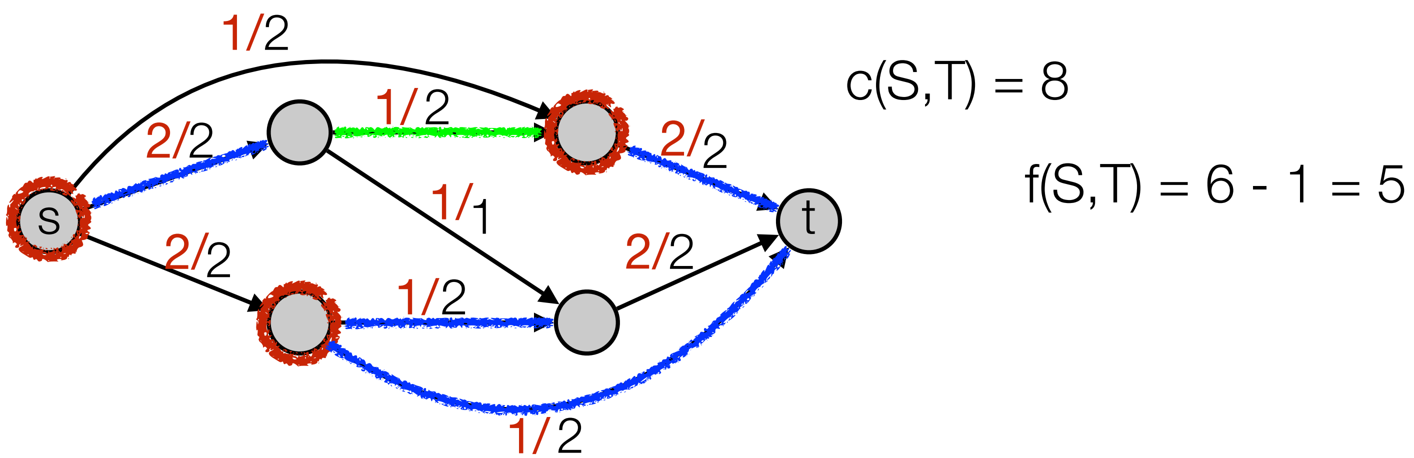

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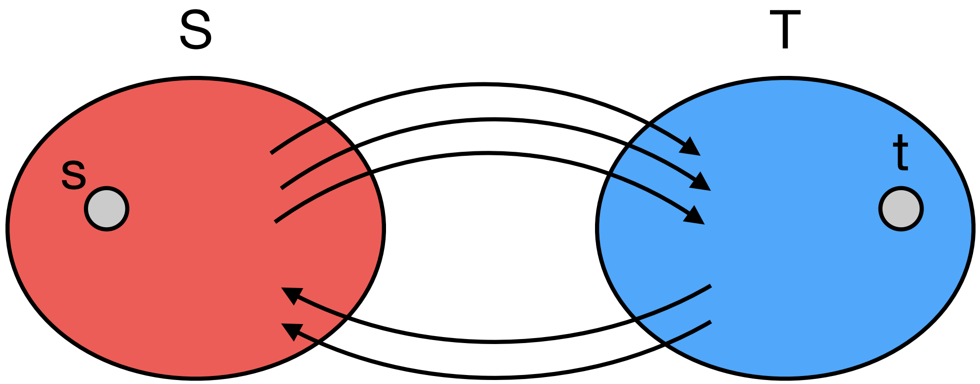

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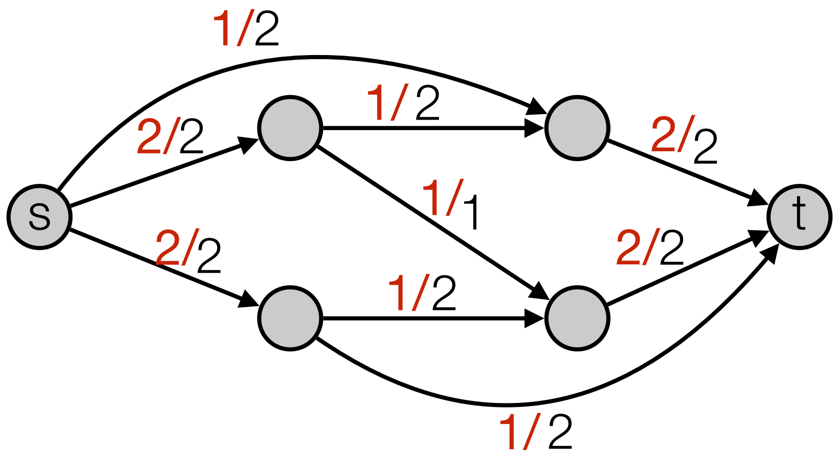

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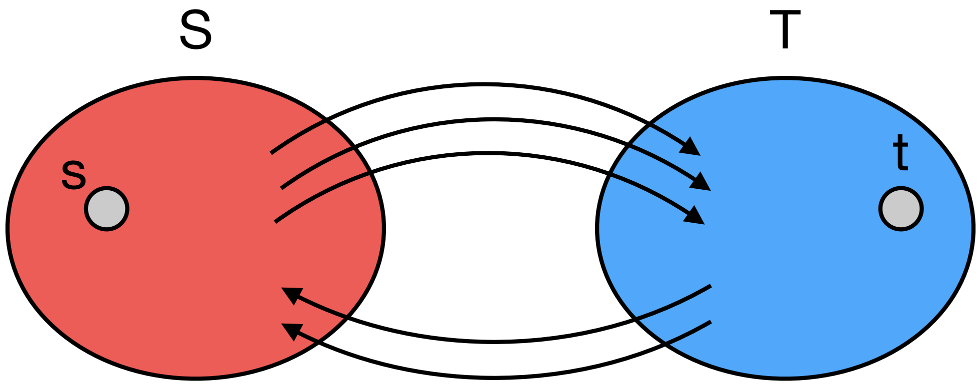

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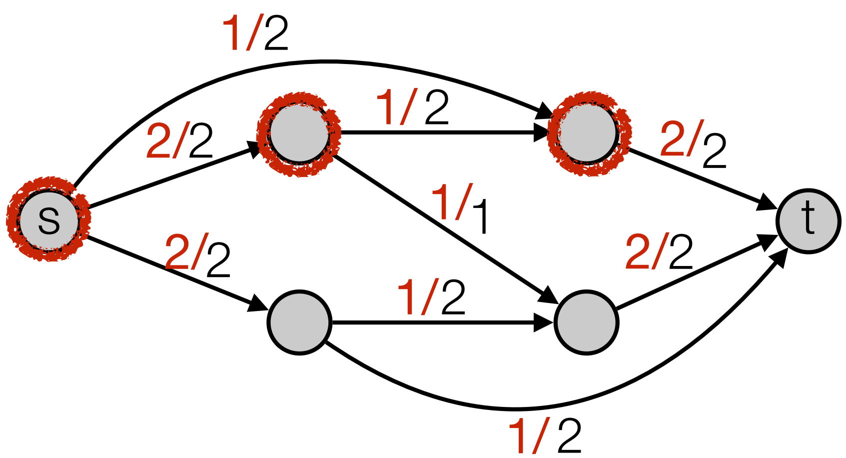

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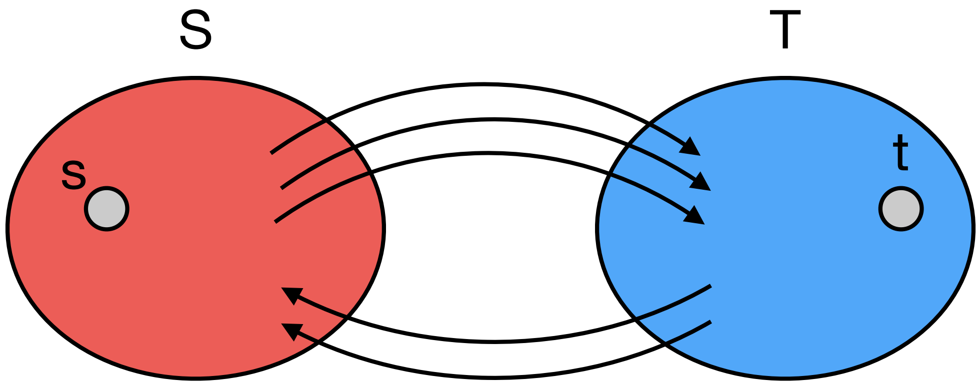

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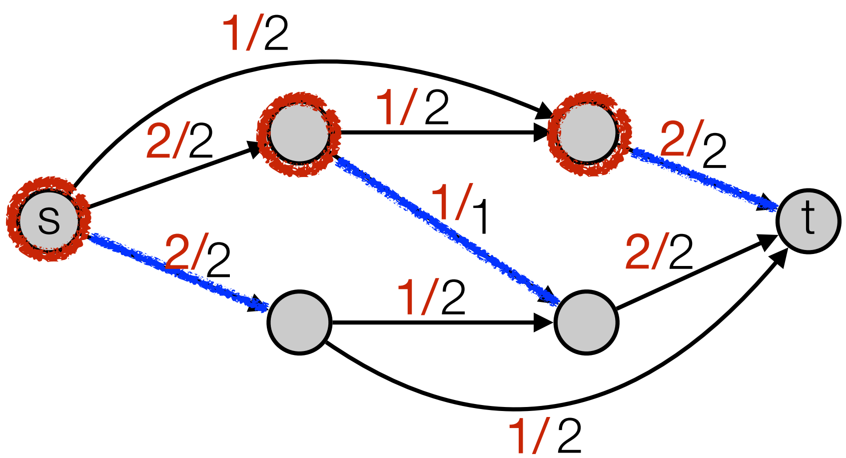

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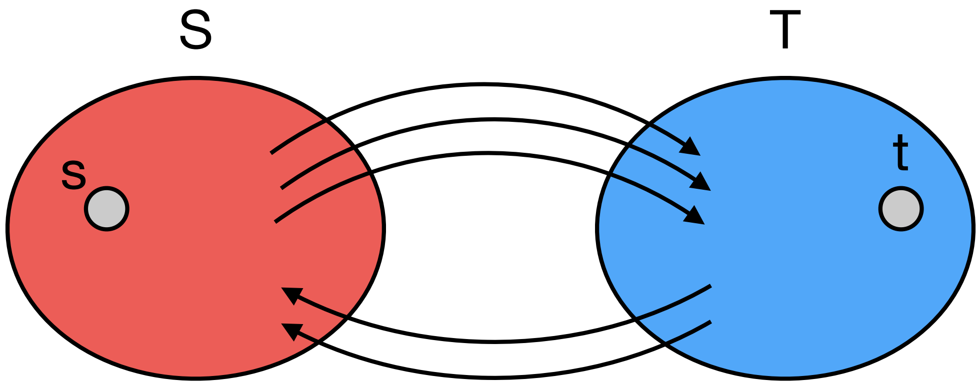

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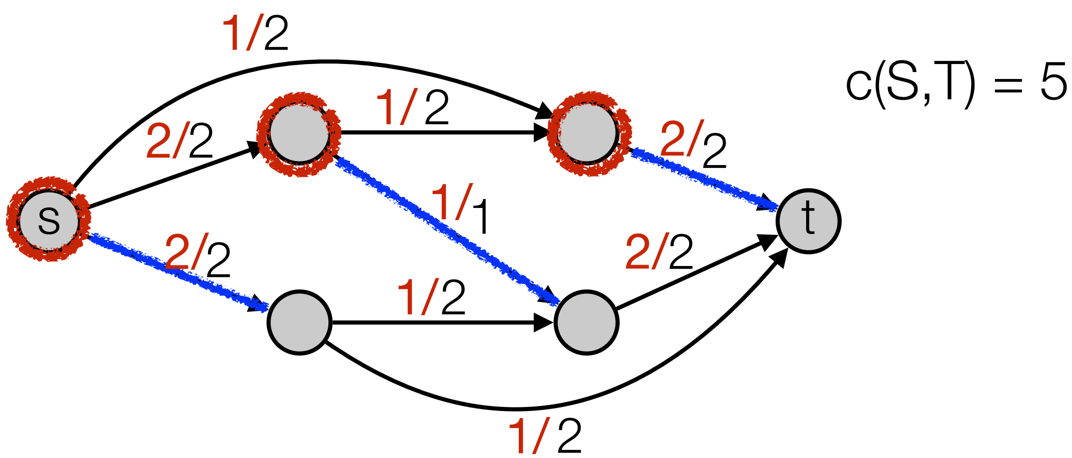

