Technical University of Denmark

Written exam, December 11, 2020.

Course name: Algorithms and data structures II.

Course number: 02110.

Aids allowed: All aids allowed. NO open internet.

Exam duration: 4 hours

Weighting: Question 1: 29% - Question 2: 12% - Question 3: 15% - Question 4: 26% - Question 5: 18%

The weighting is only an approximative weighting.

You can answer the exam in either Danish or English.

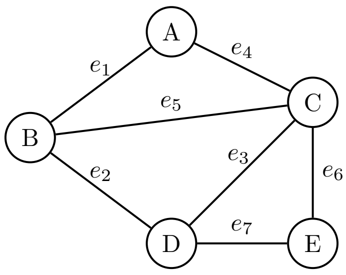
All questions should be answered by filling out the box below the question.

As exam paper just hand in this and the following pages filled out.

The exam set continues on the next page

Question 1

Question 1.1 (9%) Consider the unweighted graph G below.



Question 1.1.1 What is the size of a minimum cut in the graph? Give a minimum cut as a partitioning of the vertices. How many minimum cuts are there?

Solution:

Size of a minimum cut: 2

Minimum cut (partition of the vertices): $\{A\}, \{B, C, D, E\}$ or $\{A, B, C, D\}, \{E\}$

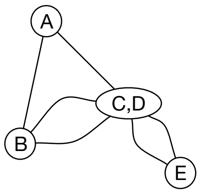
Number of minimum cuts: 2

Question 1.1.2 Consider running the randomized Min-Cut algorithm on the graph G. Assume we first contract edge e_3. What is the probability that B and C are in the same contracted node after the next step? Explain your answer.

Solution:

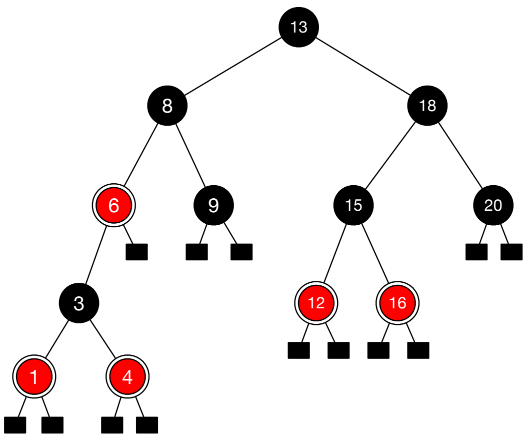
Probability: 1/3

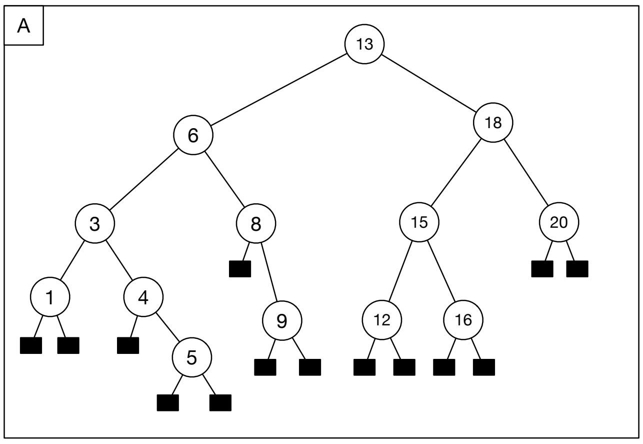
After the contraction the graph looks like this:

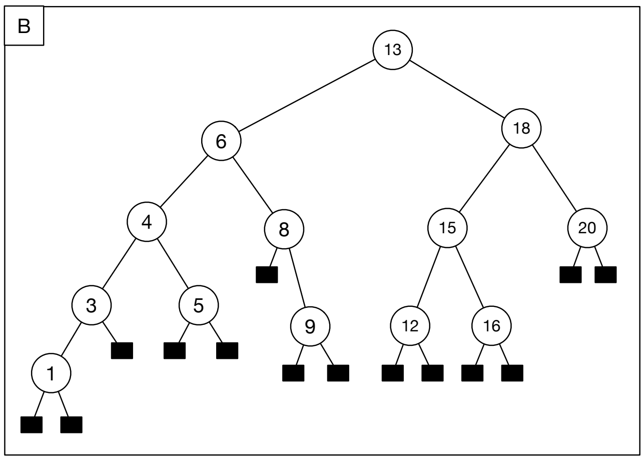


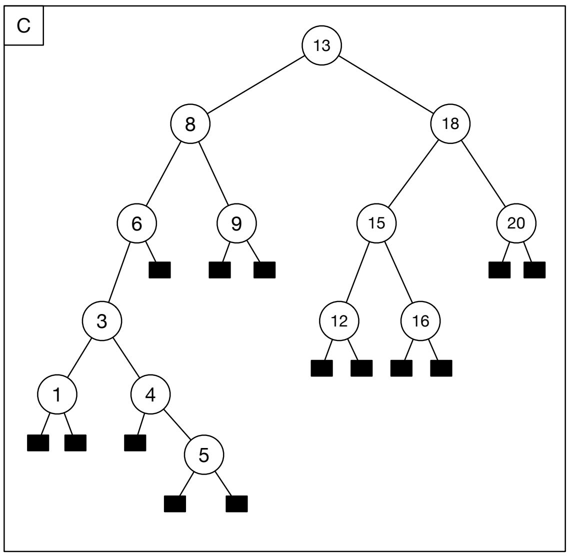
The probability that B and C are in the same component after the next step is thus the same as the probability that we contract either e_2 or e_5 in the next step. This is $2/6 = 1/3$.

Question 1.2 (7%) How does the red-black tree below look after inserting 5? Choose the correct tree and color the nodes the color they have after the insertion. (You can either color the nodes, write R or B next to each node, or clearly mark the red and black nodes in some other way.)

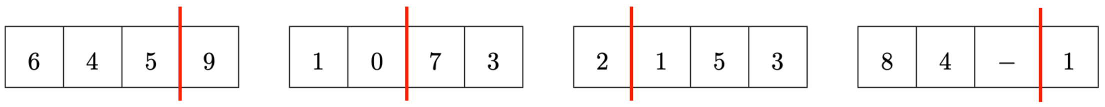






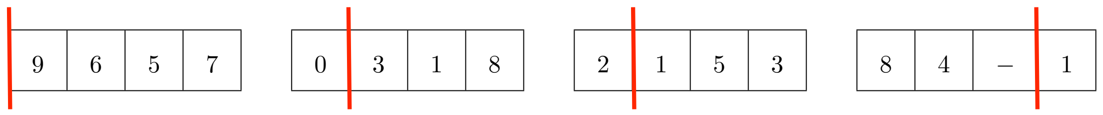


Question 1.3 (7%) Consider a 2-level rotated array data structure below.



Show the result of applying the operation INSERT(7, 8) and then the operation DELETE(2) in that order. (Recall that INSERT(i, x) inserts a new entry with value x immediately *to the left* of entry i.)

Solution:



Here I have rebuild the array that we insert/delete in when inserting/deleting, so that its offset is 0 after the rebuild.

Question 1.4 (6%) A sequence of n operations is performed on a data structure. The cost $T(i)$ of the ith operation is:

$$T(i) = \begin{cases} i^2 & \text{if } i \text{ is a power of 2} \\ 1 & \text{otherwise} \end{cases}$$

The worst case cost of an operation is:

A $\Theta(1)$ B $\Theta(n)$ [✓ C] $\Theta(n^2)$ D $\Theta(2^n)$ E $\Theta(2^{2n})$

The amortized cost of an operation is:

A $\Theta(1)$ B $\Theta(\sqrt{n})$ [✓ C] $\Theta(n)$ D $\Theta(n^2)$ E $\Theta(2^n)$