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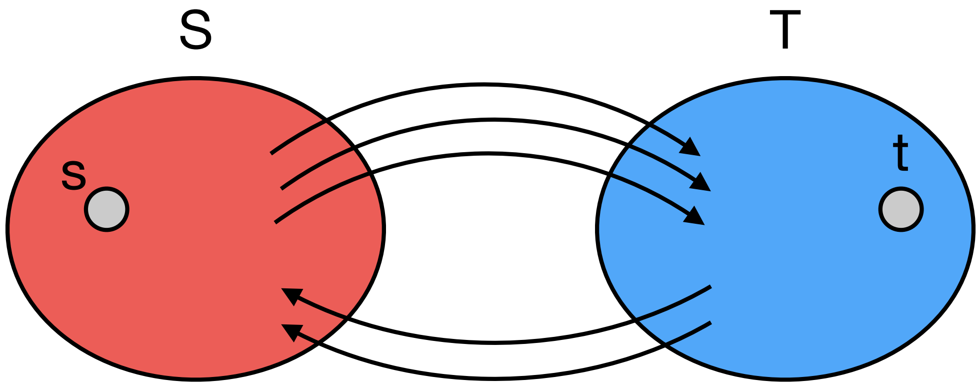

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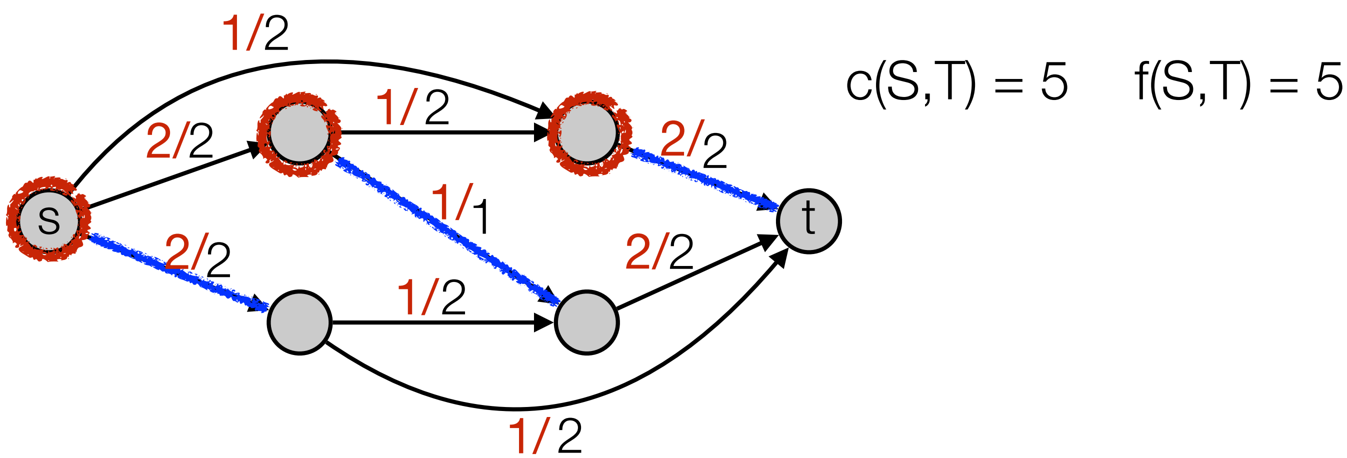

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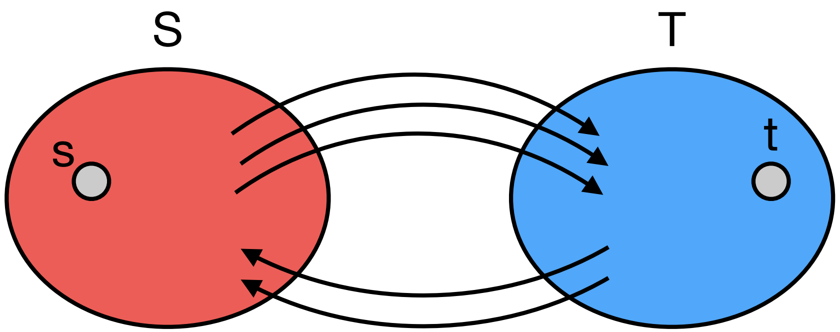

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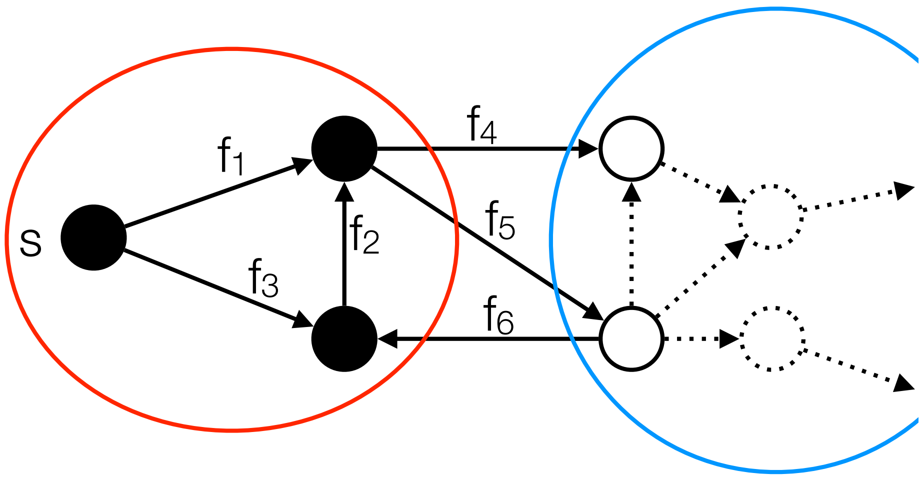

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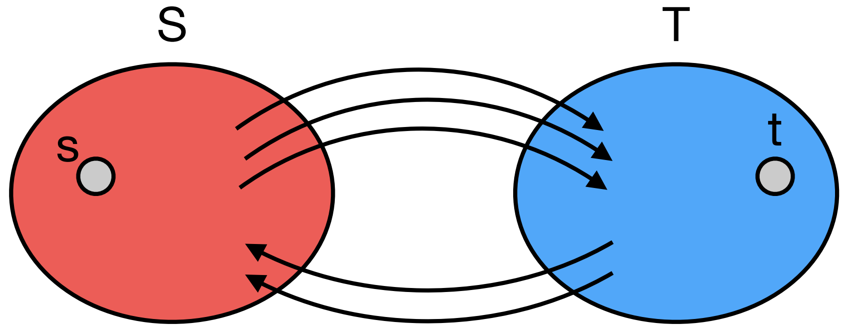
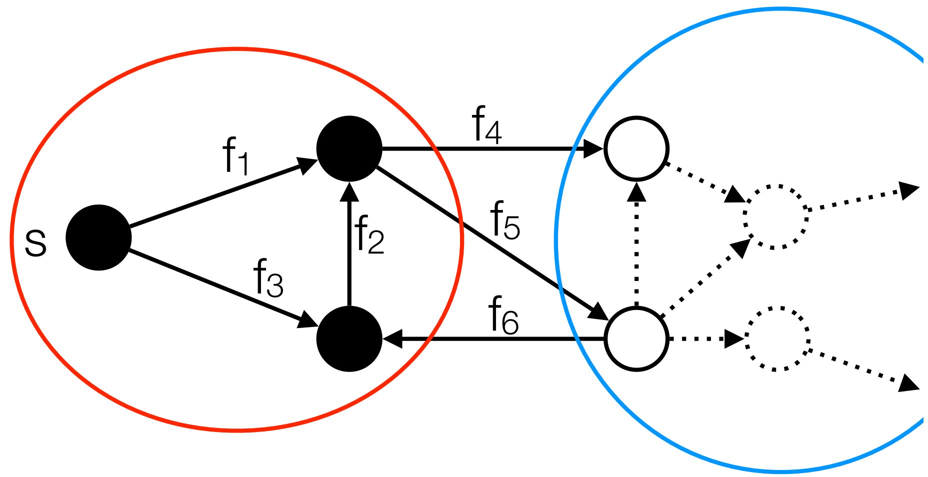


- Flow across cut = flow *from* S to T *minus flow from* T to S.
- Flow across cut: $f_4 + f_5 - f_6 = ?$



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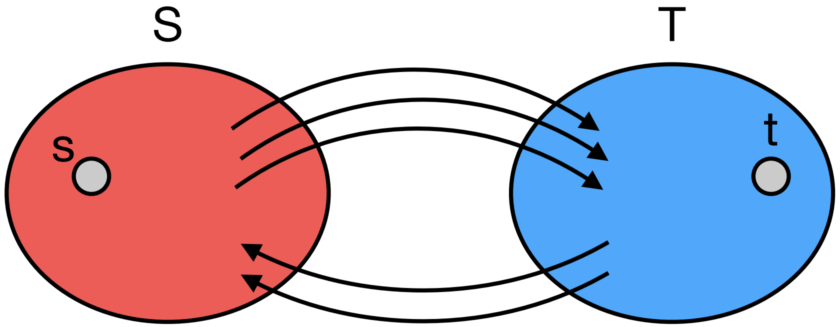
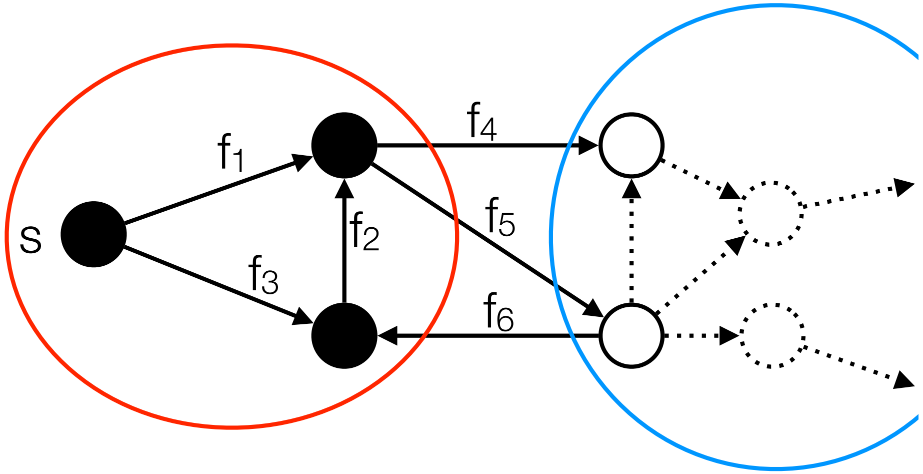


- Flow across cut = flow *from* S to T *minus flow from* T to S.
- Flow across cut: $f_4 + f_5 - f_6 = ?$
  - $f_4 + f_5 - f_1 - f_2 = 0$



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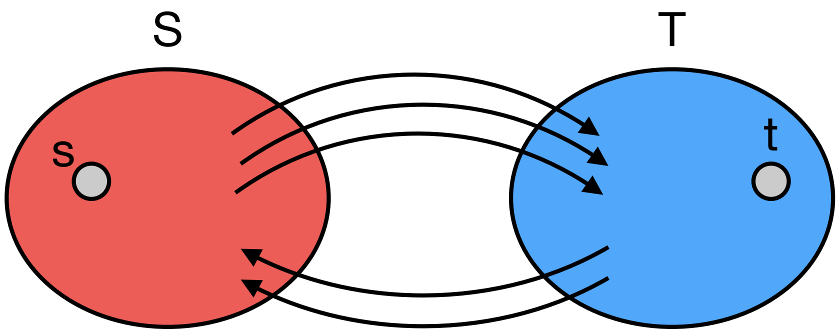