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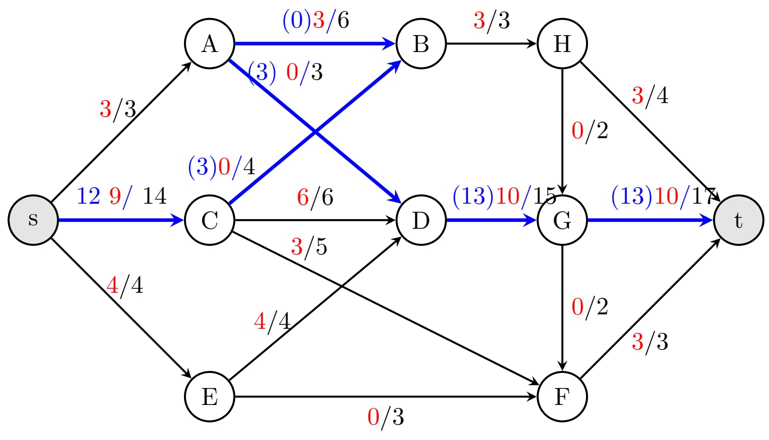
Question 2 (12%)

Consider the network N below with capacities and flow on the edges. The red numbers are the flow values and the black are the capacities.



Question 2.1 (7%) Is this a maximum $s-t$ flow? If not give an augmenting path (write the nodes of the path).

Solution:

Augmenting path (if no augmenting path exists write None): $sCBADGt$

Value you can augment with on path: 3

Question 2.2 (5%) Give a $s-t$ minimum cut in the network N.

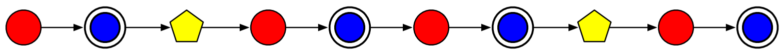
Solution:

Minimum cut (write the partition of the nodes): $\{s, C, B, F\}, \{A, D, E, H, G, t\}$ or $\{s, A, B, C, E, F\}, \{D, G, H, t\}$

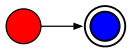
Value of the minimum cut: 19

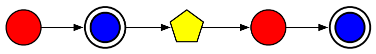
Question 3 (15%)

Paul loves pearls and has a lot of strings of pearls with pearls of different colors. Paul especially likes strings of pearls that start and end in the same way. Let S be a string of pearls of length n. If the first i pearls ($i < n$) and the last i pearls S are the same he says that the string of pearls has a *pretty end* of length i. More formally, if $S = p_1 p_2 \cdots p_n$ then there is a pretty end of length $i < n$ if and only $p_1 \cdots p_i = p_{n-i+1} \cdots p_n$. For example, the string of pearls

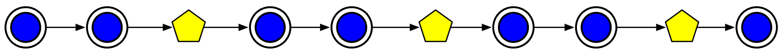


has two pretty ends: one of length 2 and one of length 5.





Question 3.1 (5%) Compute the length of all pretty ends of the string $S = bbybbybbyb$:



Solution:

S has pretty ends of length(s): $1, 4, 7$

Question 3.2 (10%) Give an efficient algorithm for computing the length of all pretty ends of a string S of length n. Analyze the running time of your algorithm in terms of n and argue that it is correct.

Solution:
Construct the KMP automaton. Follow the failure pointers backwards from the accepting state. All states reached corresponds to a pretty end.

Running time: $O(n)$. It takes $O(n)$ time to construct the automaton and $O(n)$ time to follow the failure pointers back.

Correctness: The first failure pointer we follow points to the longest prefix p_1 of S that is a suffix of S. Thus this is the longest pretty end of S. Now all other pretty ends are shorter and must therefore also be suffixes of p_1. The failure pointer from the state of p_1 therefore points to the second longest pretty end of S.

Solution 3.2 continued:

The exam set continues on the next page

Question 4 (26%)

Consider the following simple 2-player card game. At the beginning of the game, the n cards are dealt face up in a long row. Each card is worth a different number of points. After all the cards are dealt, the players take turns removing either the leftmost or rightmost card from the row, until all the cards are gone. At each turn, you can decide which of the two cards to take. The winner of the game is the player that has collected the most points when the game ends.

Question 4.1 (6%) Having never taken an algorithms class, your friend Donald follows the obvious greedy strategy—when it's his turn, Donald always takes the card with the higher point value.

Prove that you should not also use the greedy strategy when you play against Donald. That is, give an example of a sequence where you can win, but only if you do not follow the same greedy strategy as Donald.

Solution:
Consider the sequence $10, 1, 100, 20$.

If you start and use the greedy strategy you get 20 points in the first round. Now Donald pick 100, you pick 10 and he picks 1. You get 30 and Donald gets 101. If you had picked 10 first, then Donald picks 20 and you can then pick 100 getting 110 points.

Question 4.2 (5%) Let $R = [r_1, r_2, \ldots, r_n]$ be the row of cards, that is $R[i]$ is the ith card from the left. In this exercise we want to compute the maximum number of points that you can collect playing against Donald.

Let $L[i, j]$ be the maximum number of points that you can get if you play against Donald on the sequence of cards $R[i \ldots j]$ if it is your turn first.

E.g. if $R = [5, 3, 4, 7, 9, 1, 2, 6]$ then $L[3, 5] = 13$ as $R[3 \ldots 5] = [4, 7, 9]$ and you get $9 + 4 = 13$ points if you pick 9 in your first turn, but only $4 + 7 = 11$ if you pick 4 first.

Fill out the table below for the row $R = [7, 4, 3, 5, 6]$.

$i \setminus j$	1	2	3	4	5
1	7	7	10	11	15
2		4	4	8	10
3			3	5	9
4				5	6
5					6

Question 4.3 (5%) Which of the following recurrences correctly computes $L[i, j]$:

A
$$L[i,j] = \begin{cases} 0 & \text{if } i > j \\ R[i] & \text{if } i = j \\ \max\{R[i], R[j]\} & \text{if } i+1 = j \\ R[i] + \max\{L[i+2, j], L[i, j-1]\} & \text{if } R[i] > R[j] \text{ and } i < j-1 \\ R[j] + \max\{L[i+1, j-1], L[i, j-2]\} & \text{if } R[i] < R[j] \text{ and } i < j-1 \end{cases}$$

B
$$L[i,j] = \begin{cases} 0 & \text{if } i > j \\ R[i] & \text{if } i = j \\ \max\{R[i], R[j]\} & \text{if } i+1 = j \\ \max\{R[i] + L[i+1, j-1], R[j] + L[i, j-2]\} & \text{if } R[i+1] < R[j-1] < R[j] \text{ and } i < j-1 \\ \max\{R[i] + L[i+1, j-1], R[j] + L[i+1, j-1]\} & \text{if } R[j-1] < R[i+1] < R[j] \text{ and } i < j-1 \\ \max\{R[i] + L[i+2, j], R[j] + L[i, j-2]\} & \text{if } R[j-1] < R[i] < R[i+1] \text{ and } i < j-1 \\ \max\{R[i] + L[i+2, j], R[j] + L[i+1, j-1]\} & \text{if } R[i] < R[j-1] < R[i+1] \text{ and } i < j-1 \end{cases}$$

[✓ C]
$$L[i,j] = \begin{cases} 0 & \text{if } i > j \\ R[i] & \text{if } i = j \\ \max\{R[i] + L[i+1, j-1], R[j] + L[i, j-2]\} & \text{if } R[i+1] \leq R[j], R[i] \leq R[j-1] \text{ and } i < j \\ \max\{R[i] + L[i+1, j-1], R[j] + L[i+1, j-1]\} & \text{if } R[i+1] \leq R[j], R[i] > R[j-1] \text{ and } i < j \\ \max\{R[i] + L[i+2, j], R[j] + L[i, j-2]\} & \text{if } R[i+1] > R[j], R[i] \leq R[j-1] \text{ and } i < j \\ \max\{R[i] + L[i+2, j], R[j] + L[i+1, j-1]\} & \text{if } R[i+1] > R[j], R[i] > R[j-1] \text{ and } i < j \end{cases}$$