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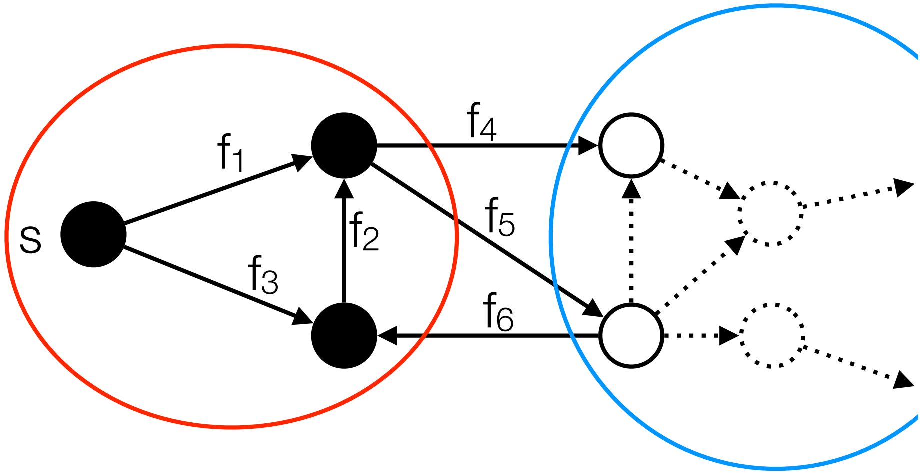

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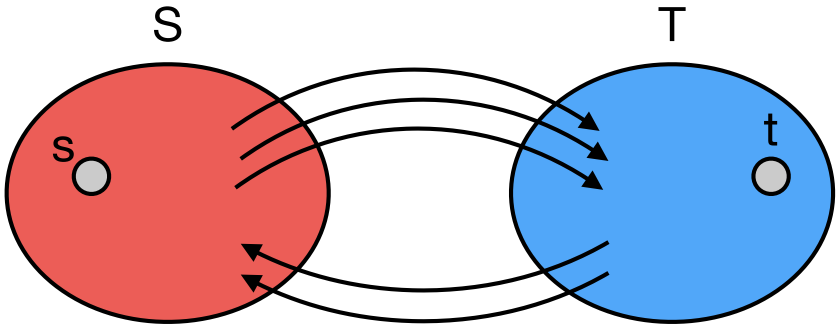

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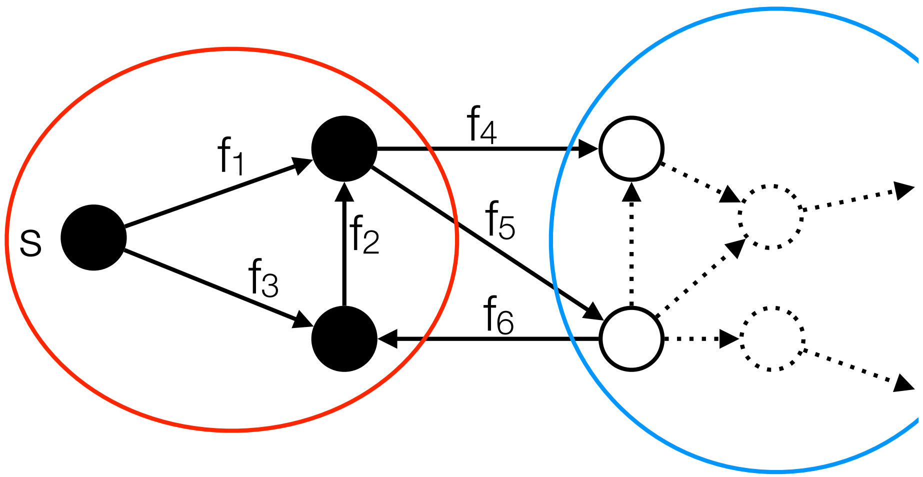
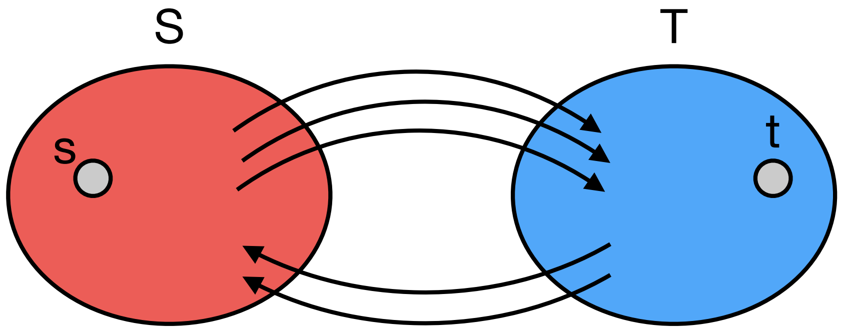
  - $f_1 + f_3 = |f|$



s-t Cuts

- Cut: Partition of vertices into S and T, such that $s \in S$ and $t \in T$.



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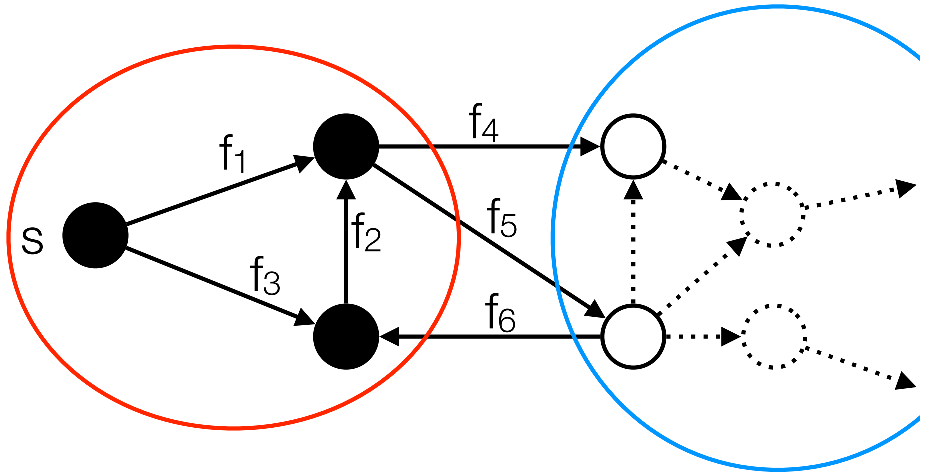
  - $f_2 - f_6 - f_3 = 0$
  - $f_1 + f_3 = |f|$
  - $(f_4 + f_5 - f_1 - f_2) + (f_2 - f_6 - f_3) + (f_1 + f_3) = |f|$



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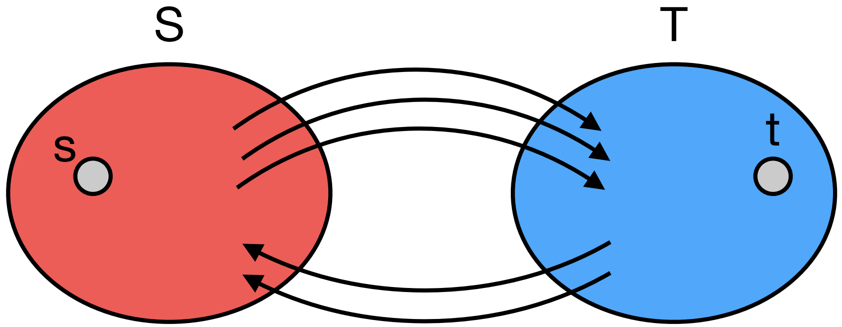

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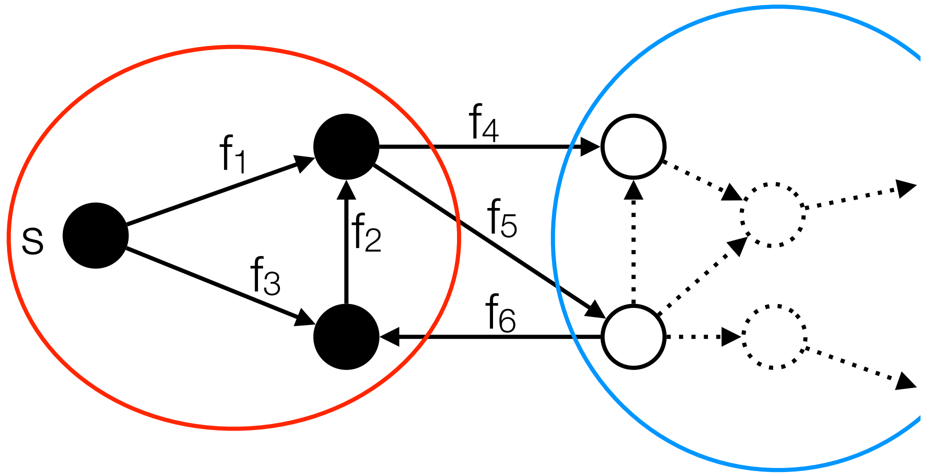
  - $f_1 + f_3 = |f|$
  - $(f_4 + f_5 - \cancel{f_1} - f_2) + (f_2 - f_6 - f_3) + (\cancel{f_1} + f_3) = |f|$



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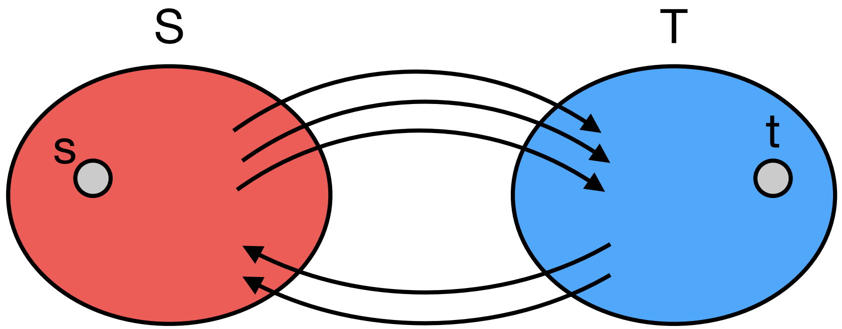
- Cut: Partition of vertices into S and T, such that $s \in S$ and $t \in T$.



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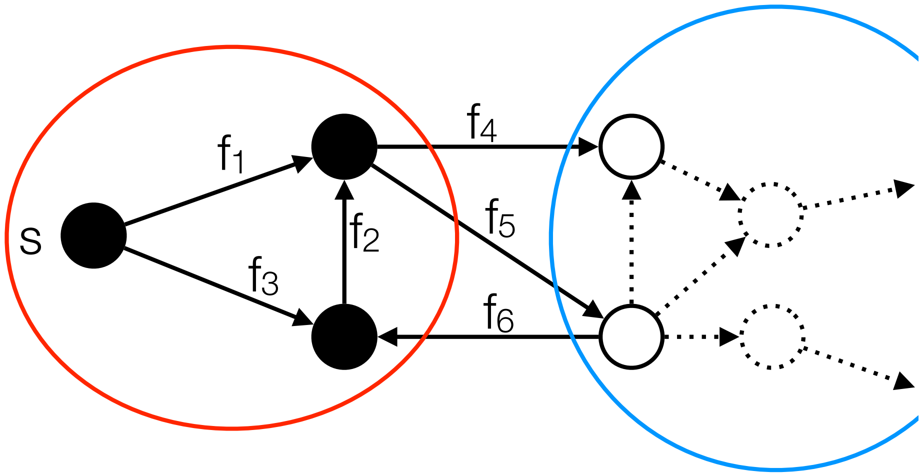
  - $f_1 + f_3 = |f|$
  - $(f_4 + f_5 - f_1 - f_2) + (f_2 - f_6 - f_3) + (f_1 + f_3) = |f|$



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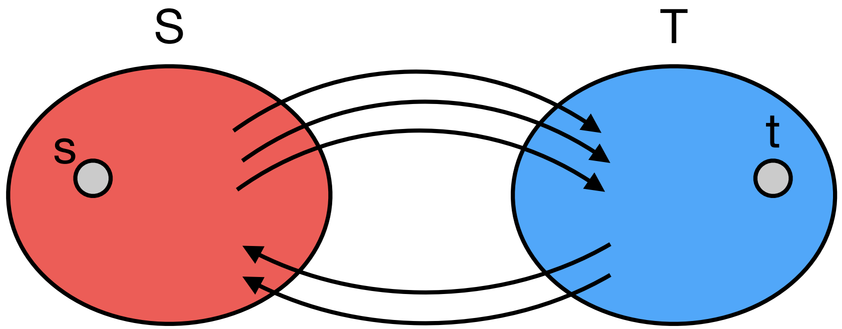
- Cut: Partition of vertices into S and T, such that $s \in S$ and $t \in T$.