Question 4.3 (10%) Write pseudocode for an algorithm *based on dynamic programming and the recurrence you chose in Question 4.2* that given a row R computes the maximum number of points you can get playing against Donald if you take the first turn. Analyze the space usage and running time of your algorithm in terms of n.

```
COMPUTE-MAXWEIGHT(R[1...n])
   L := $n \times n$ table
   for i=1 to n do
      L[i, i] := R[i]
   for i = 1 to n do
      L[i,i+1] = max(R[i],R[i+1])
   for i = n downto 1 do
      for j = i+2 to n do
         if R[i+1] $\leq$ R[j] and R[i] $\leq$ R[j-1] do
            L[i,j] := max(R[i] + L[i+1,j-1], R[j] + L[i,j-2])
         else if R[i+1] $\leq$ R[j] and R[i] > R[j-1] do
            L[i,j] := max(R[i] + L[i+1,j-1], R[j] + L[i+1,j-1])
         if R[i+1] < R[j] and R[i] $\leq$ R[j-1] do
            L[i,j] := max(R[i] + L[i+2,j], R[j] + L[i,j-2])
         else if R[i+1] < R[j] and R[i] > R[j-1] do
            L[i,j] := max(R[i] + L[i+2,j], R[j] + L[i+1,j-1])
         end if
      end for
   end for
   return L[1,n]
```

Solution:
Running time $O(n^2)$. Two for loops in linear time then two nested for-loops, where the body takes constant time. Everything else takes constant time.
Spacec $O(n^2)$.

Solution 4.3 continued:

The exam set continues on the next page

Question 5 (18%)

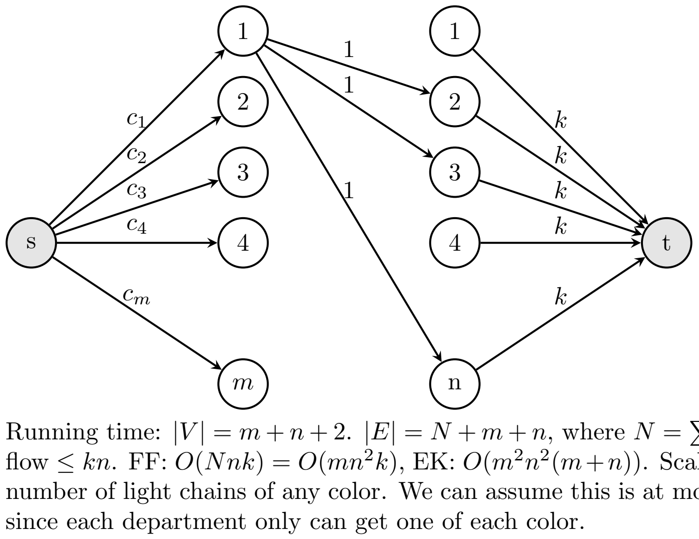
The yearly light fest is coming up. Every year all n departments at the university puts different colored light chains on a tree outside the department. Each department i needs k light chains of different colors and they each have a list L_i of the colors they want to use. Each department should get at most one light chain of each color and they should only receive light chains that has a color that is on their list. Your friends are in charge of distributing the light chains to the departments. The university has a limited number of light chains c_j of each color j and their job is to make all the departments happy. Suppose that $k = 2$, there are $n = 4$ departments and $m = 4$ colors with $c_1 = c_2 = c_3 = c_4 = 2$, that is there are two light chains of each color. The departments lists are $L_1 = L_2 = \{1, 2, 3\}$ and $L_3 = L_4 = \{1, 3, 4\}$. Then one possible way to distribute the light chains such that each department gets at least 2 light chains of different colors are:

- department 1 gets light chains of color c_1 and c_2,
- department 2 gets light chains of color c_2 and c_3,
- department 3 gets light chains of color c_3 and c_4,
- department 4 gets light chains of color c_1 and c_4,

Question 5.1 (10%)

Give an efficient algorithm that takes as input the number of light chains of each color $c_1, c_2, \ldots, c_m$, the lists L_i for each of the n departments, and the parameter k, and decides whether there is a way to distribute the light chains such that each department gets k light chains of different colors. Analyze the time and space usage of your algorithm in terms of n, m, and k. Remember to argue that your algorithm is correct.