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  - $f_1 + f_3 = |f|$
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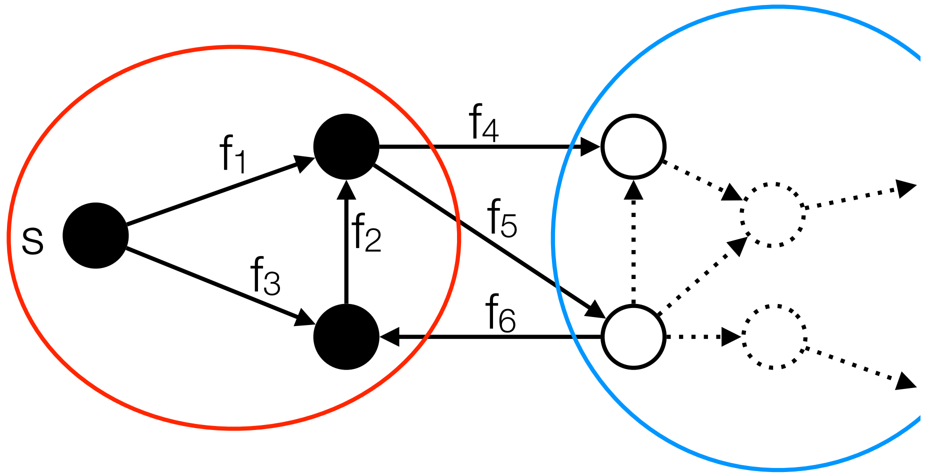

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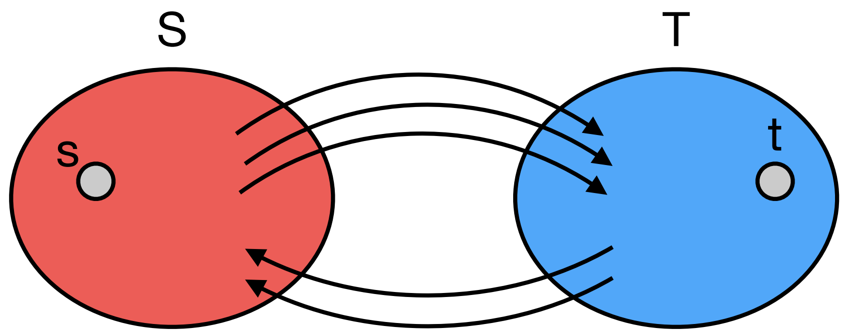

- Flow across cut = flow *from* S to T *minus flow from* T to S.
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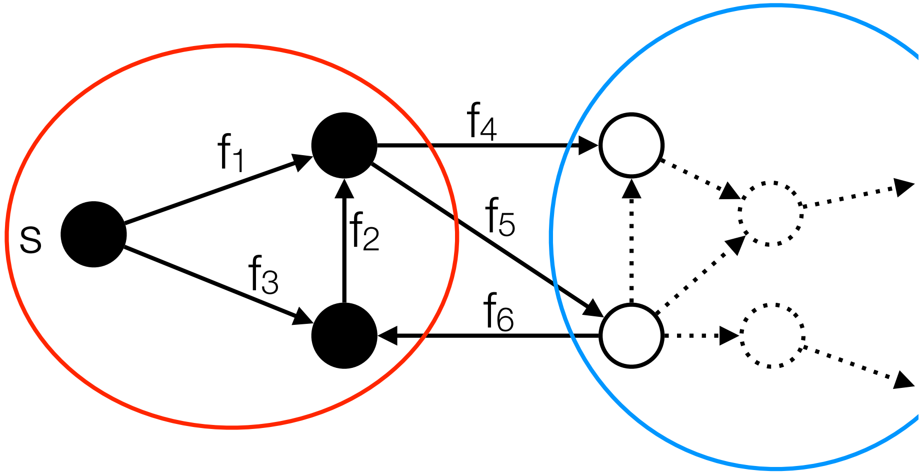

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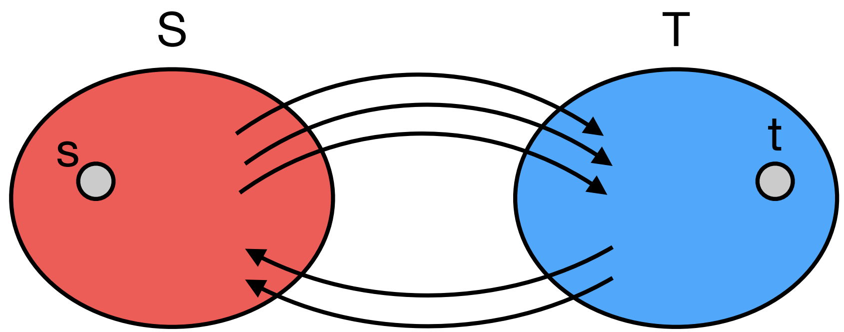

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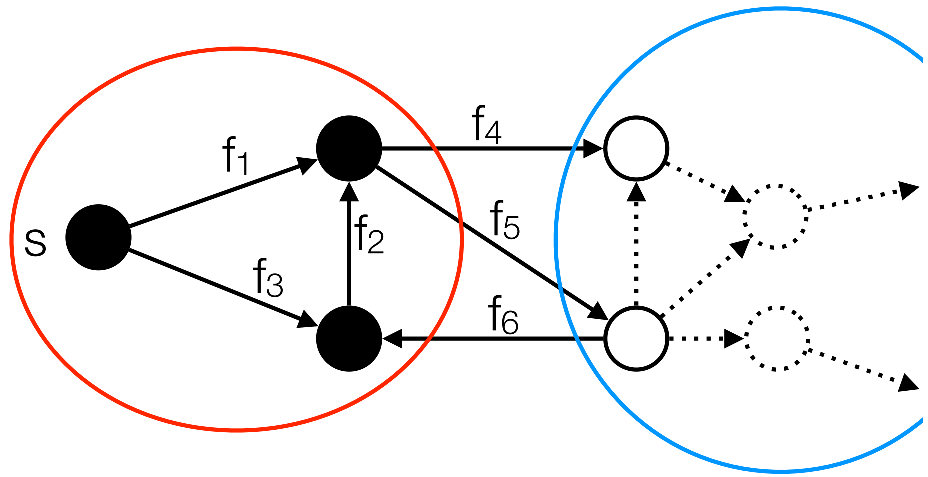

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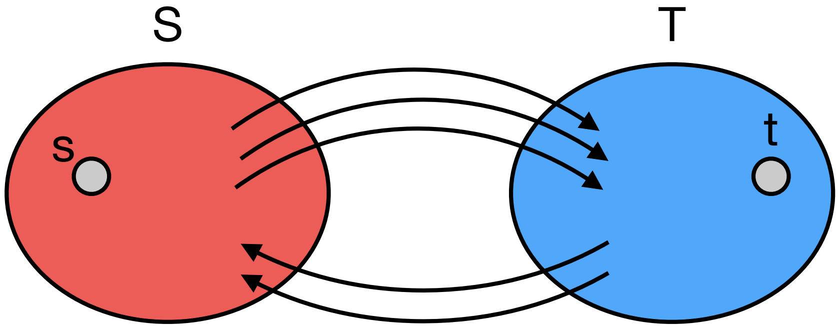

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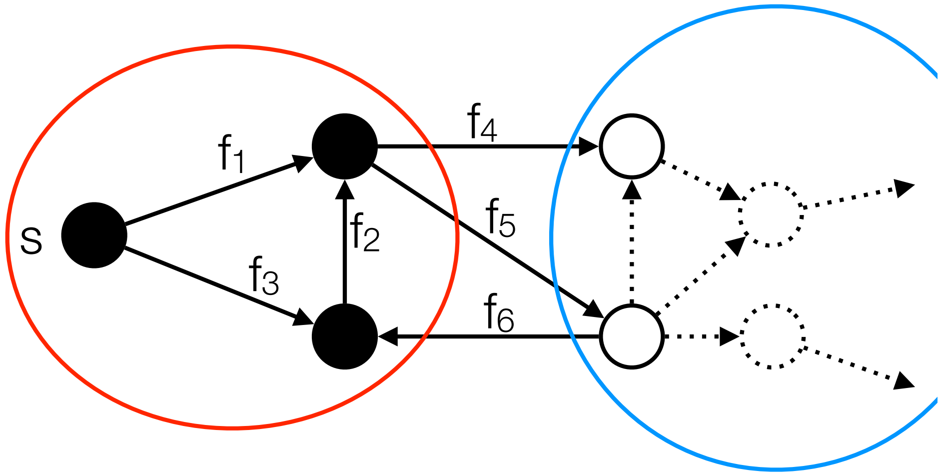

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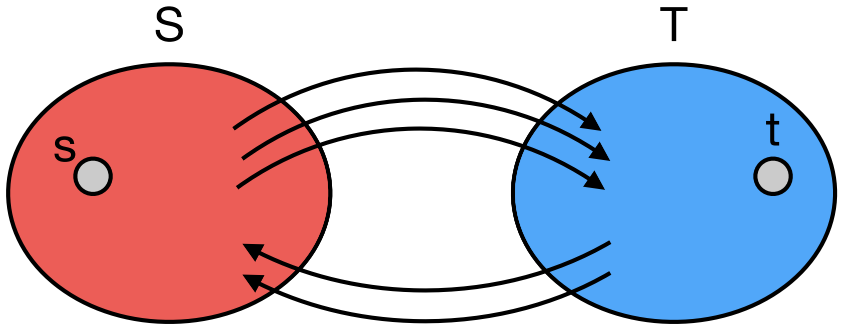

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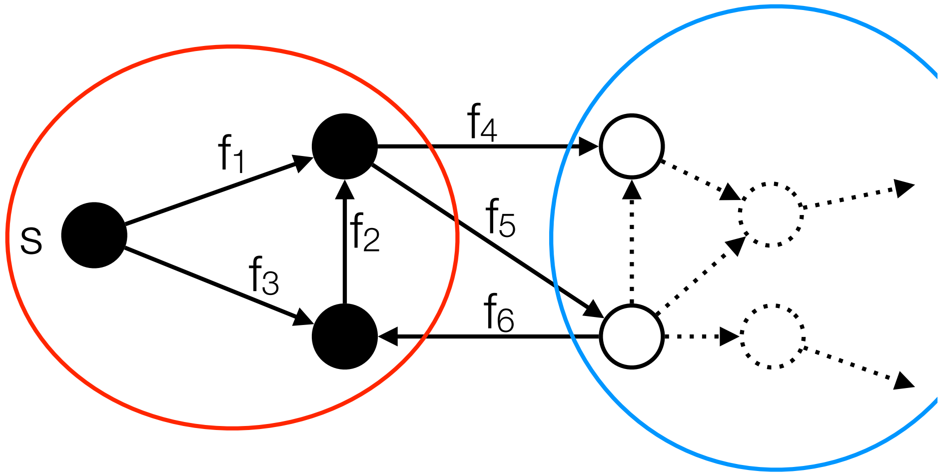

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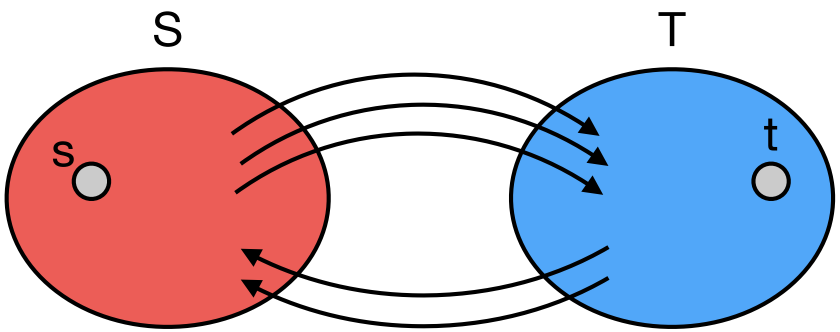

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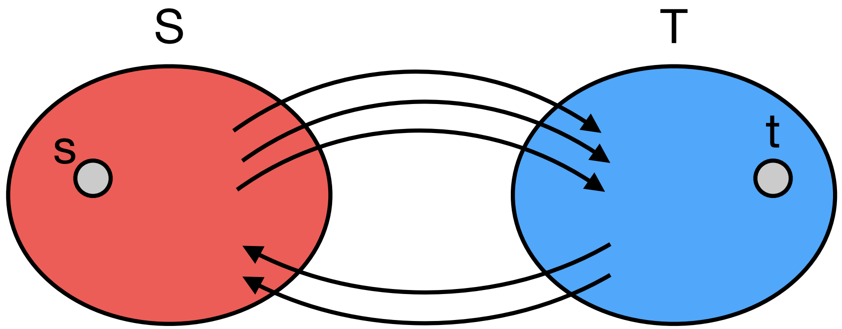

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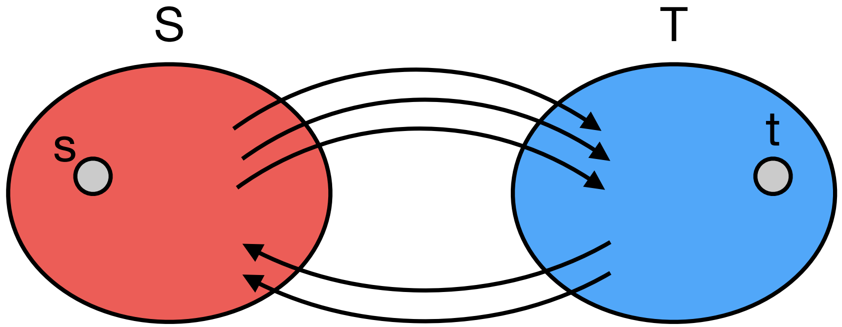

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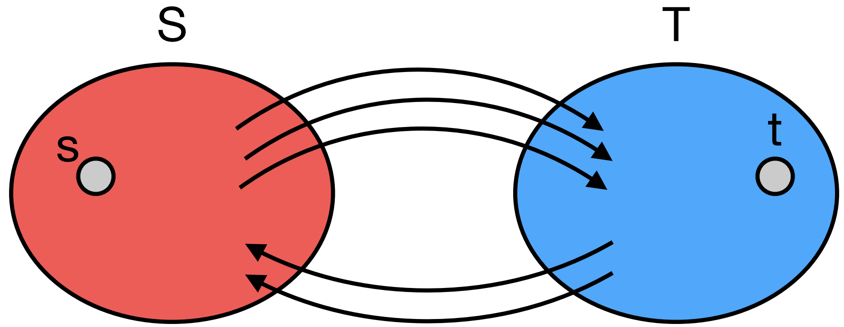

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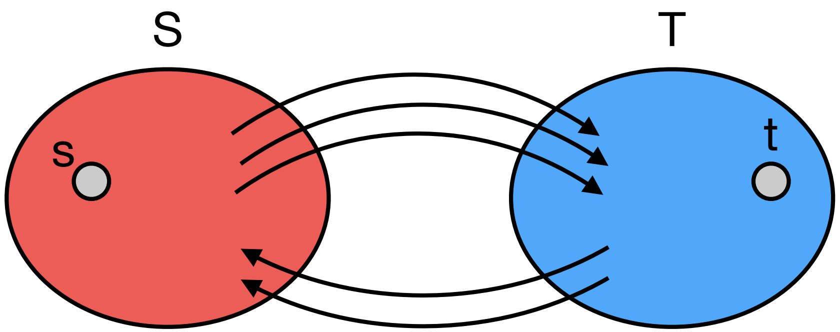

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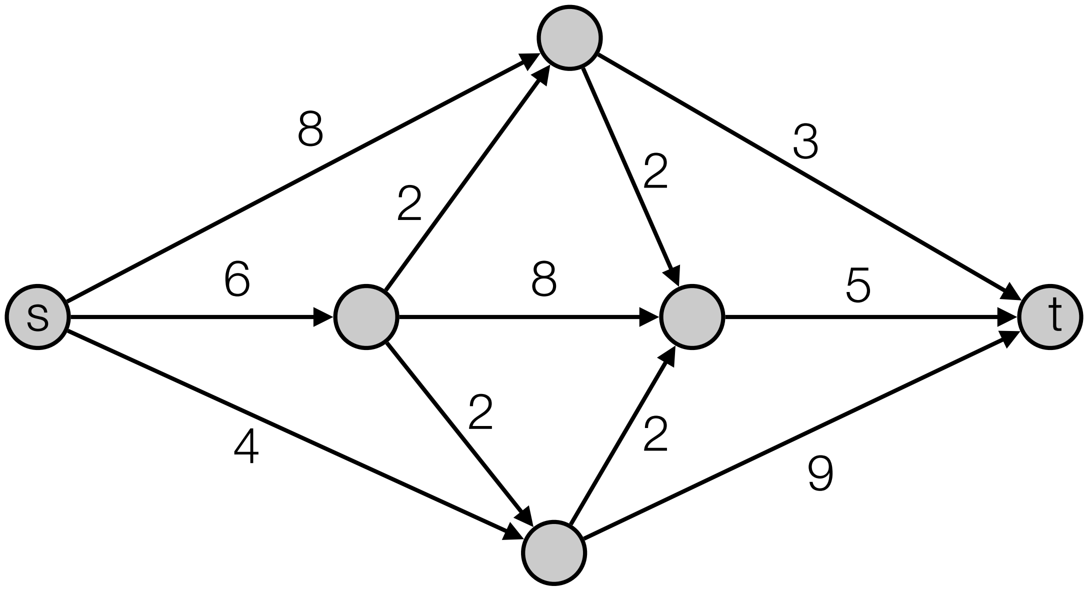
Finding minimum cuts



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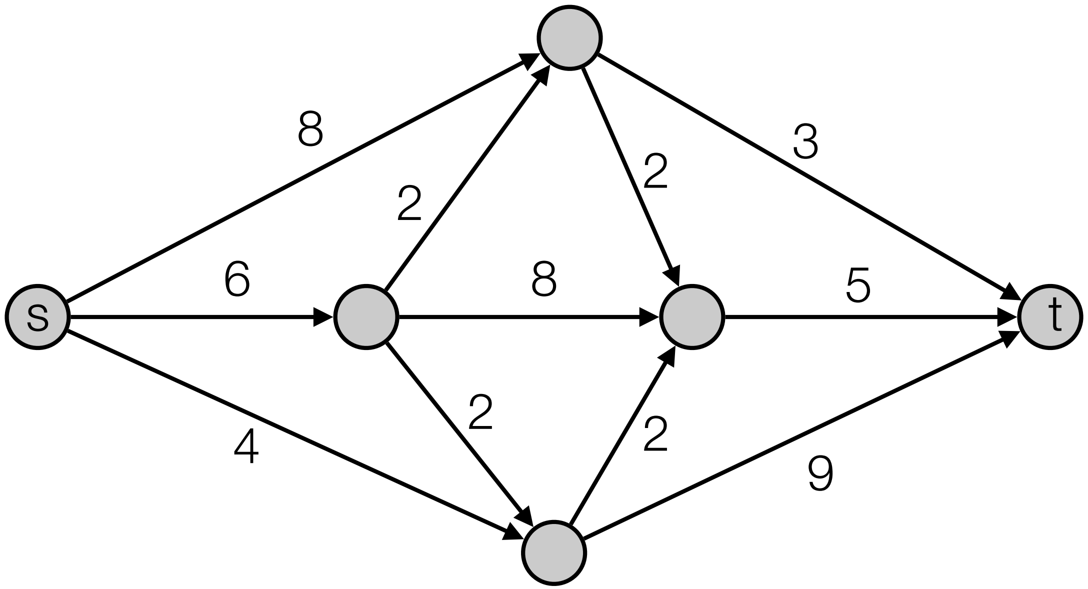

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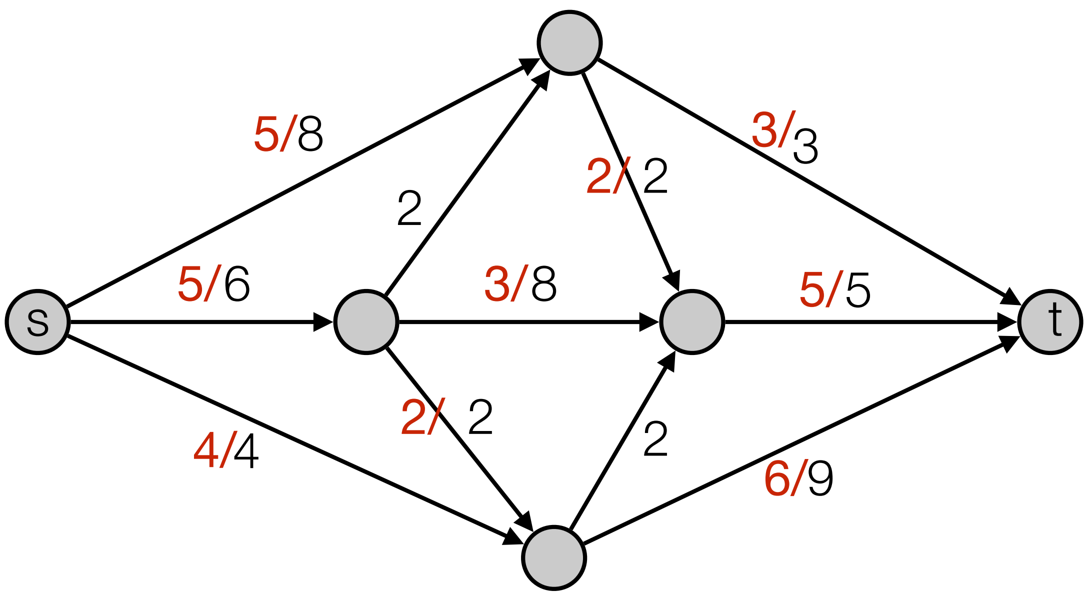

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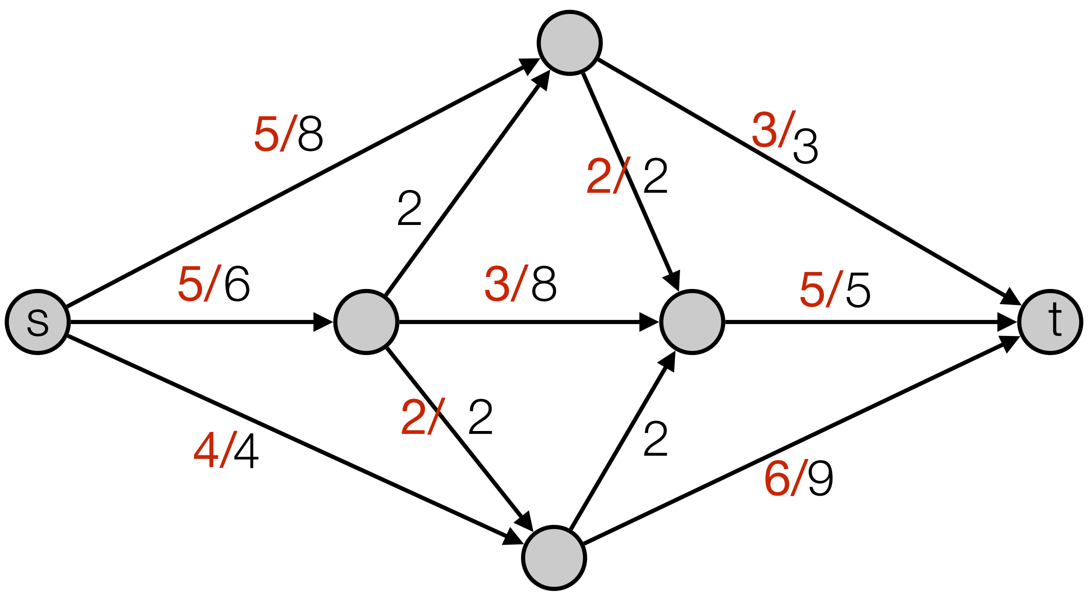

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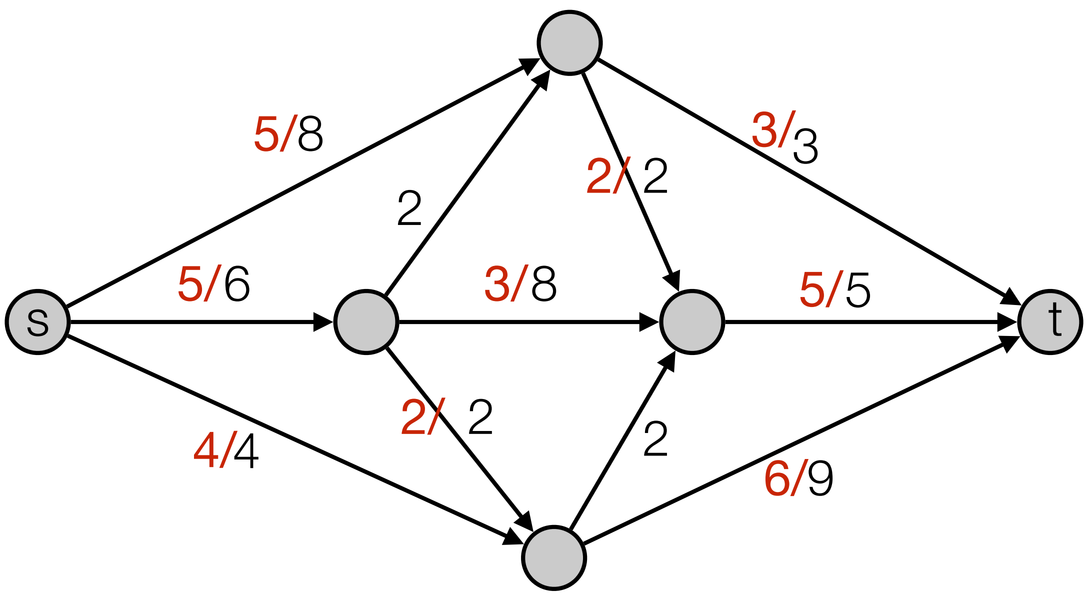

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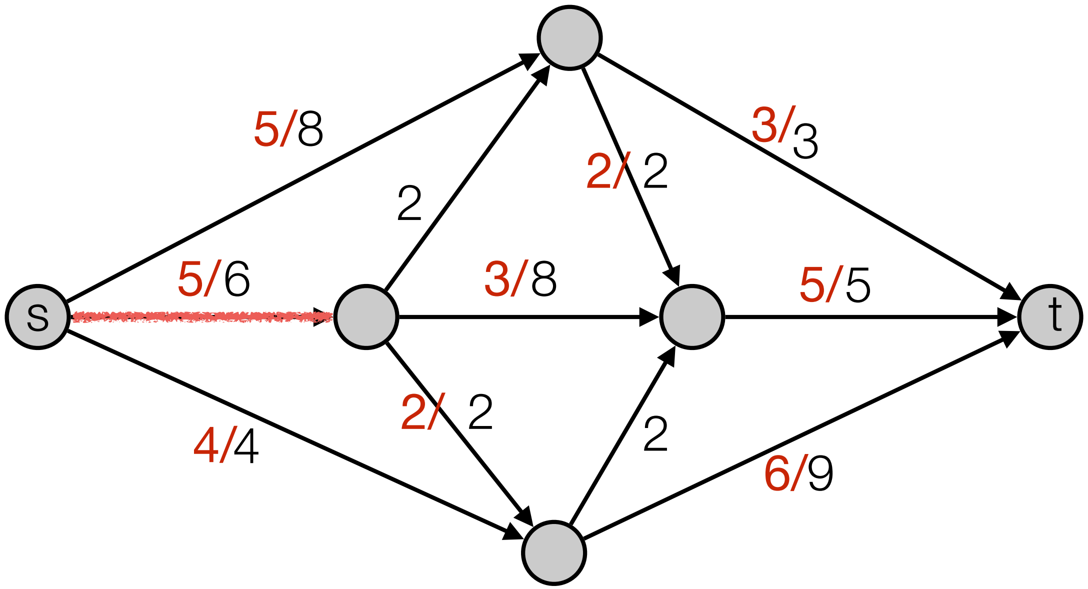

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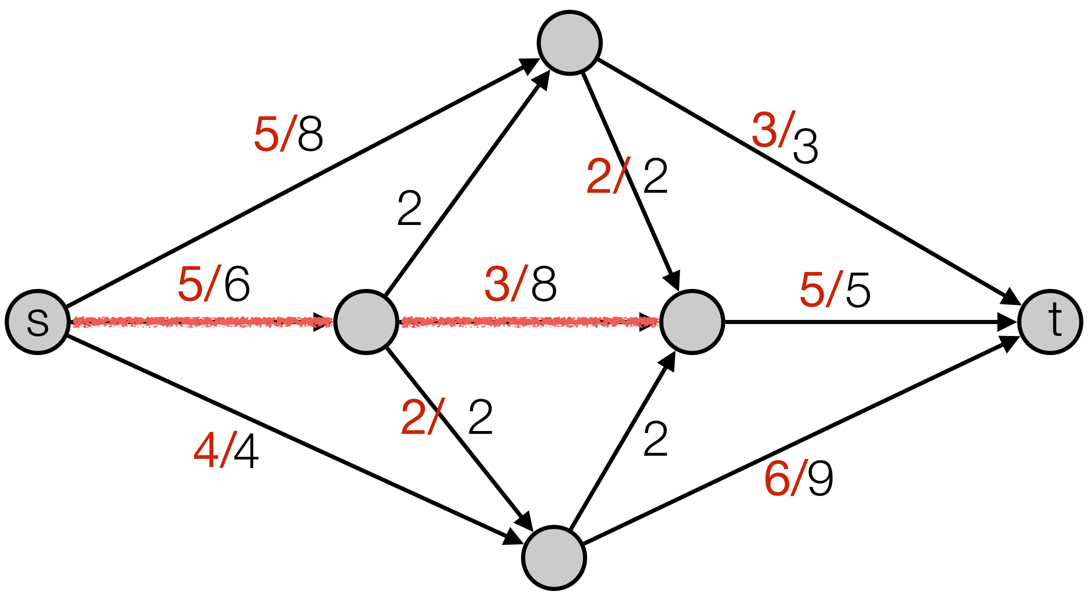

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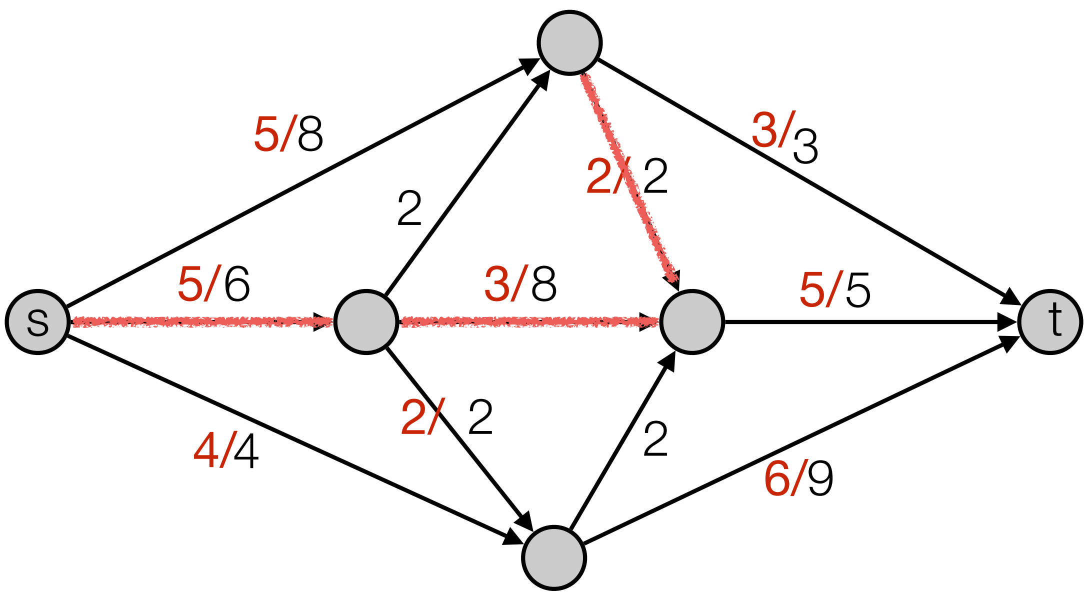

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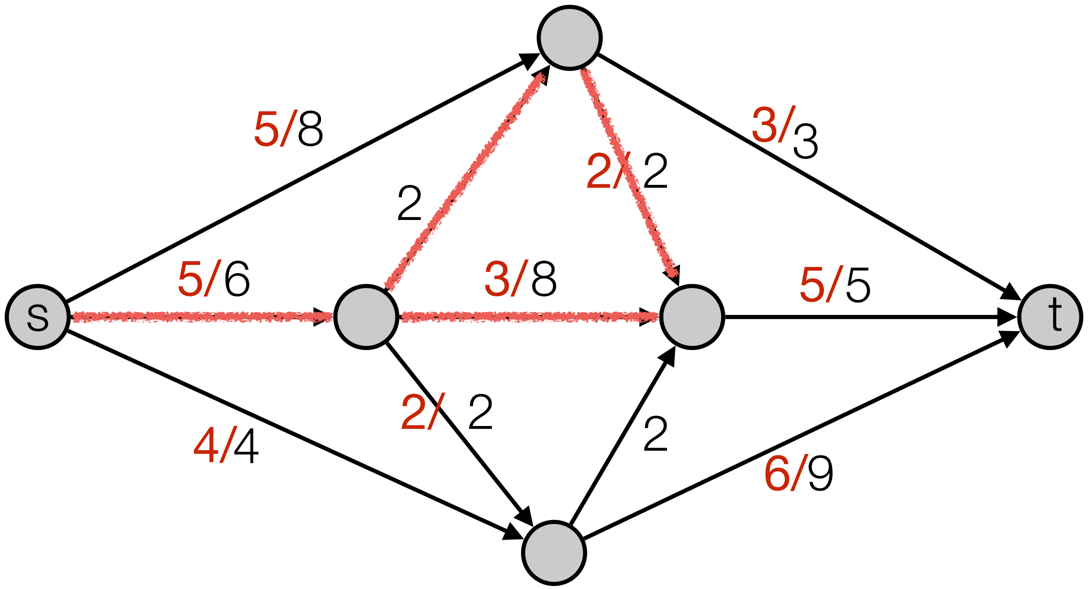

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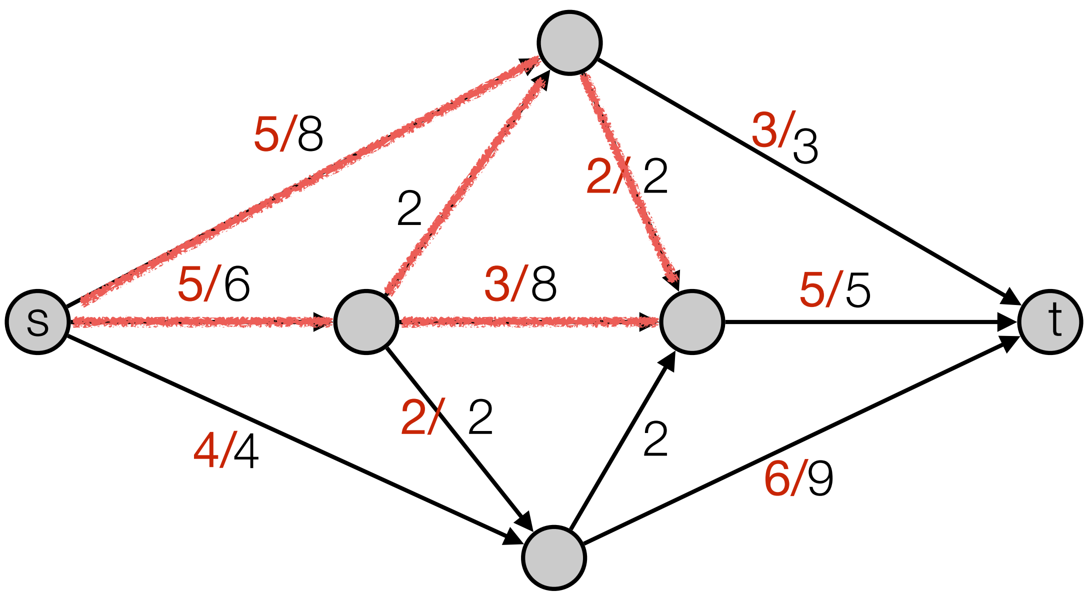

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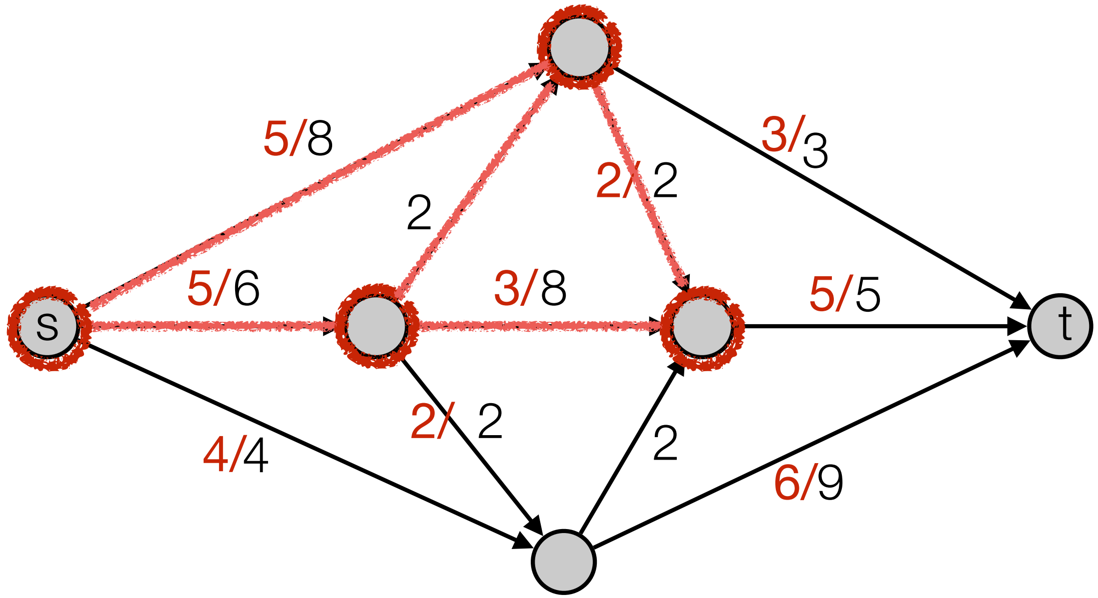

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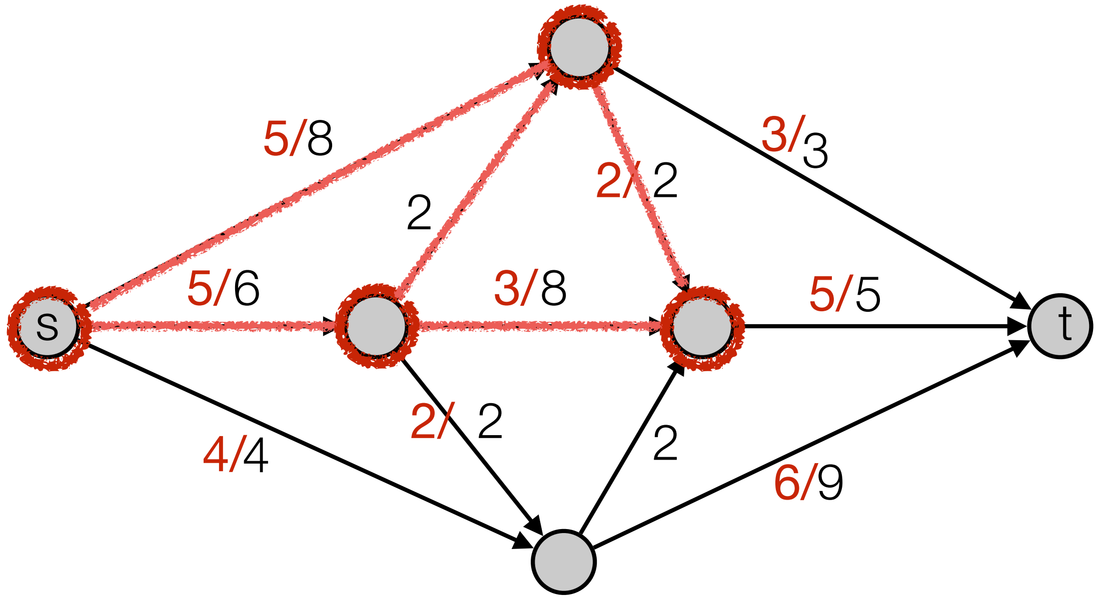

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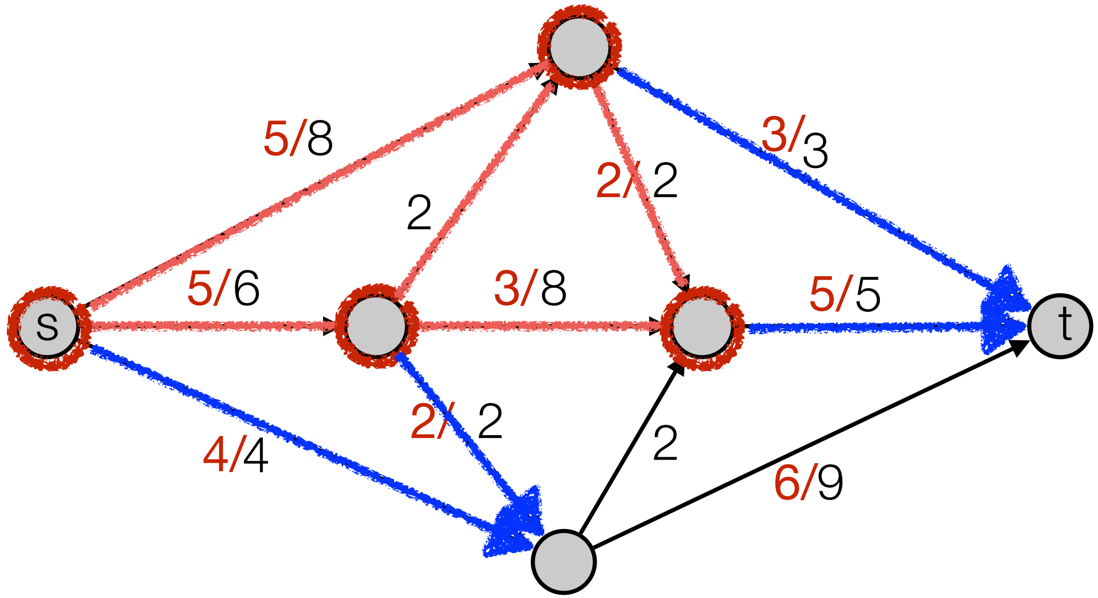

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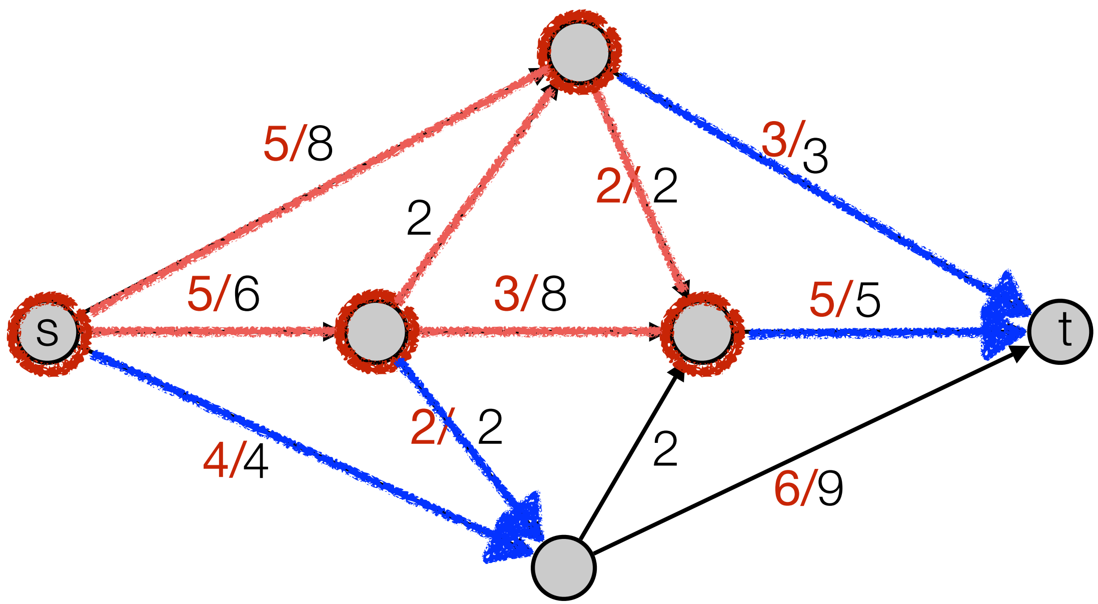

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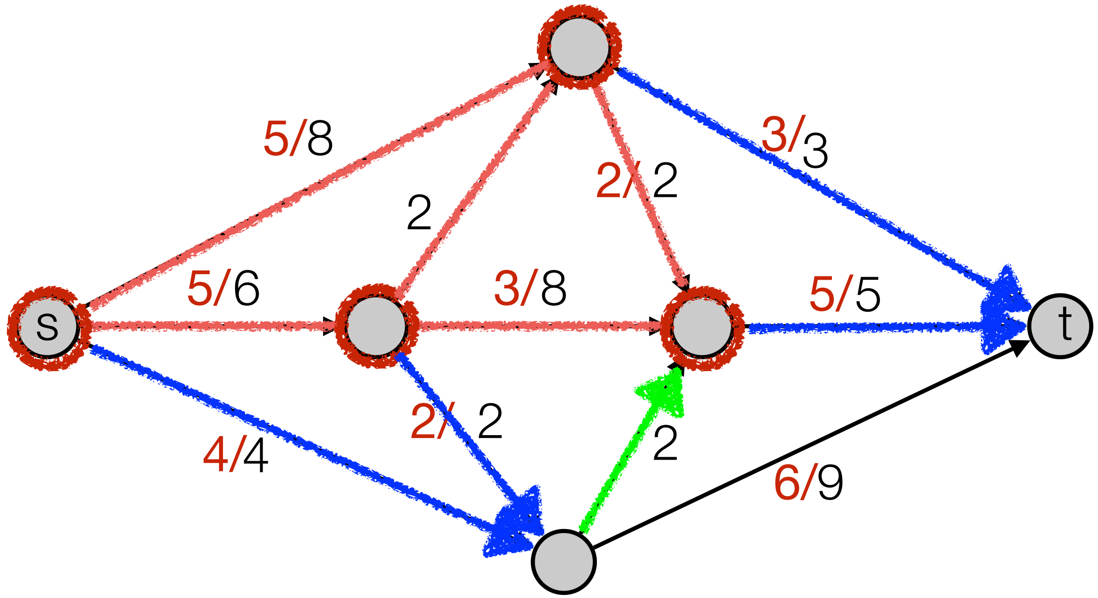

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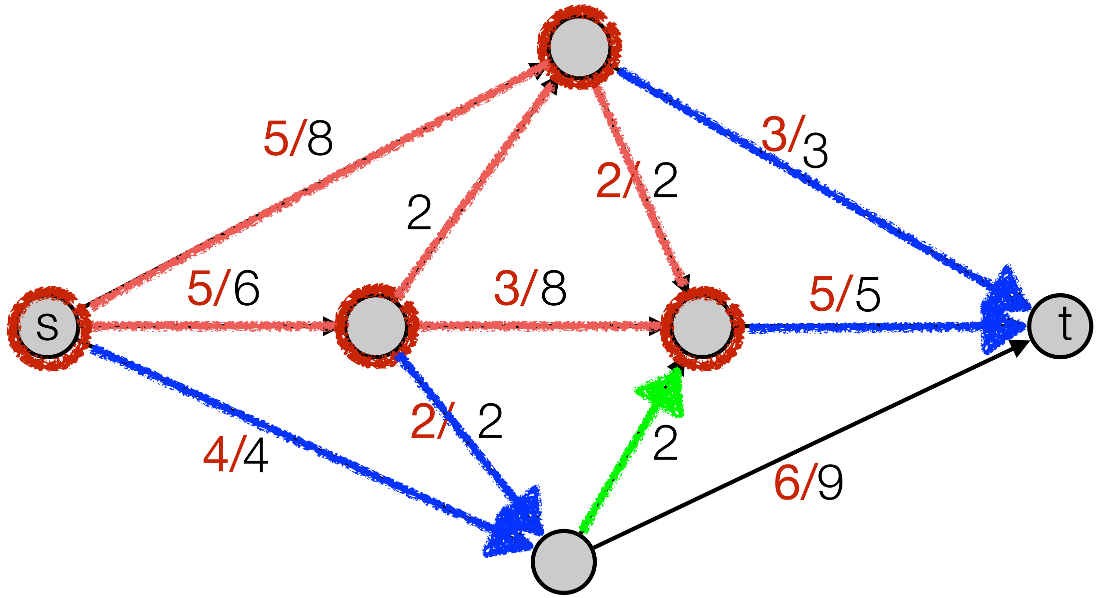

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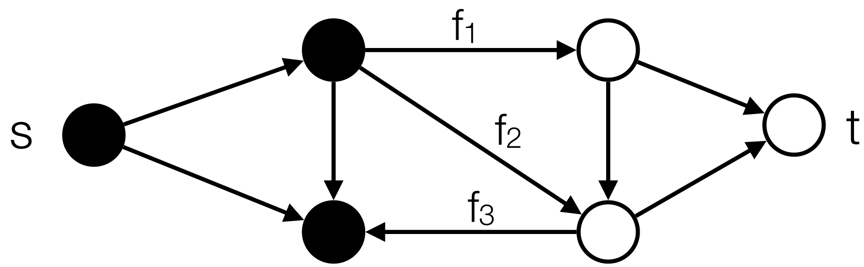

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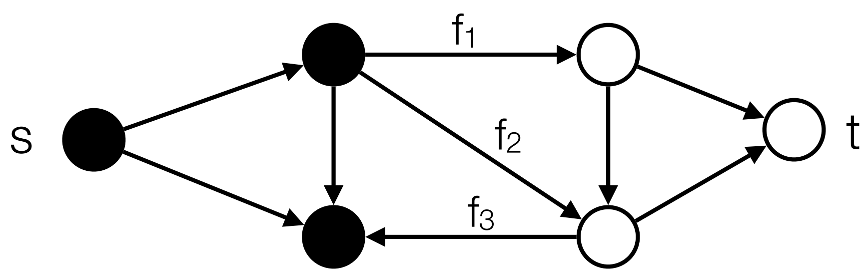

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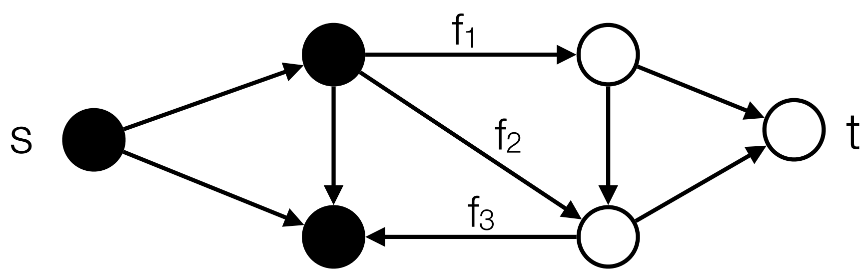

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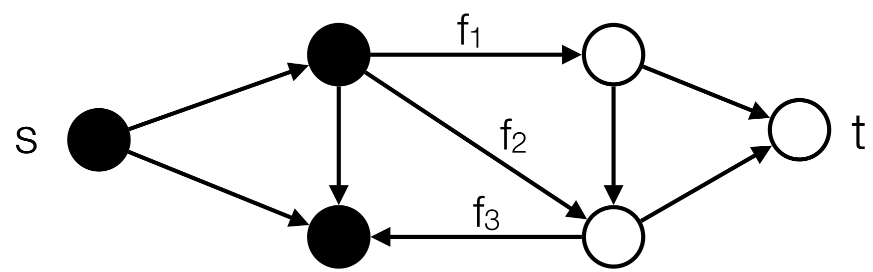

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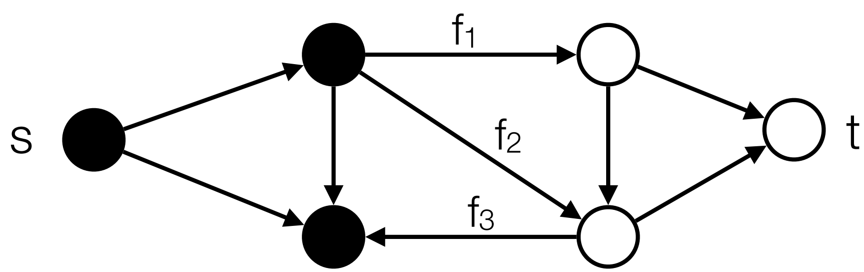

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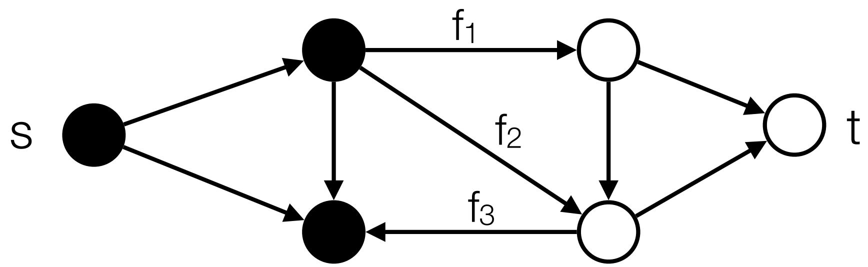


Removing assumptions

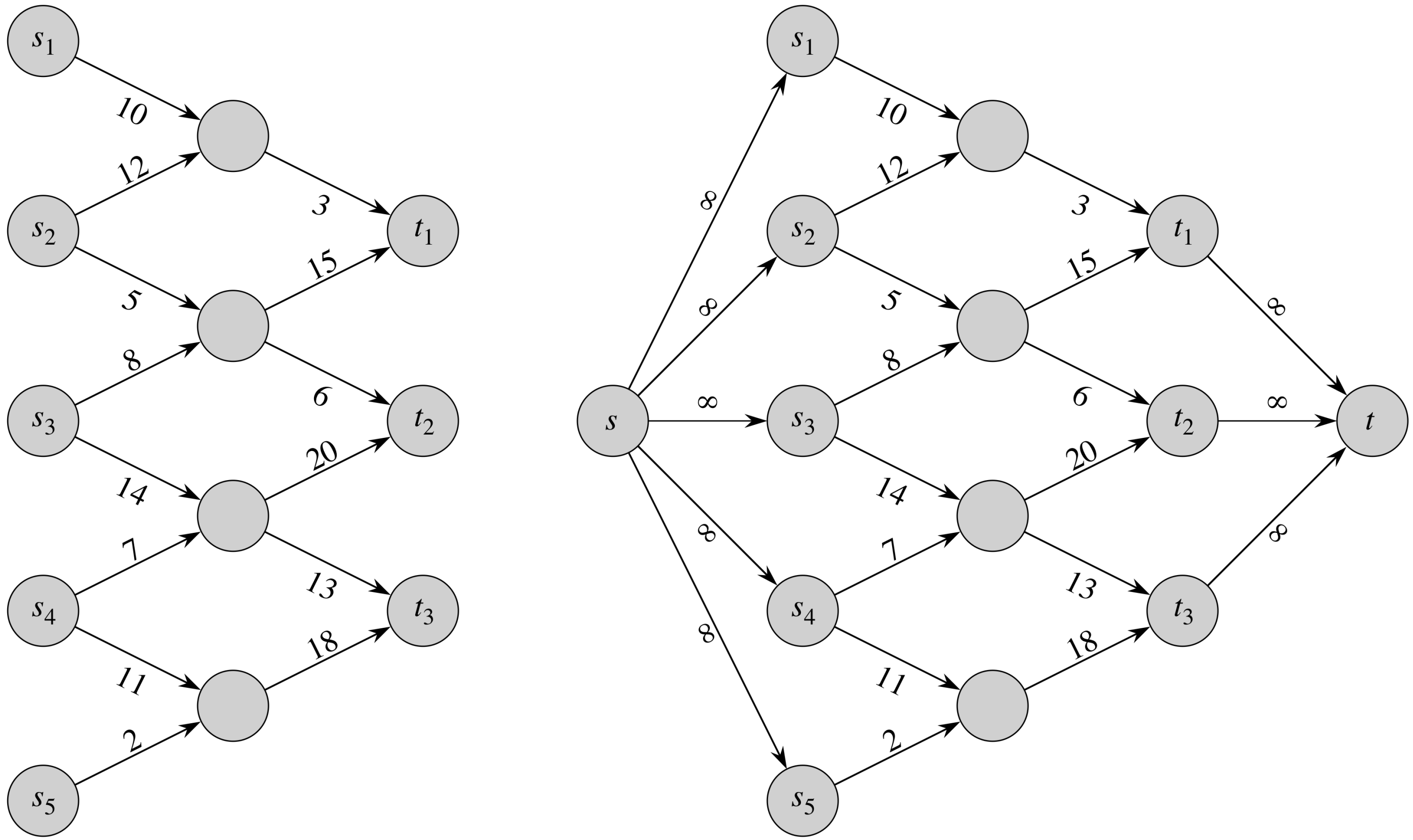
- Edges into s and out of t:

$$v(f) = f^{out}(s) - f^{in}(s)$$

- Capacities not integers.

Network Flow

- Multiple sources and sinks:



(a) (b)