Solution:
Construct a flow network. A node u_c for each color c and a node v_d for each department d. There is an edge from u_c to v_d of capacity one if and only if color c is on the list L_d. Construct a source node s with edges to all nodes correspoding to colors. The capacity of the edge (s, u_i) is set to c_i (the number of light chains of color i). Construct a sink node t with edges of capacity k from all nodes corresponding to departments. Now it is possible to distribute the light chains such that each department get k light chains of different colors if and only if there is a flow of value kn. We can use a max flow algorithm to solve the problem.



Running time: $|V| = m + n + 2$. $|E| = N + m + n$, where $N = \sum_{i=1}^{n} |L_i| \leq nm$. The value of the maximum flow $\leq kn$. FF: $O(Nnk) = O(mn^2k)$, EK: $O(m^2n^2(m+n))$. Scaling $O(m^2n^2 \log C)$, where C is the maximal number of light chains of any color. We can assume this is at most n as we can otherwise just scale it down, since each department only can get one of each color.

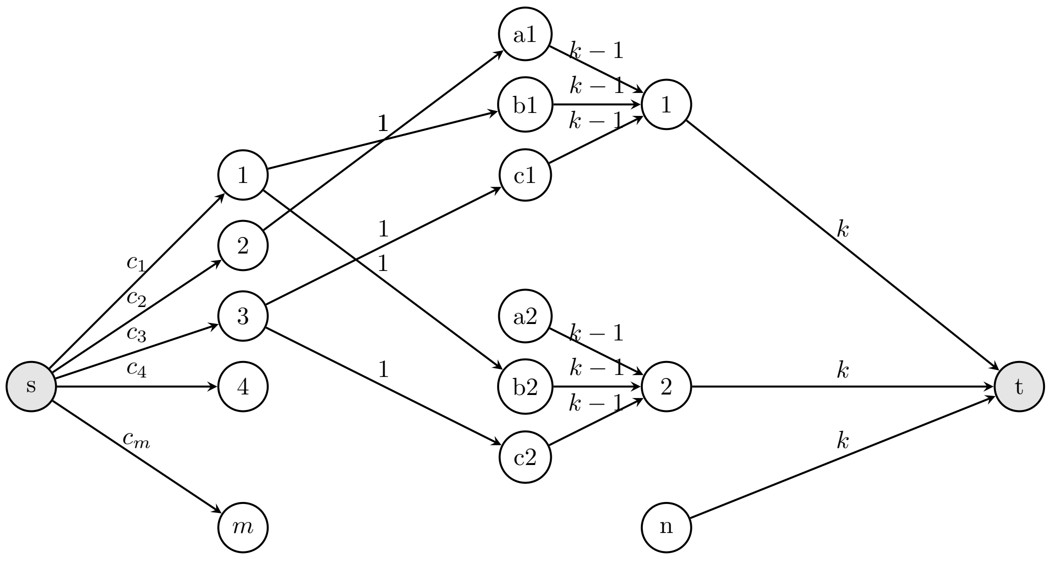
Solution 5.1 continued:

Question 5.2 (8%)

You show your friends a solution computed by your algorithm from Question 5.1, and to your big surprise they tell you that this solution is not at all feasible. It turns out that there are three different types of light chains (star shaped, heart shaped and round) and no department should get light chains of only a single type. Each color only comes in one shape (fx. are all the red light chains heart shaped), but every type is made in many different colors. For example, suppose color 1 is star shaped, color 2 and 3 are heart shaped, and color 4 is round. Then the solution from before does not work as department 2 only gets heart shaped light chains. However, we could give both department 1 and 2 light chains of color 1 and 2, and department 3 and 4 light chains of color 3 and 4.

Give an efficient algorithm that decides whether there exists a solution satisfying all the conditions from Question 5.1, plus the new requirement about types. Analyze the time and space usage of your algorithm in terms of n, m, and k. Remember to argue that your algorithm is correct.

Solution:
Use network from before, but add nodes as below. For each department node construct three type nodes. From each of these three type nodes there is an edge to the department node with capacity $k-1$. The edges from the color nodes now go to the correct type nodes instead of the department nodes. That is, if we had an edge from u_c to v_d and color c chains are of type i then the edge should go to type node i of v_d.



Solution 5.2 continued: