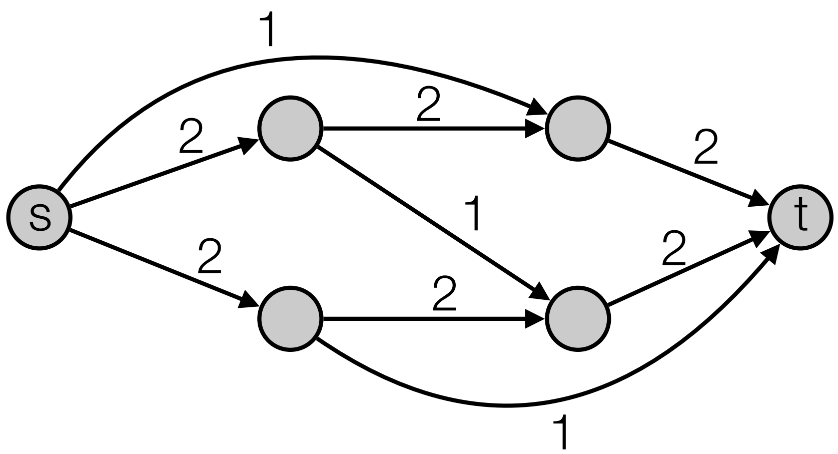
Network Flow II

Inge Li Gørtz

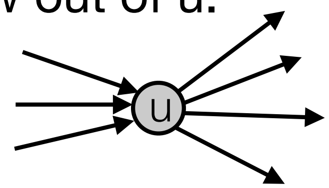
KT 7.3, 7.5, 7.6

Network Flow

- Network flow:
  - graph G=(V,E).
  - Special vertices s (source) and t (sink).
  - Every edge e has a capacity c(e) ≥ 0.
  - Flow:
    - **capacity constraint:** every edge e has a flow 0 ≤ f(e) ≤ c(e).
    - **flow conservation:** for all u ≠ s, t: flow into u equals flow out of u.



$$\sum_{v:(v,u)\in E} f(v,u) = \sum_{v:(u,v)\in E} f(u,v)$$



- Value of flow f is the sum of flows out of s minus sum of flows into s:

$$v(f) = \sum_{v:(s,v)\in E} f(e) - \sum_{v:(v,s)\in E} f(e) = f^{out}(s) - f^{in}(s)$$

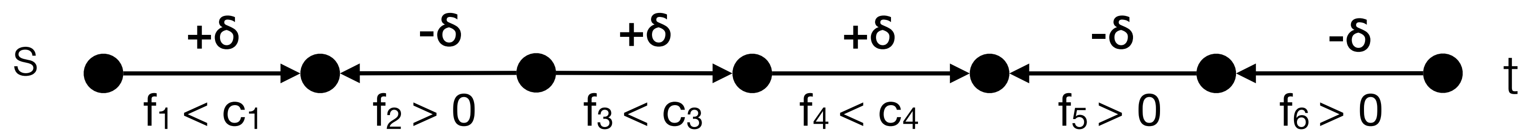
- **Maximum flow problem:** find s-t flow of maximum value

Today

- Applications
  - Bipartite matching: Hospital have to schedule doctors for the holidays.
    - Doctors have constraints on how many and on which holidays they can work.
    - Hospital needs a certain amount on doctors at work.
  - Disjoint paths:
- Finding good augmenting paths. Edmonds-Karp and scaling algorithm.

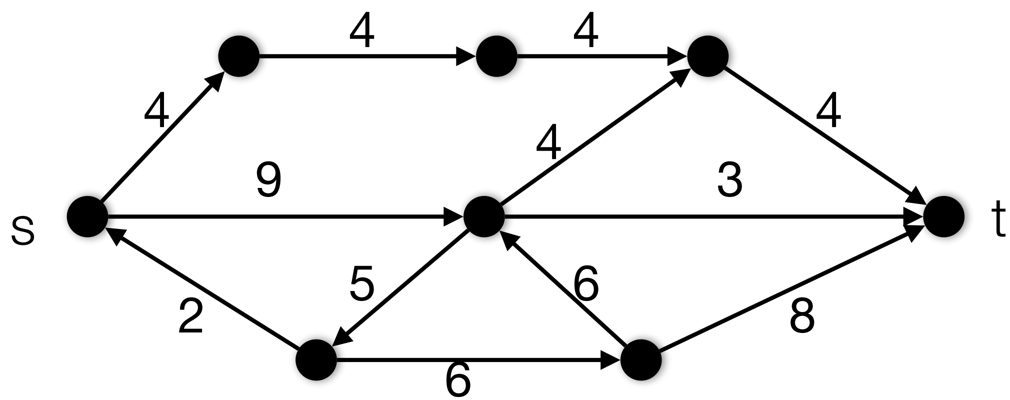
Ford-Fulkerson

- Find (any) augmenting path and use it.
- Augmenting path (definition different than in CLRS): s-t path where
  - forward edges have leftover capacity
  - backwards edges have positive flow



  - Can add extra flow: $\min(c_1 - f_1, f_2, c_3 - f_3, c_4 - f_4, f_5, f_6) = \delta$

- To find augmenting path use DFS or BFS:



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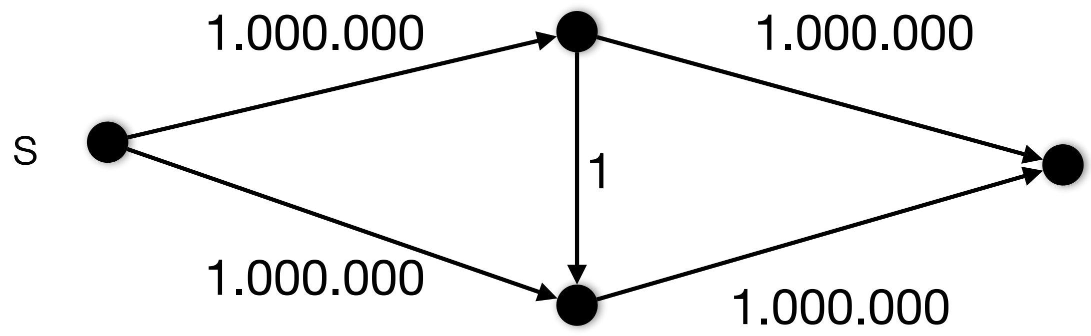

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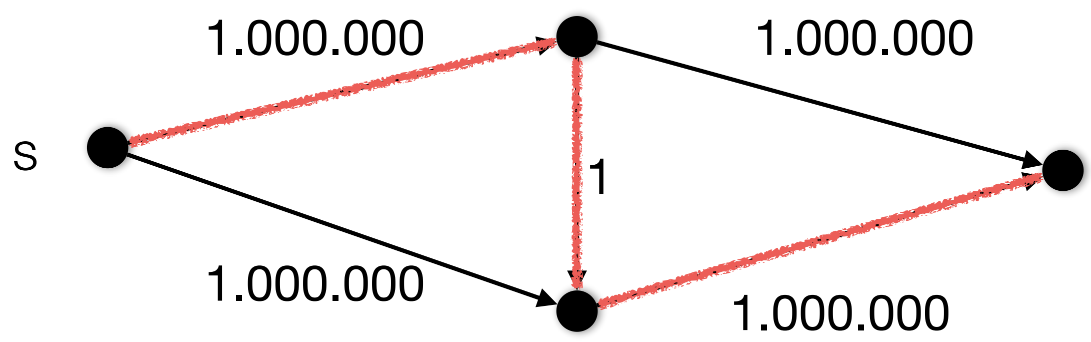

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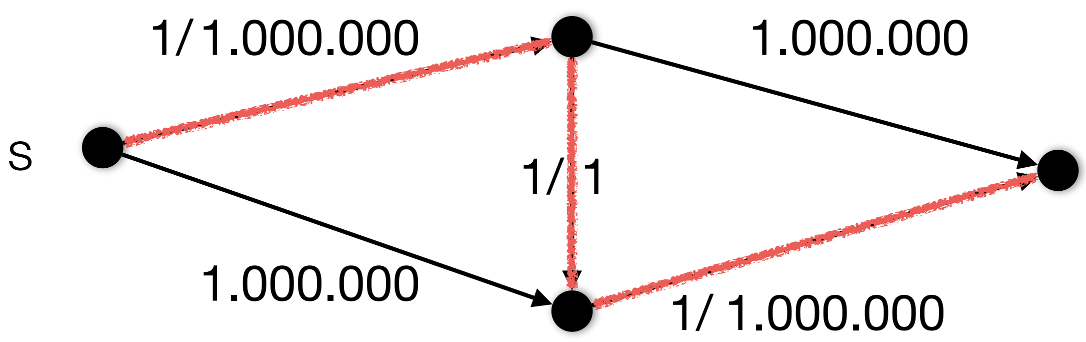

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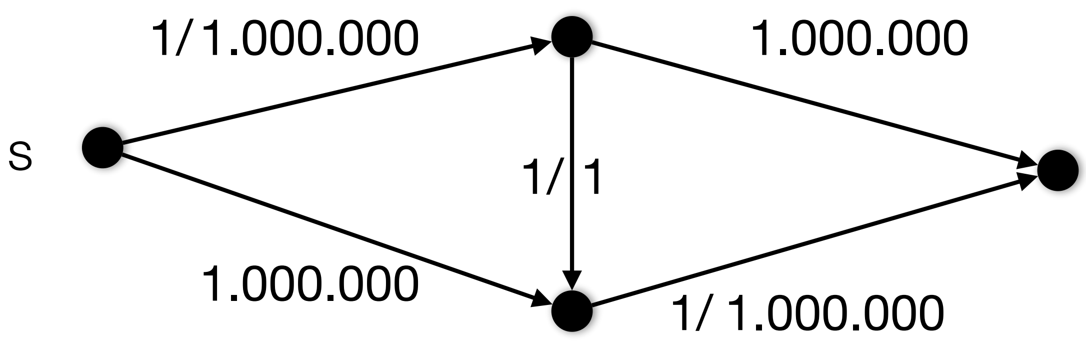

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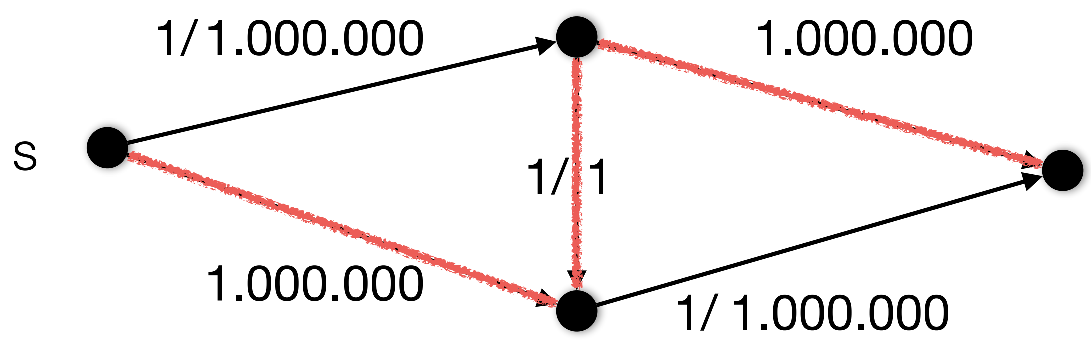

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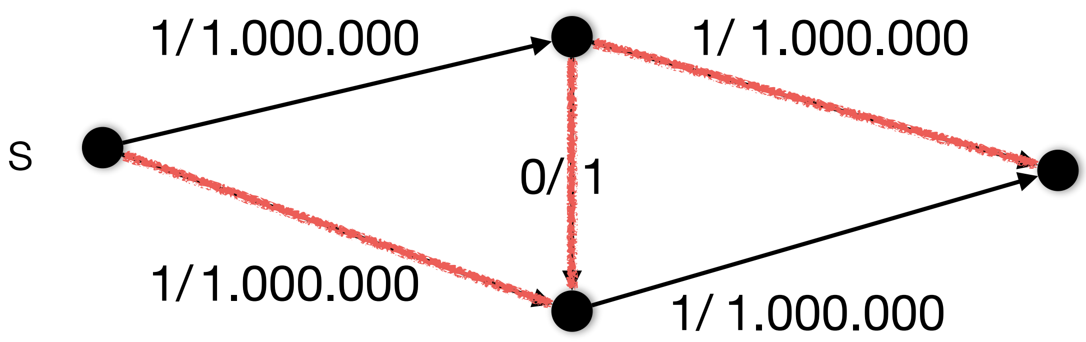

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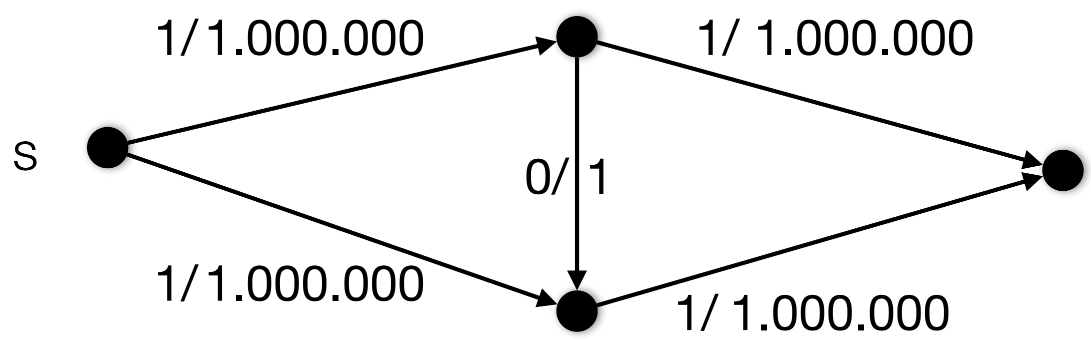

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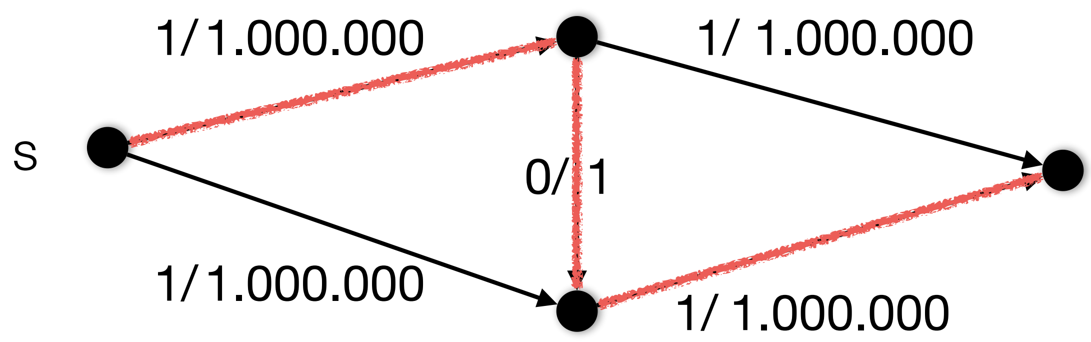

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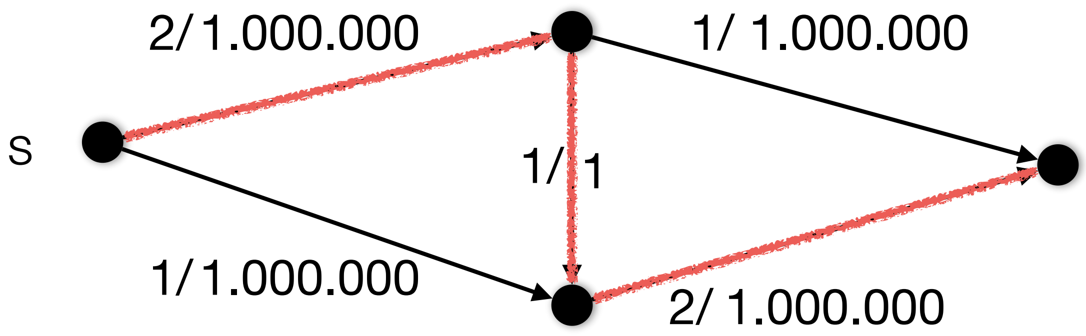

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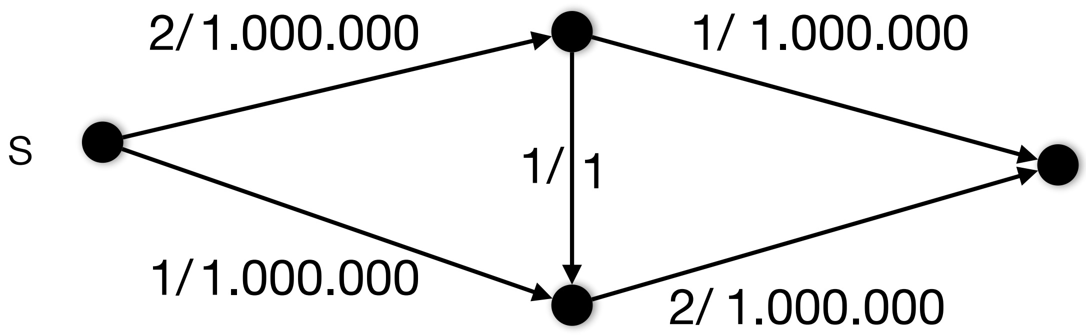

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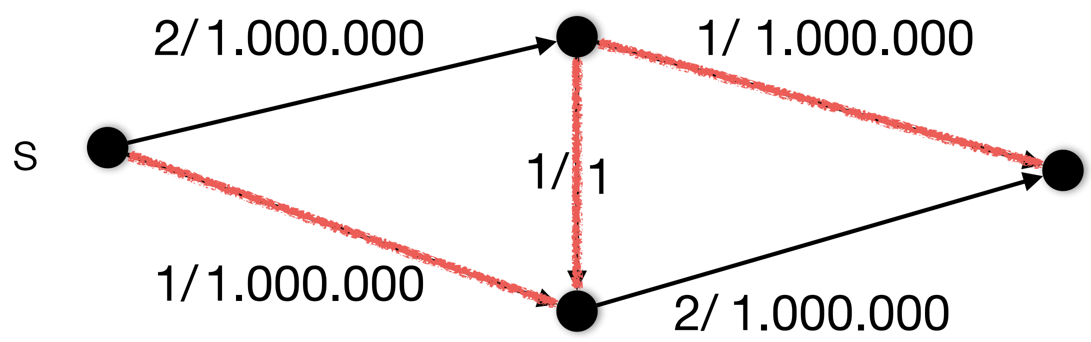

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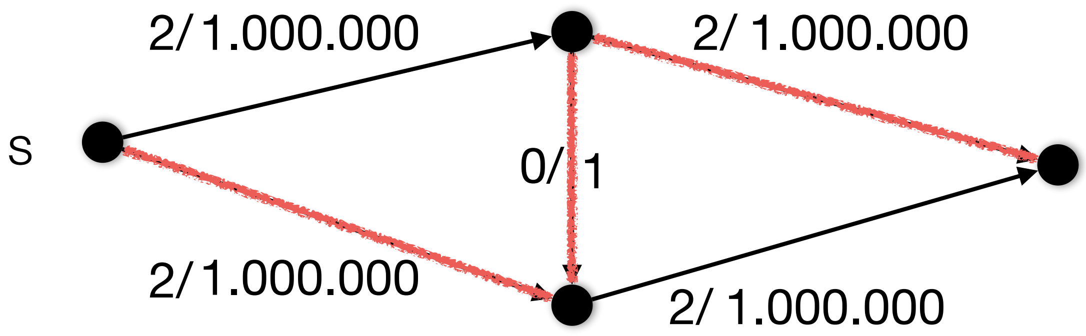

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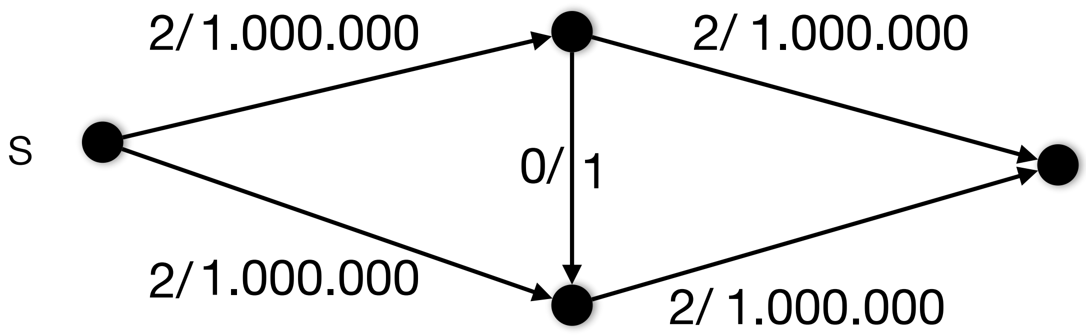

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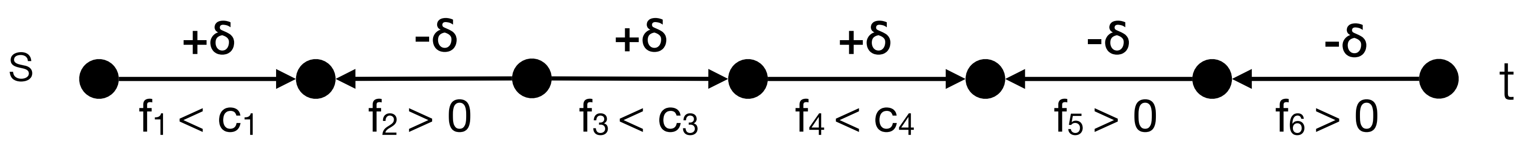

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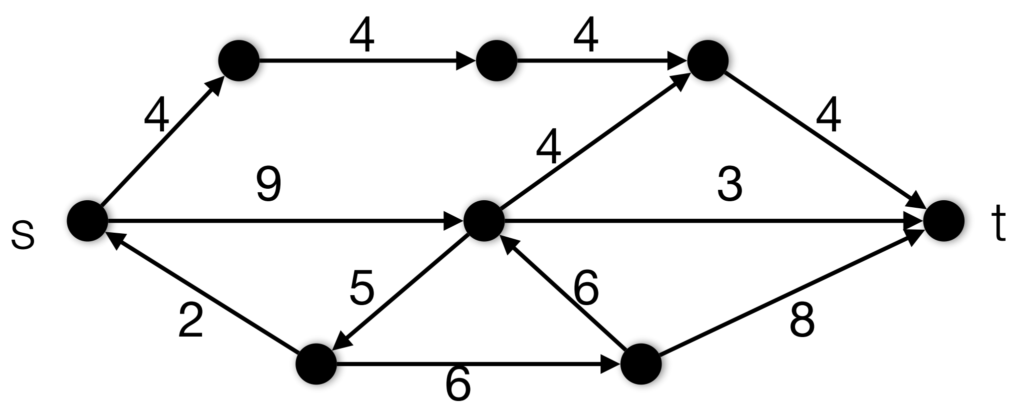

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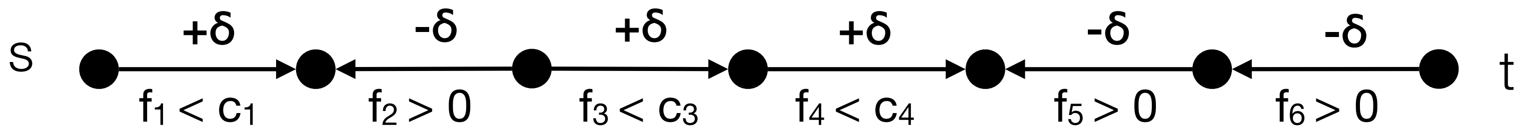

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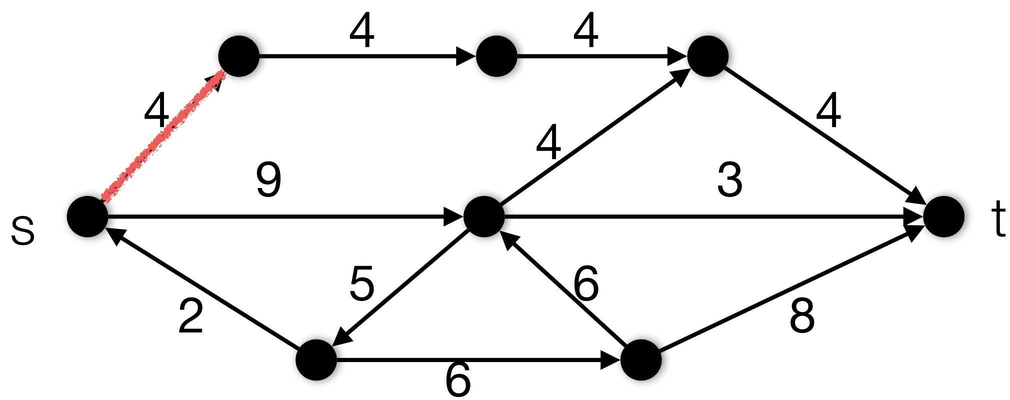

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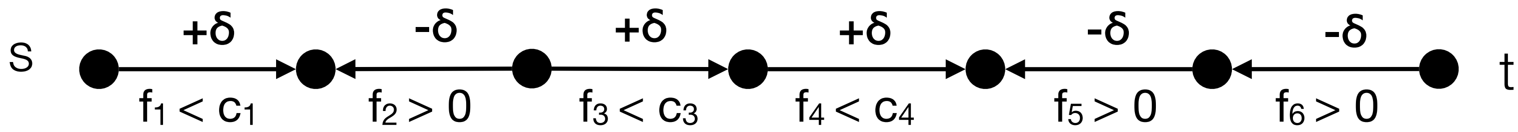

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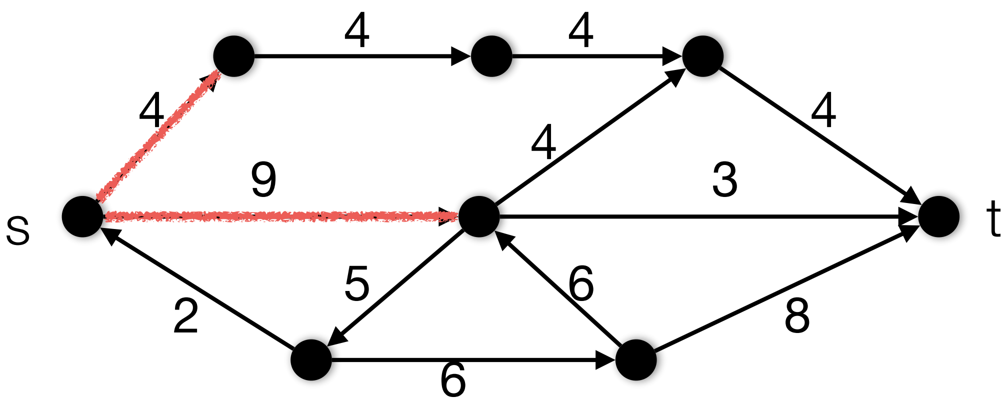

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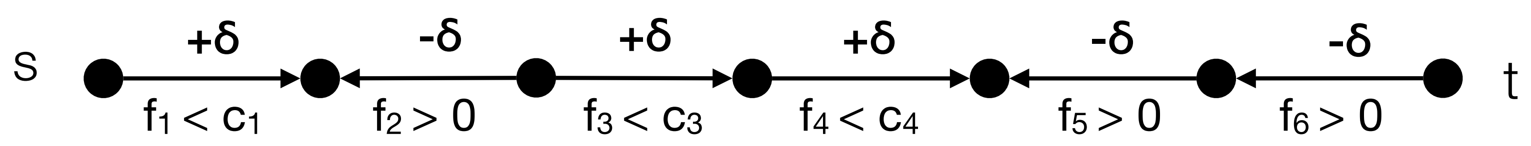

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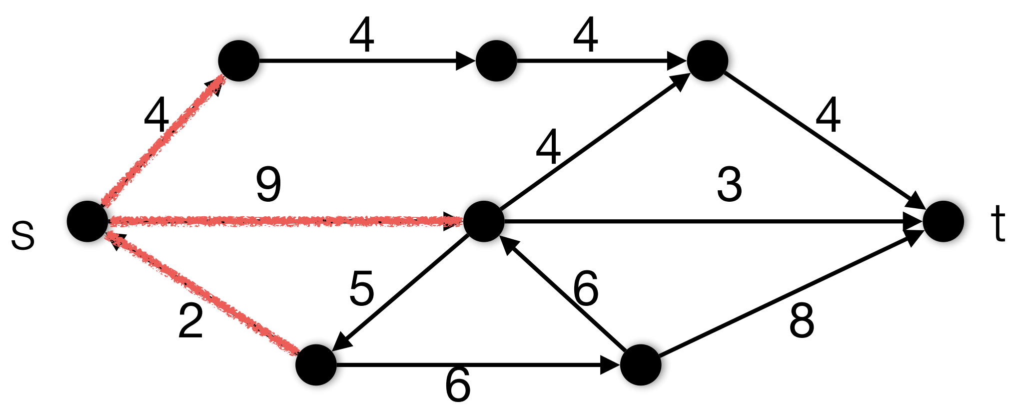

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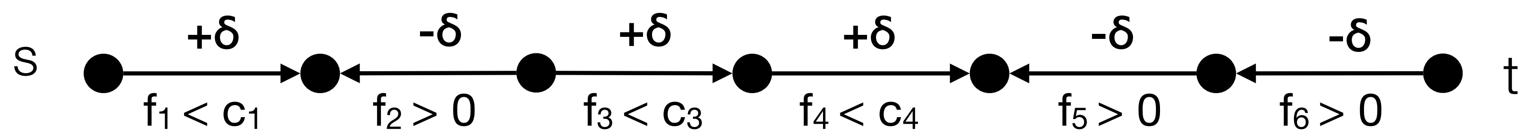

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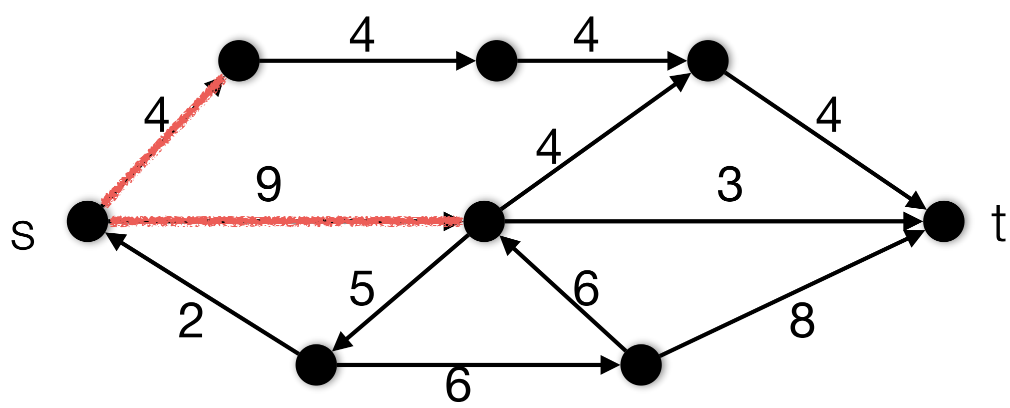

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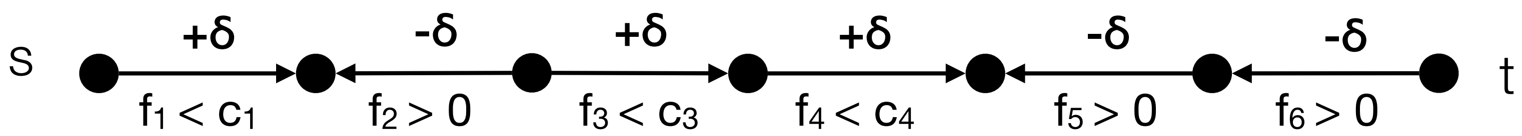

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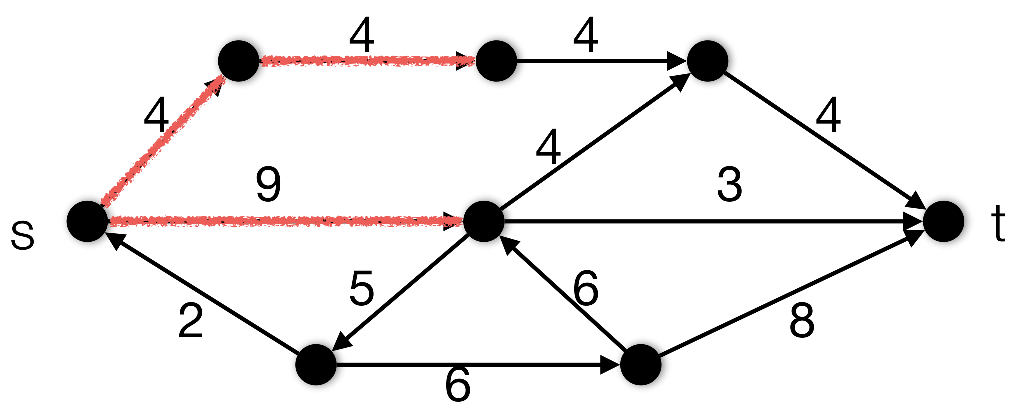

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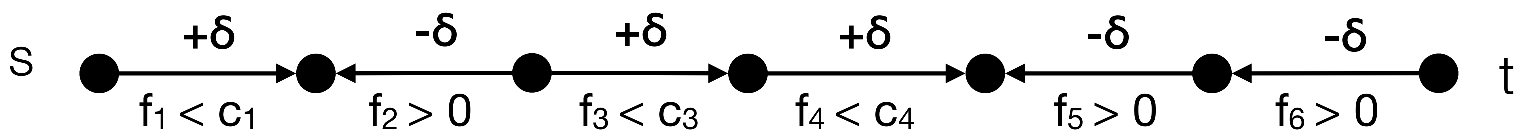

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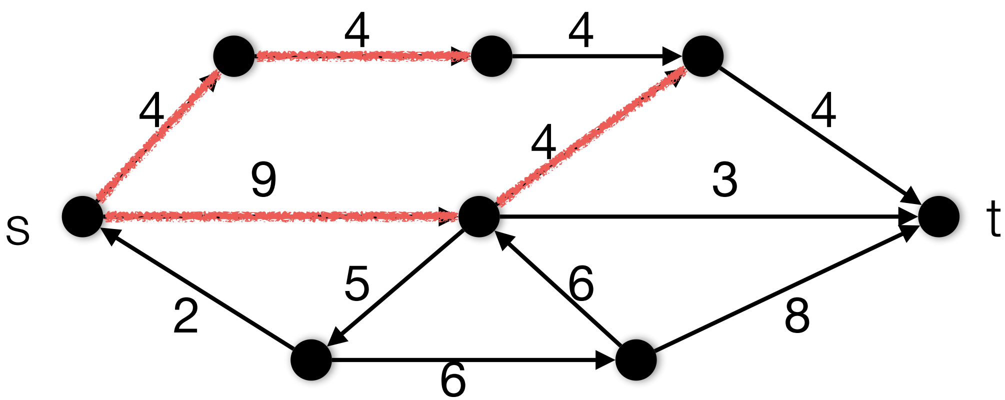

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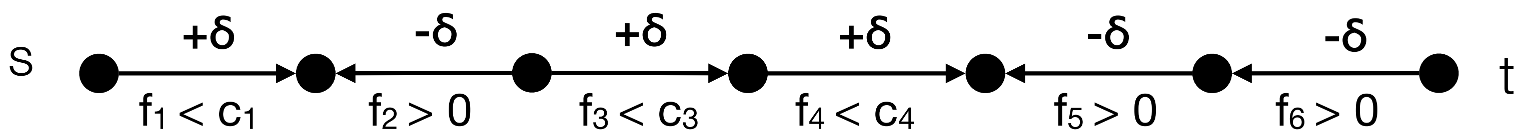

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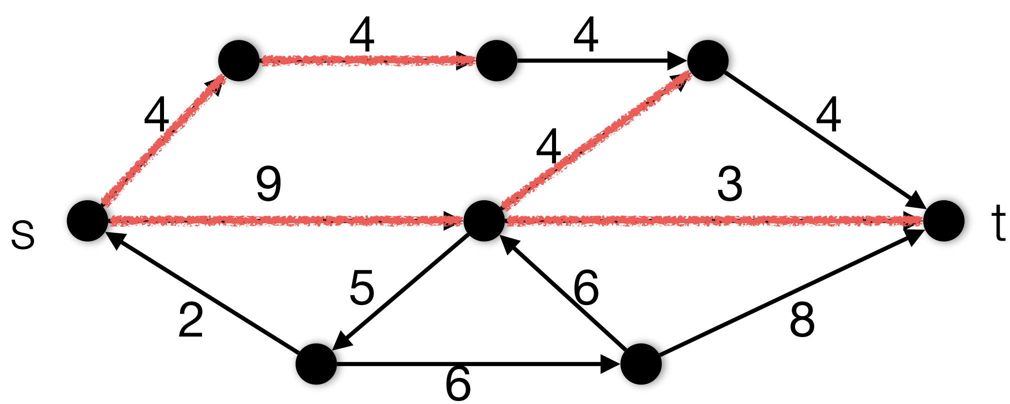

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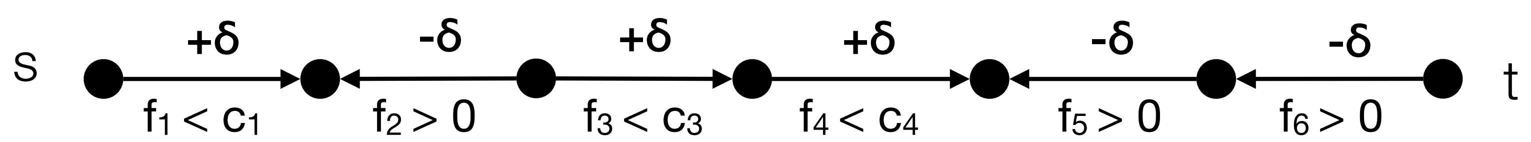

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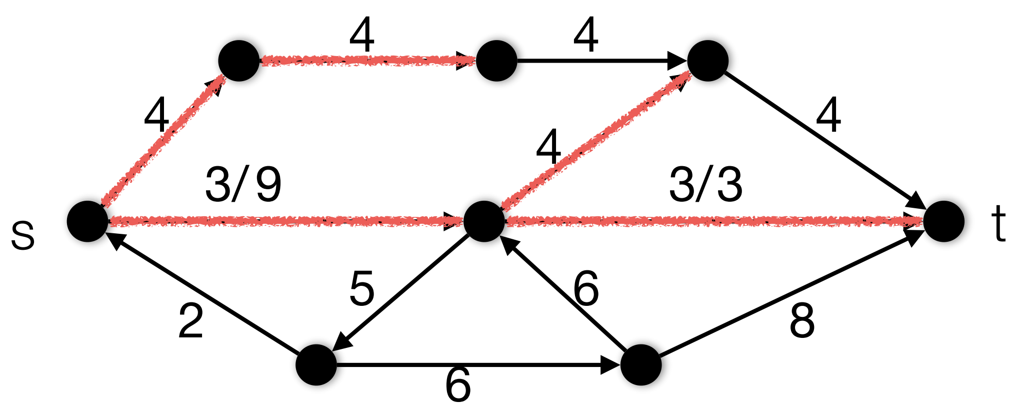

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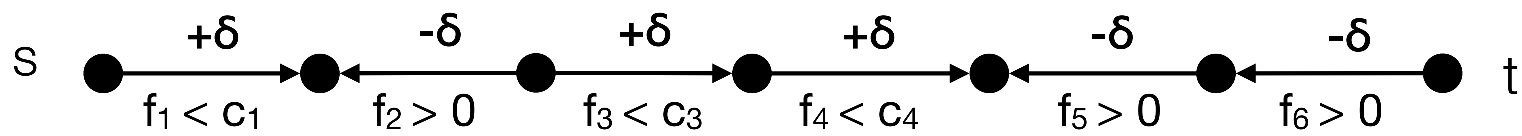

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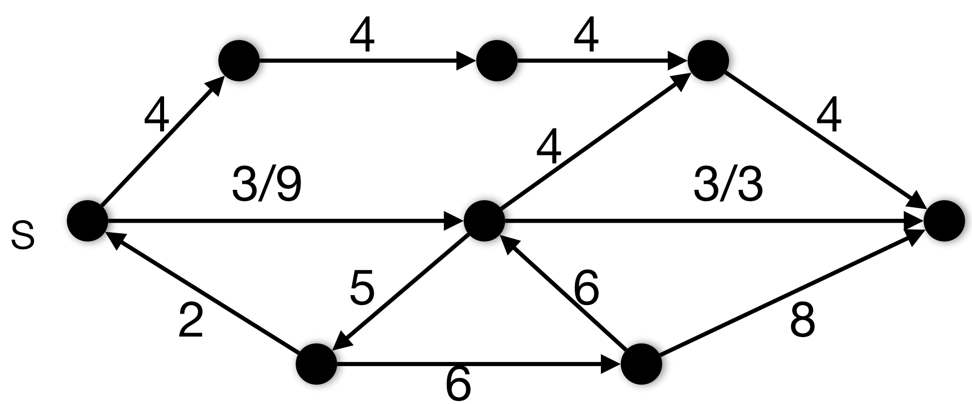

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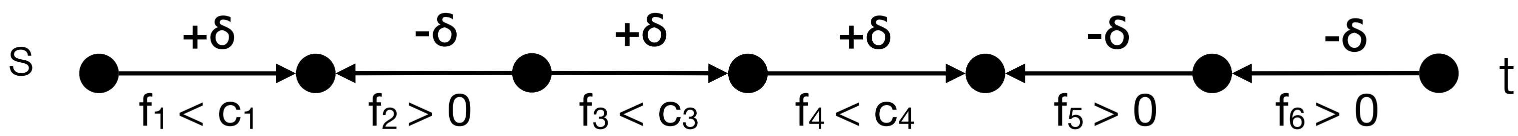

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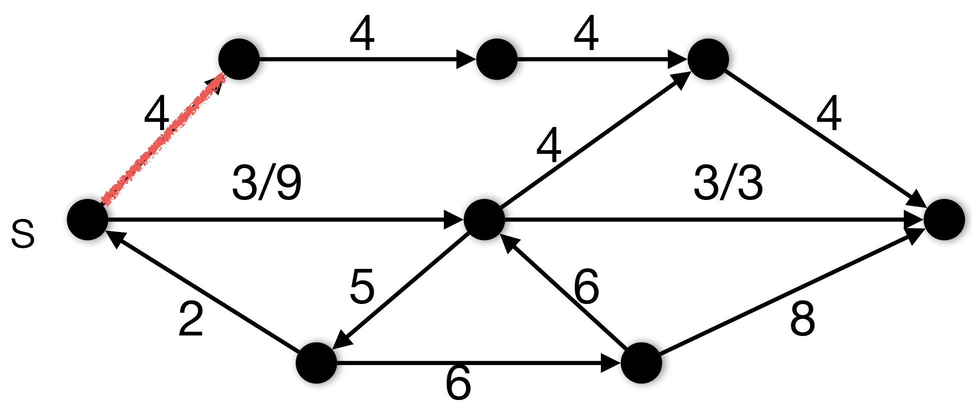

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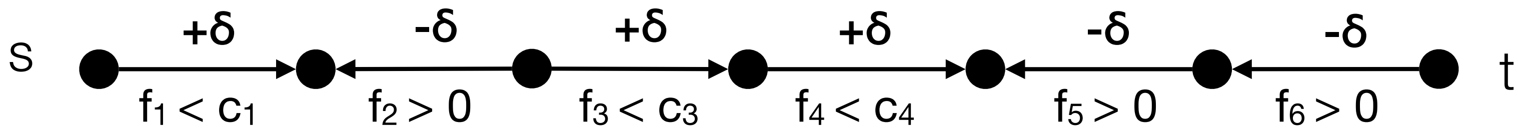

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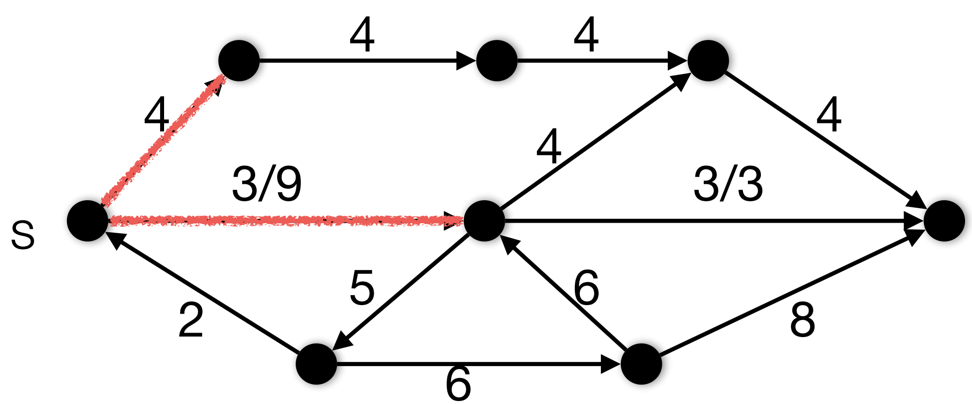

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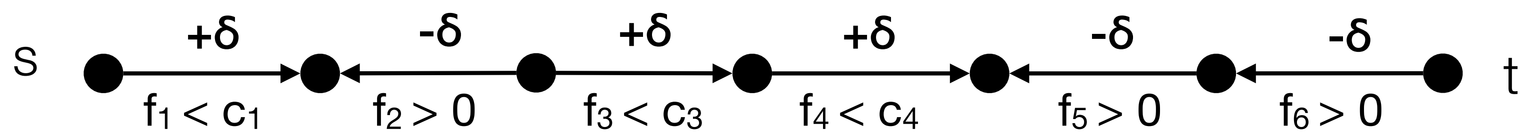

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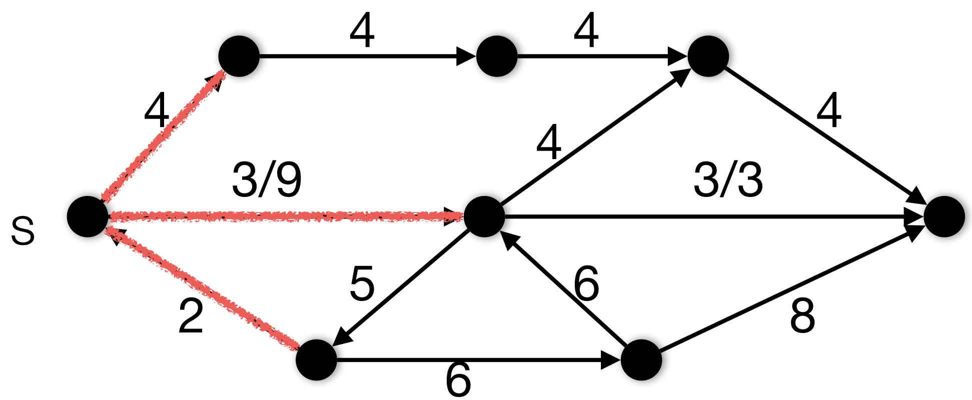

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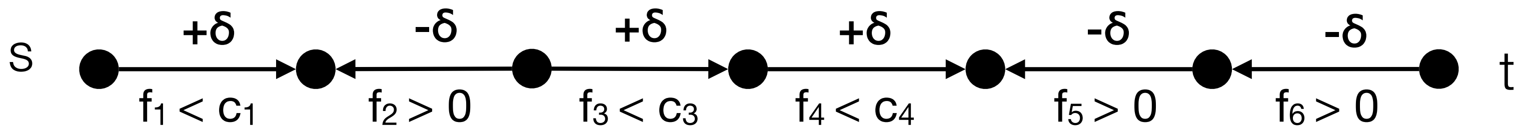

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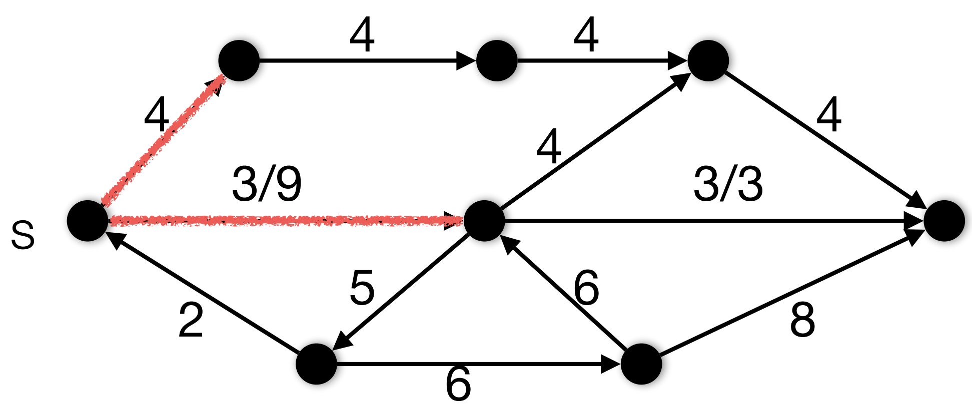

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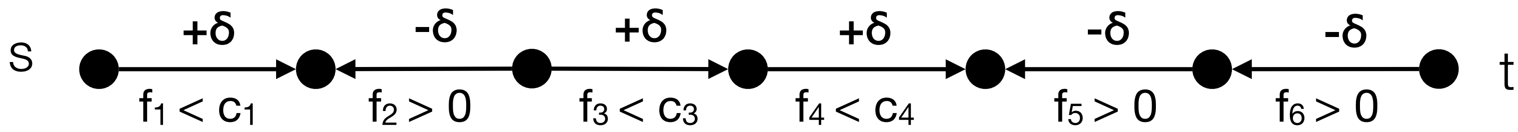

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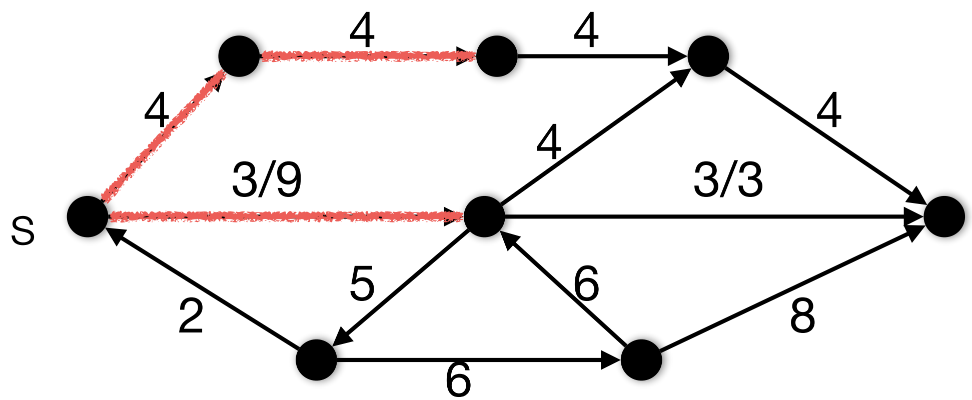

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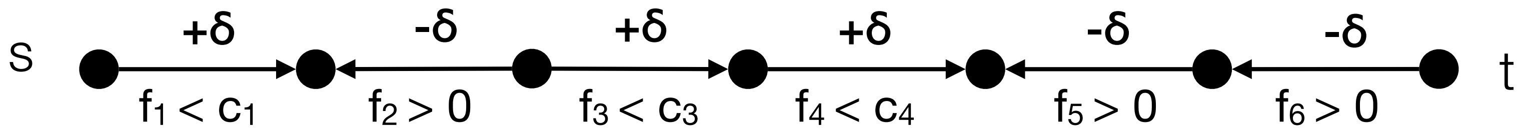

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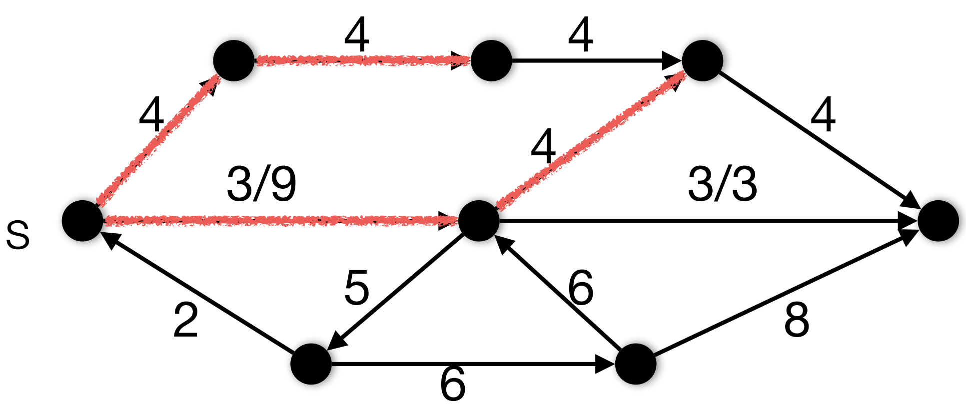

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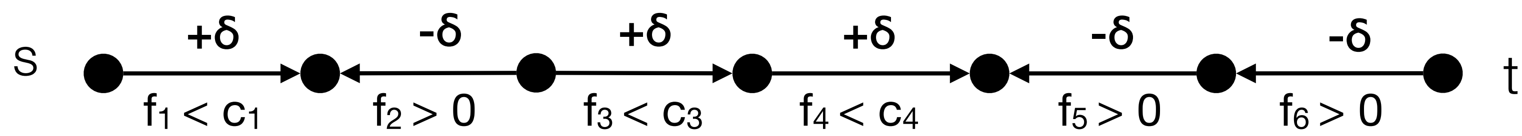

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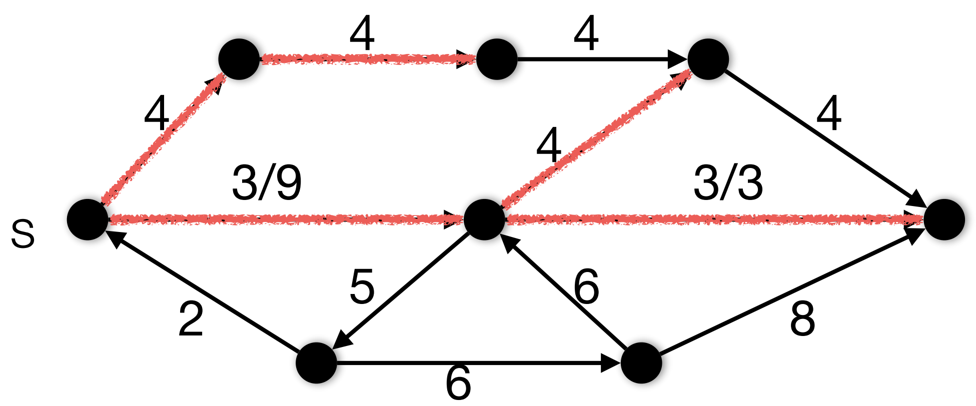

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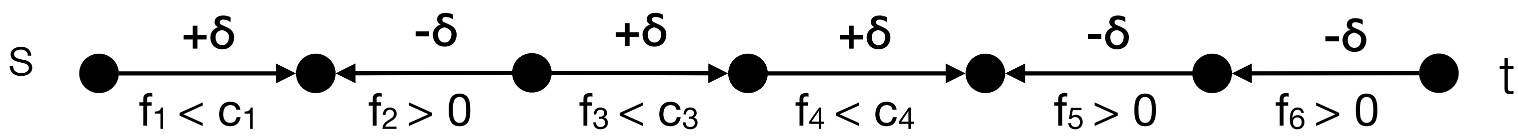

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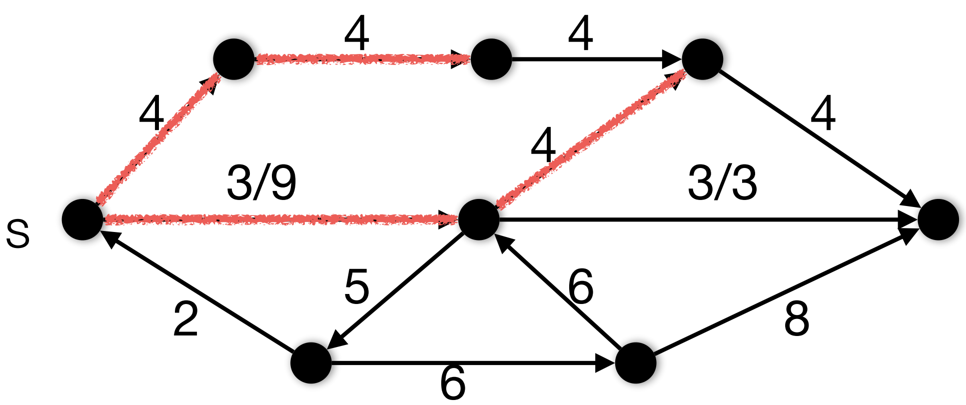

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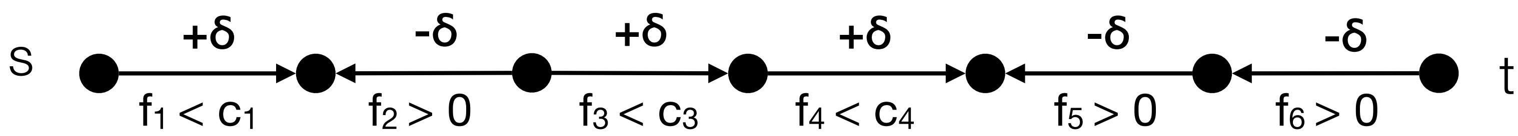

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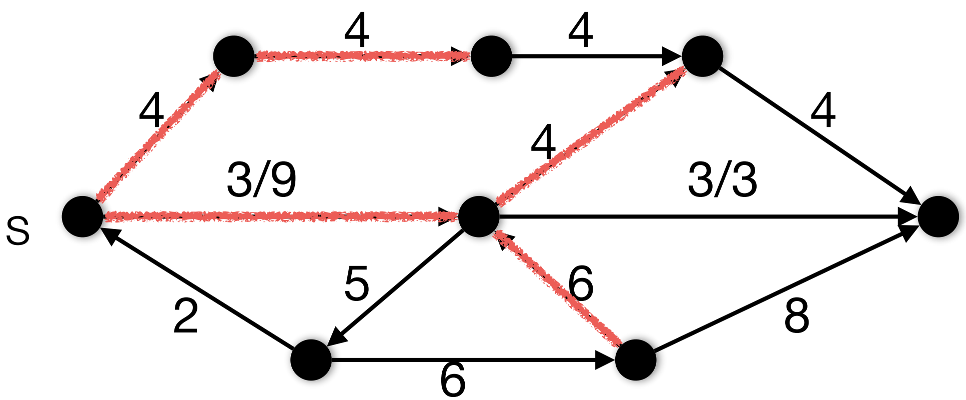

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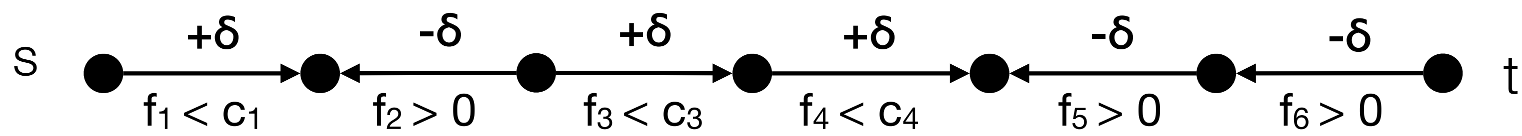

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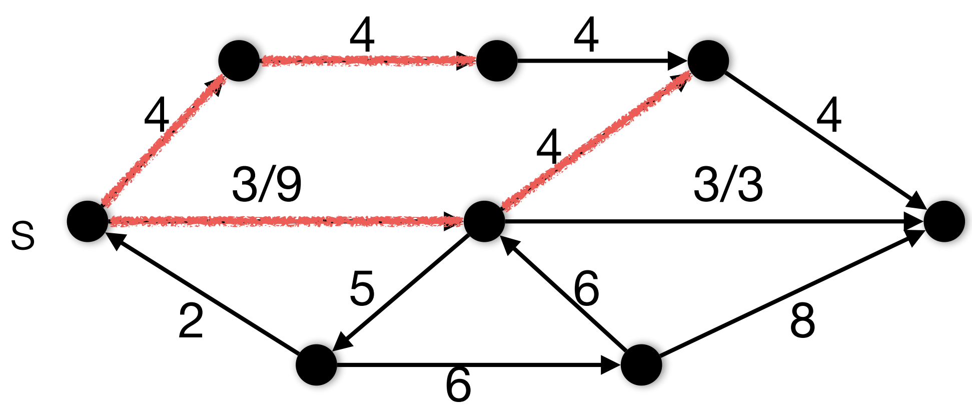

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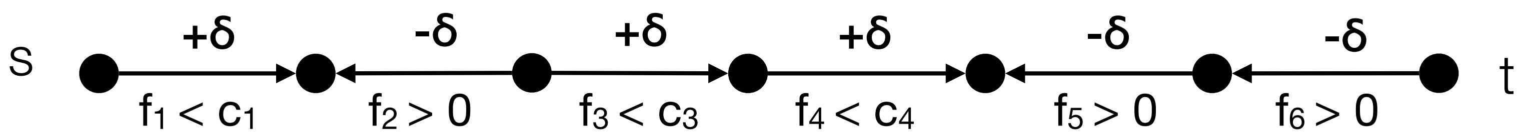

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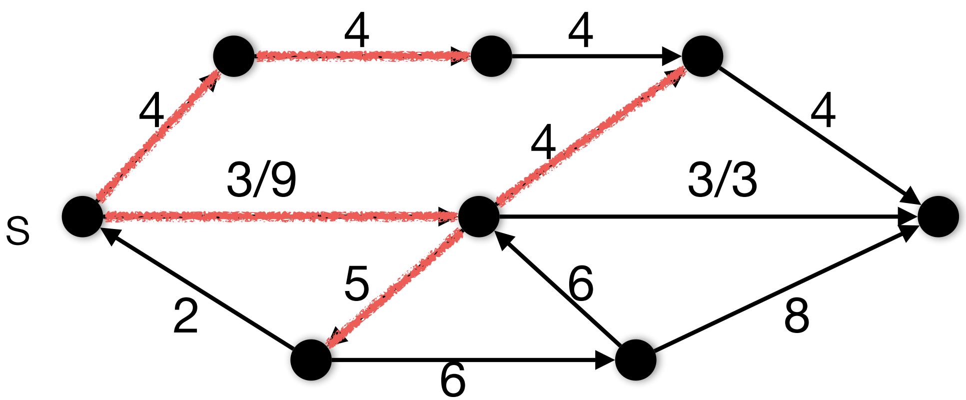

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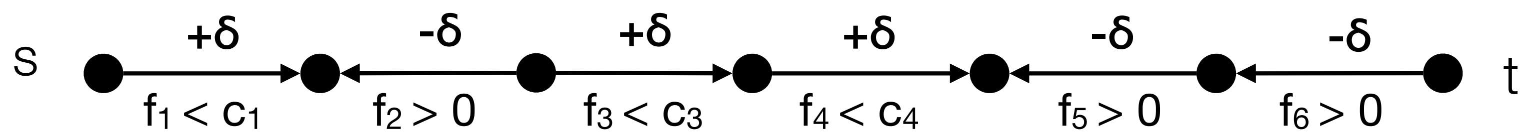

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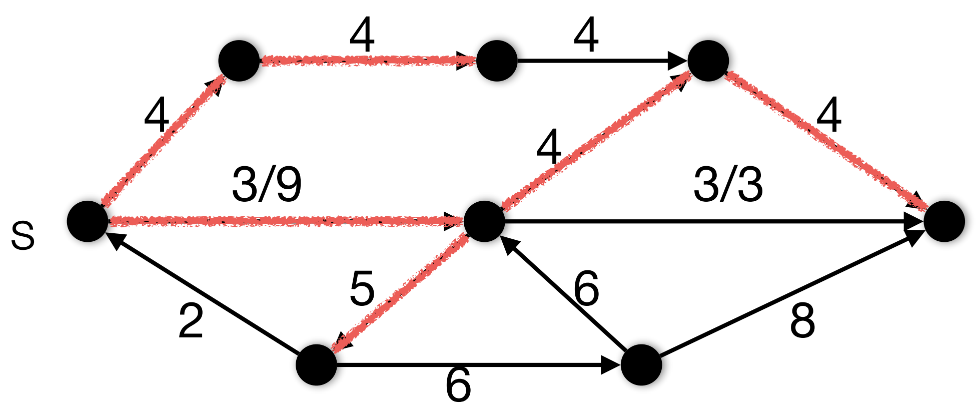

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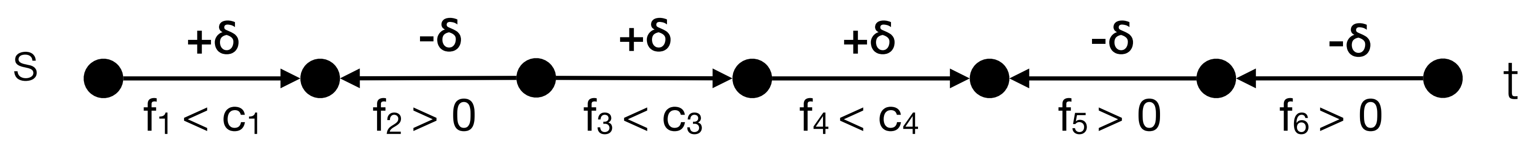

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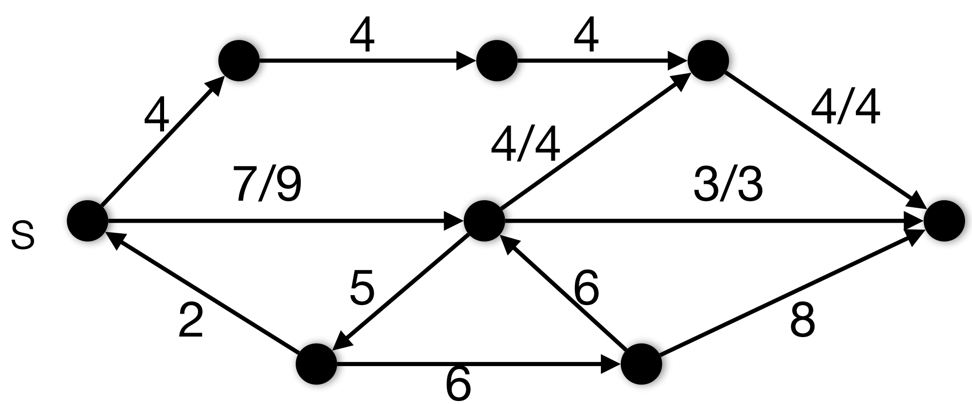

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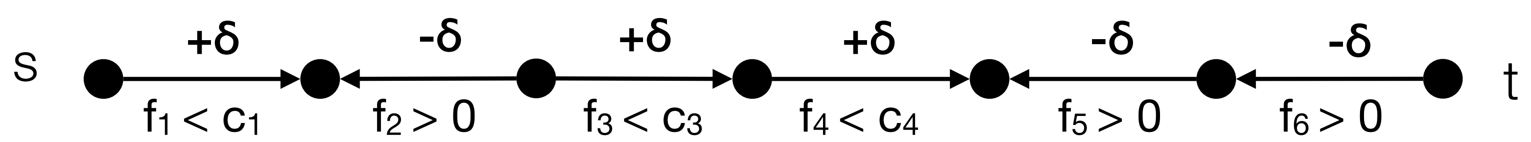

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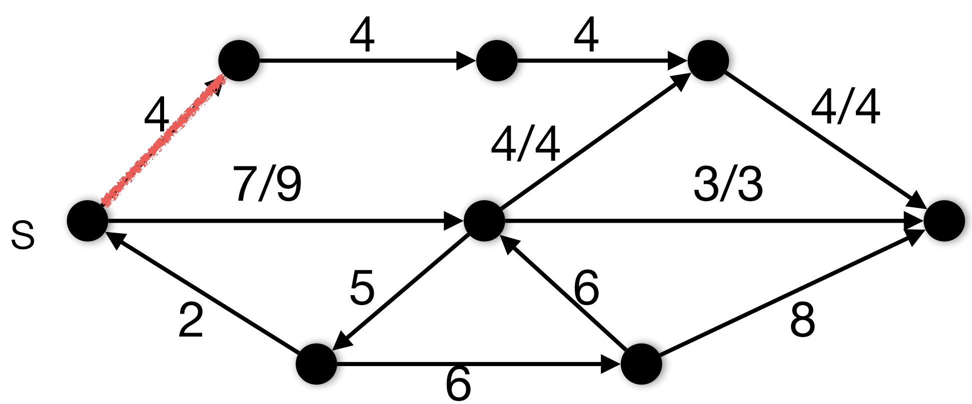

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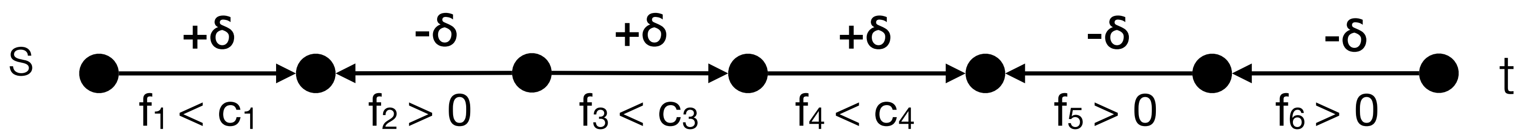

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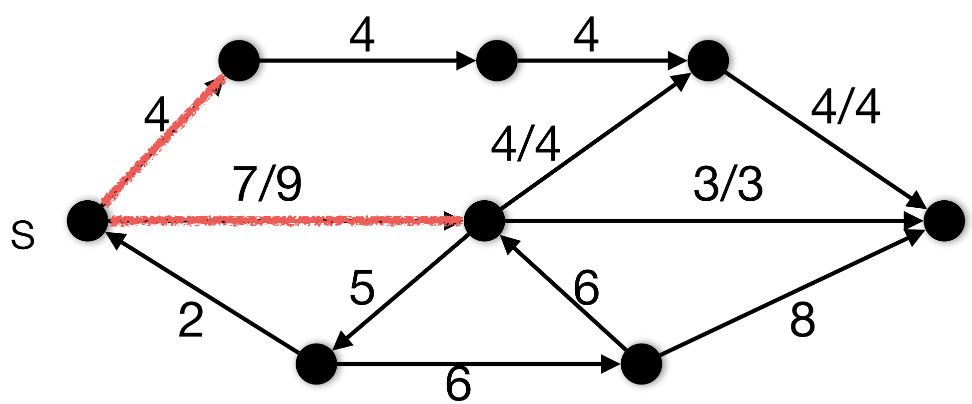

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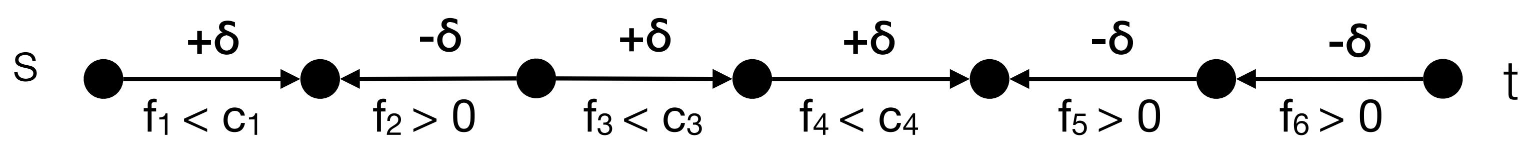

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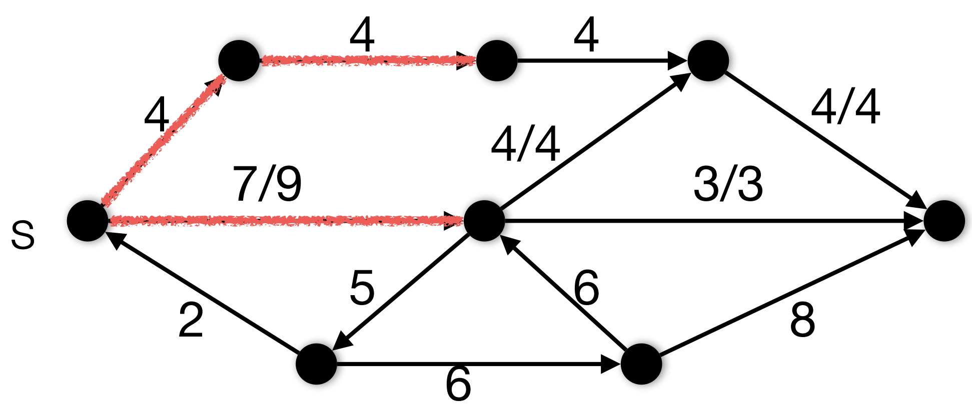

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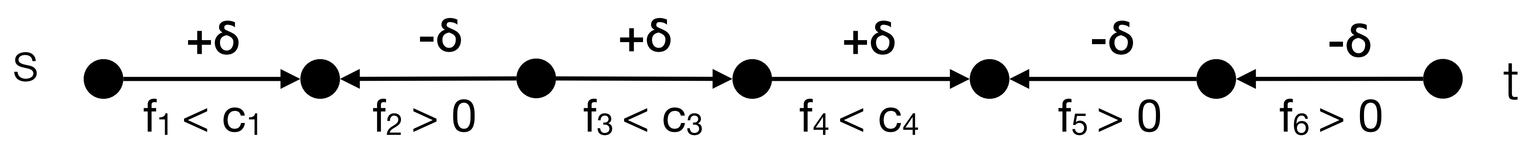

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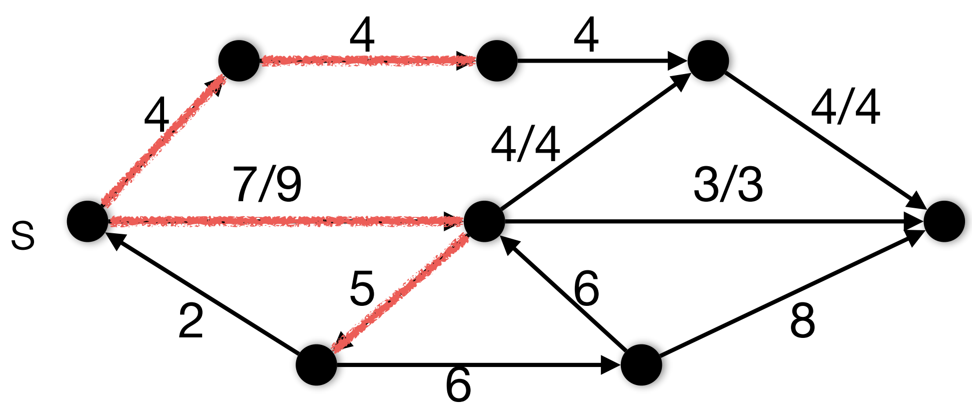

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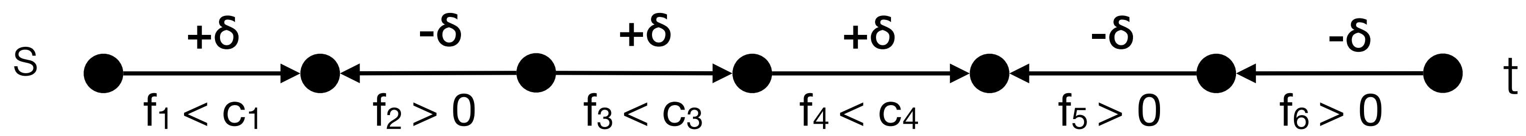

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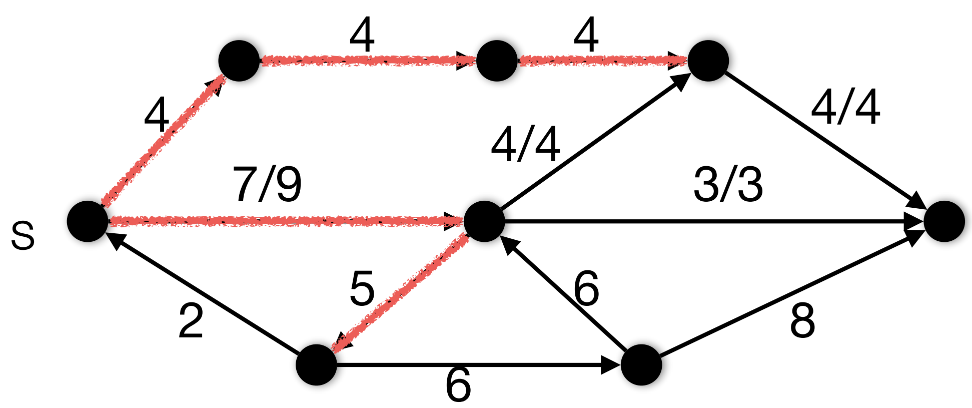

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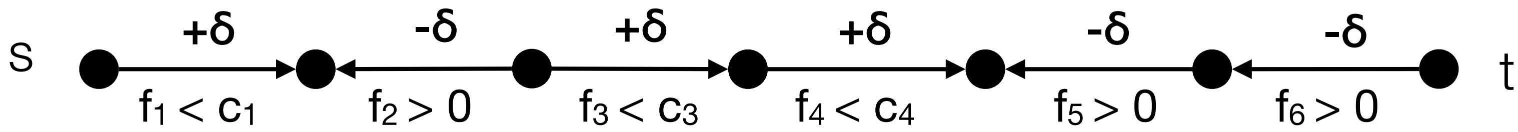

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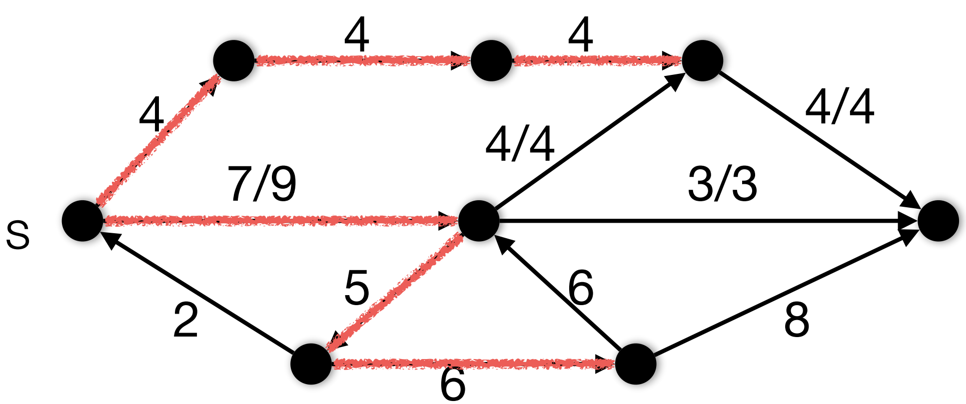

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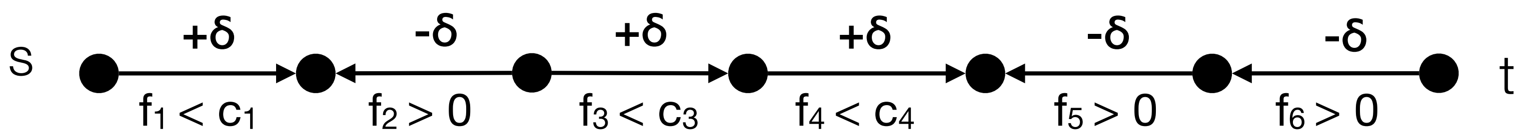

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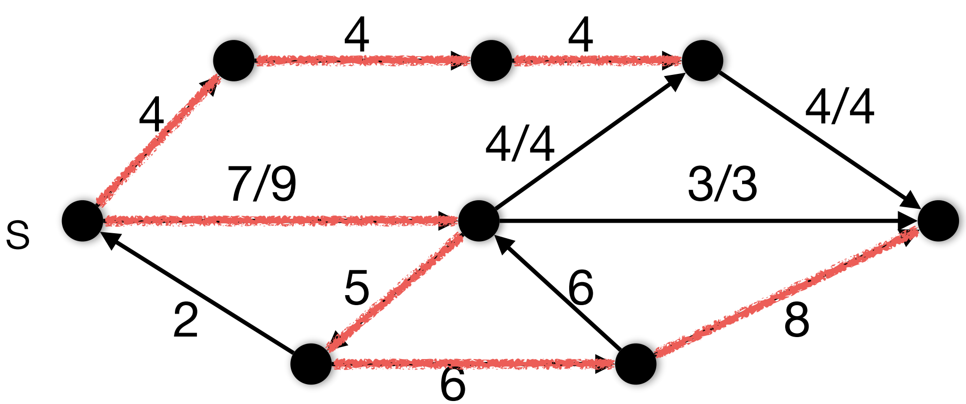

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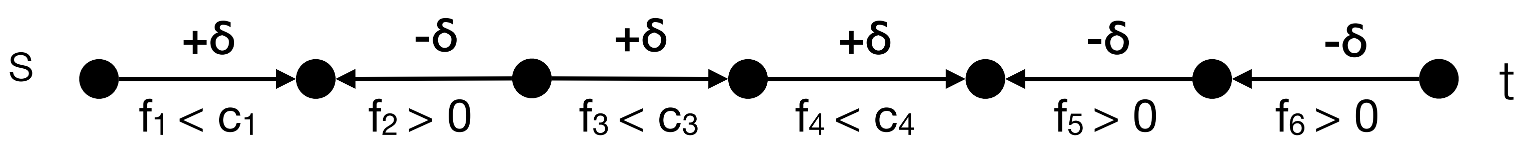

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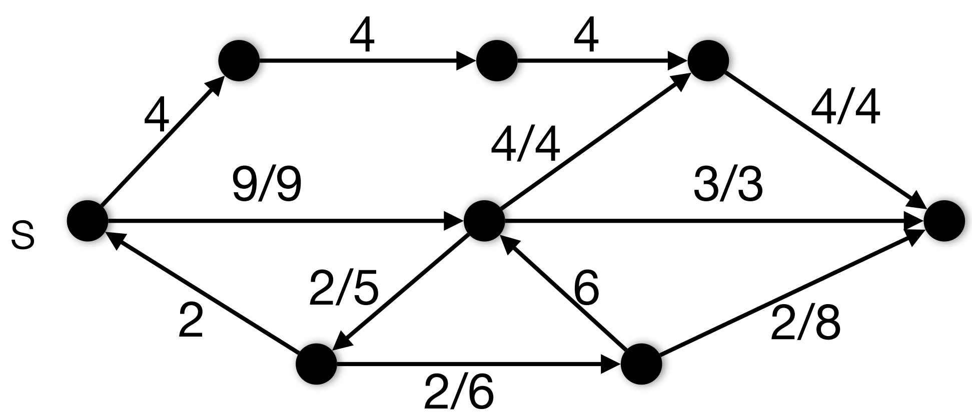

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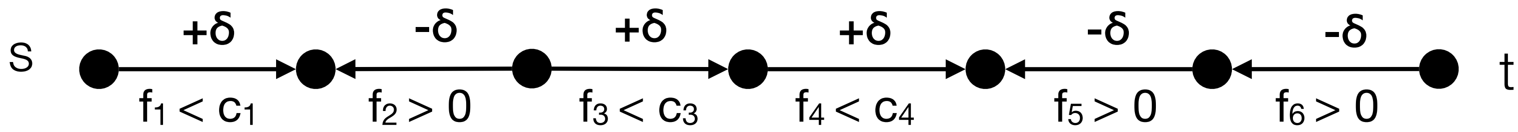

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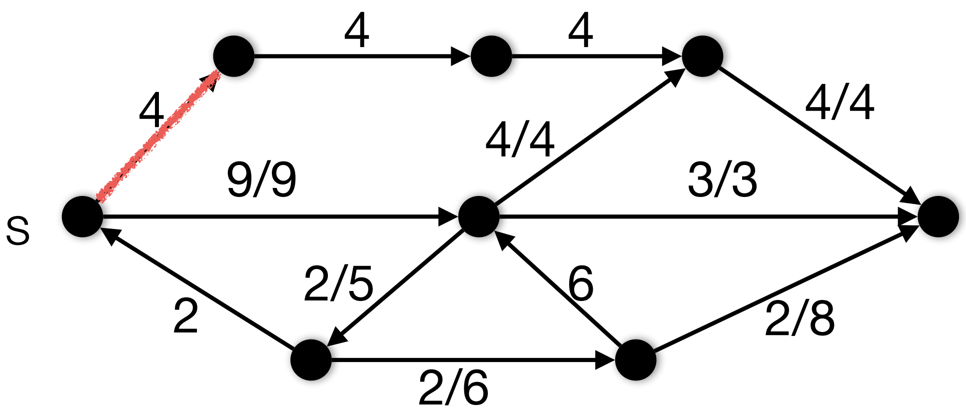

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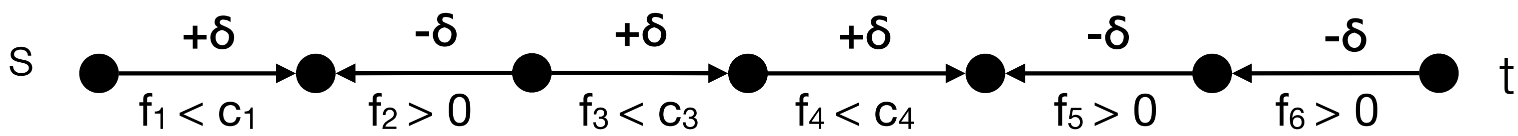

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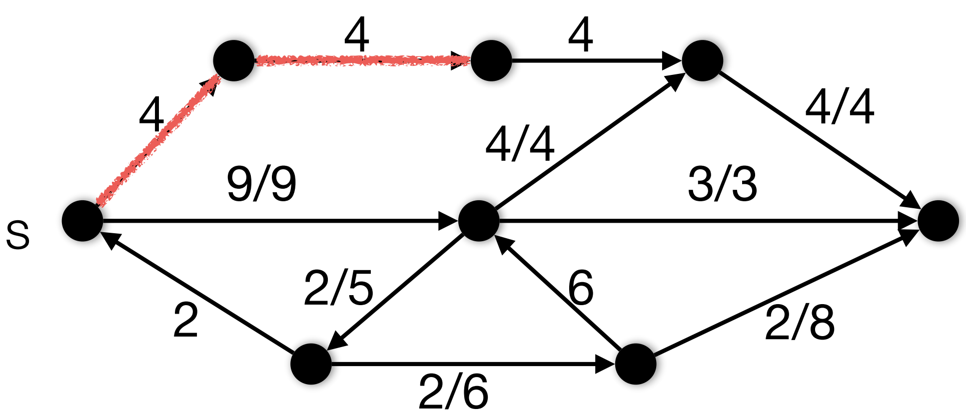

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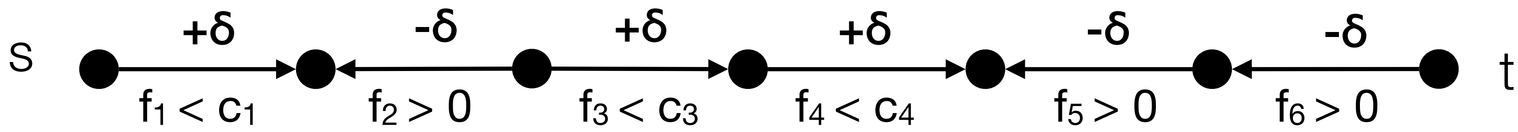

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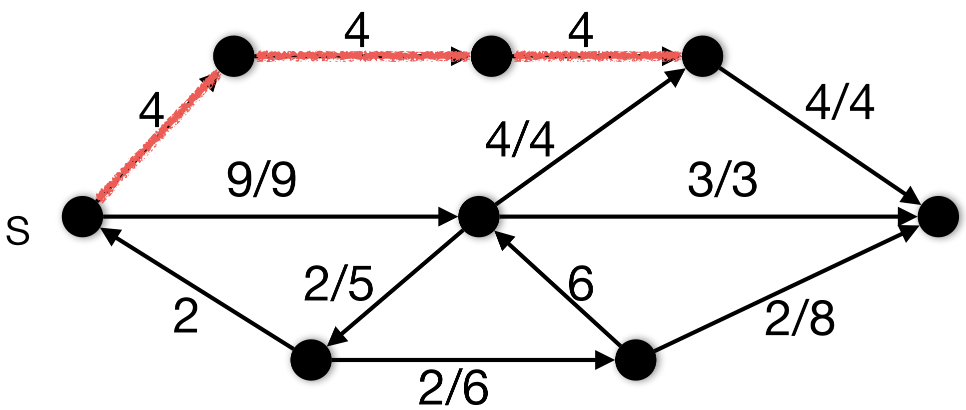

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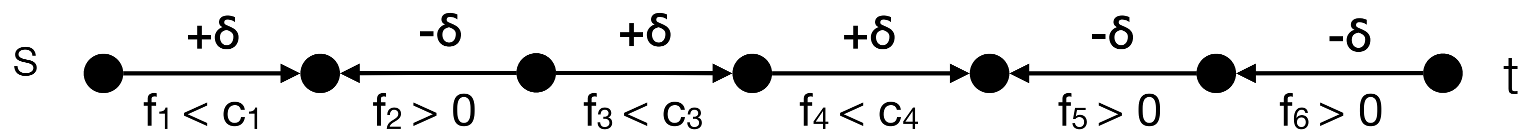

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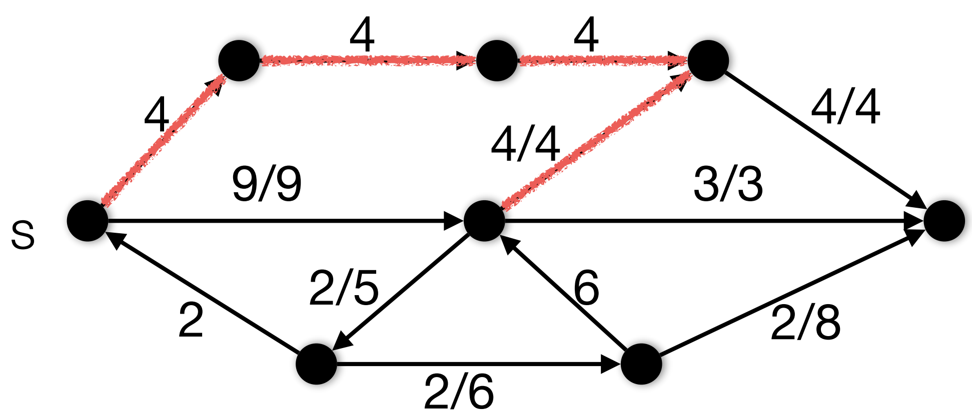

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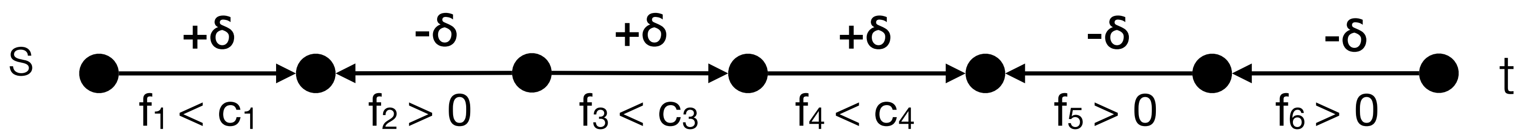

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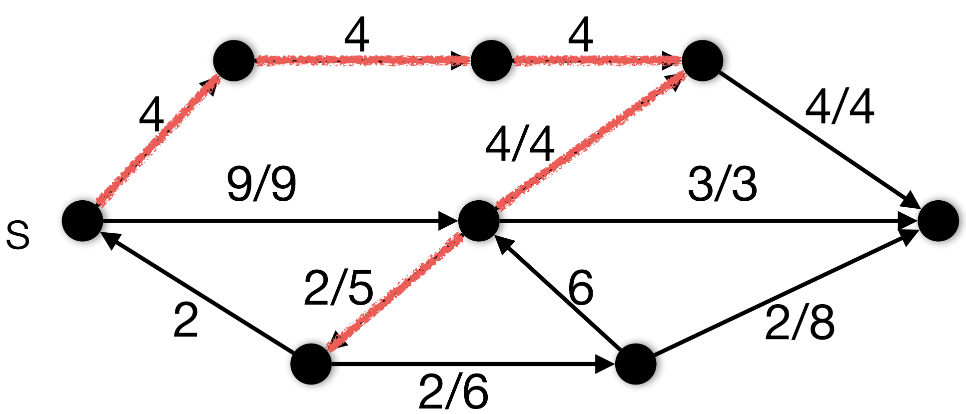

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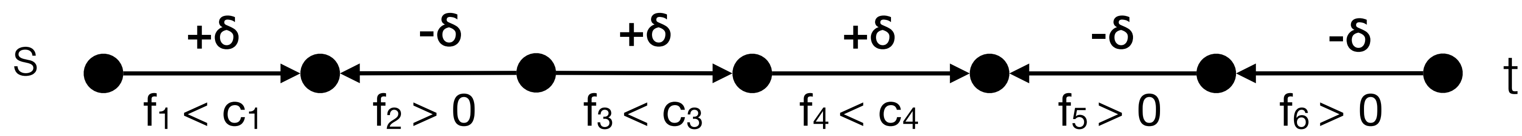

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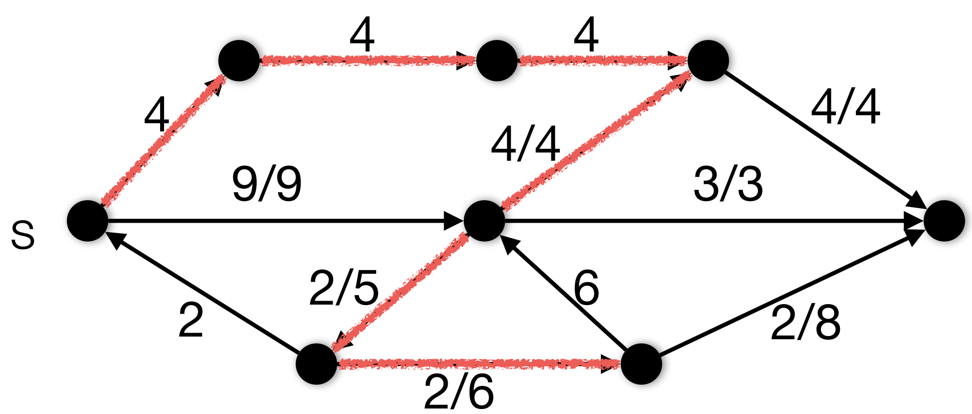

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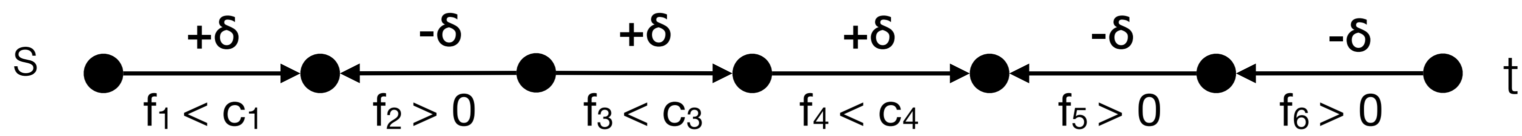

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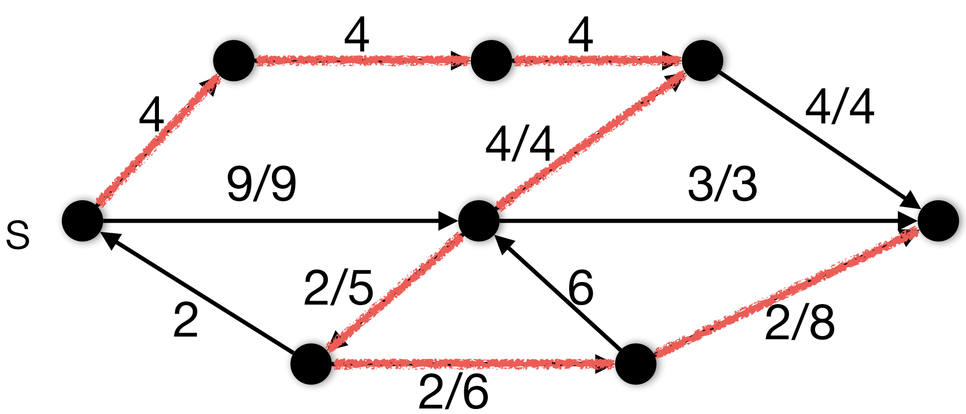

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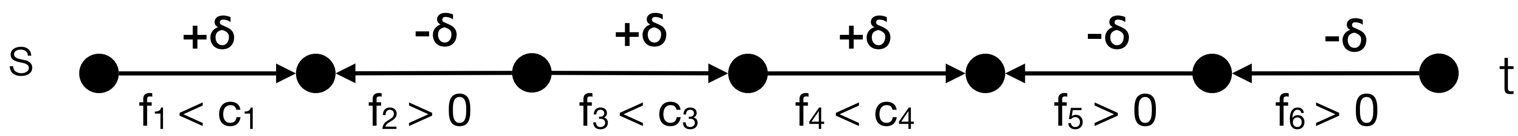

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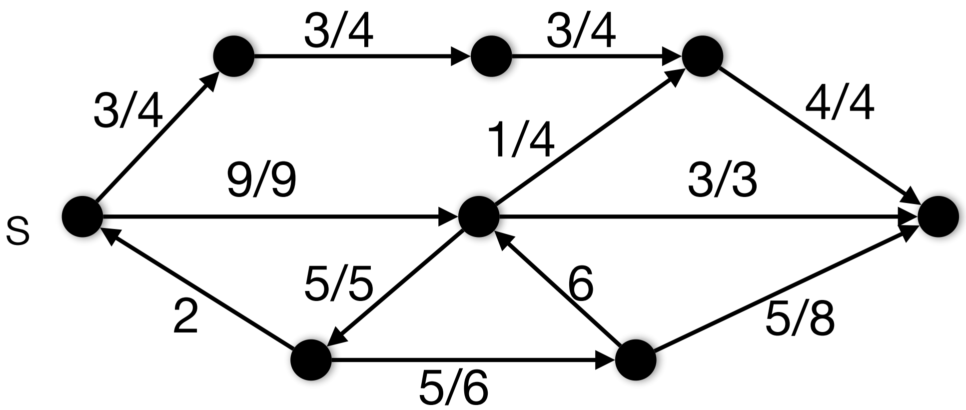

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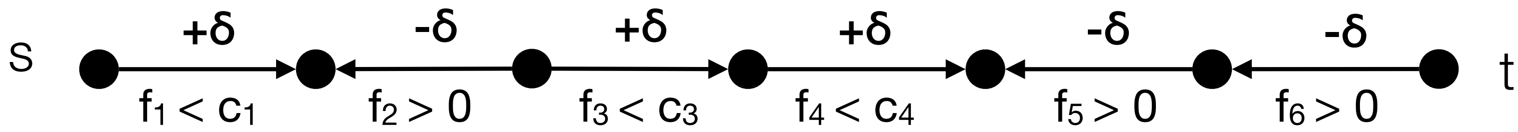

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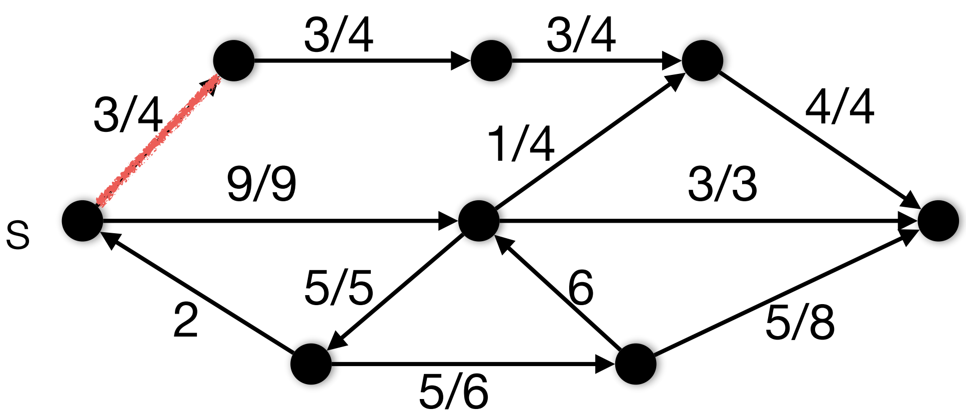

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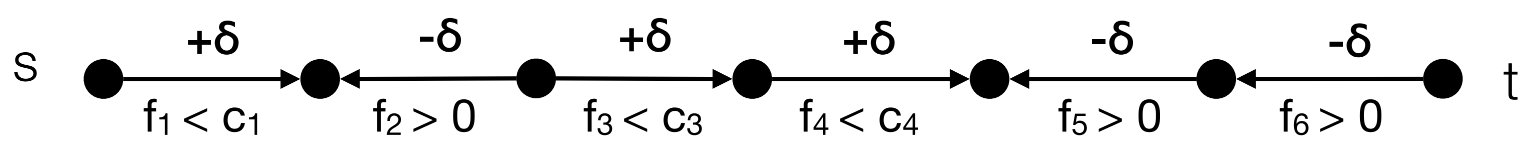

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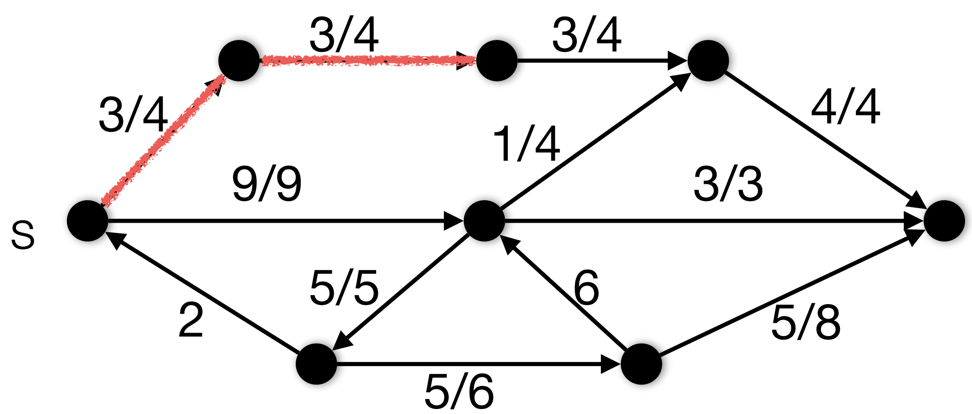

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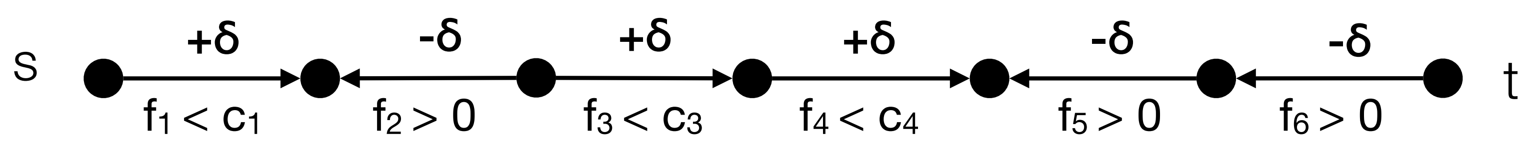

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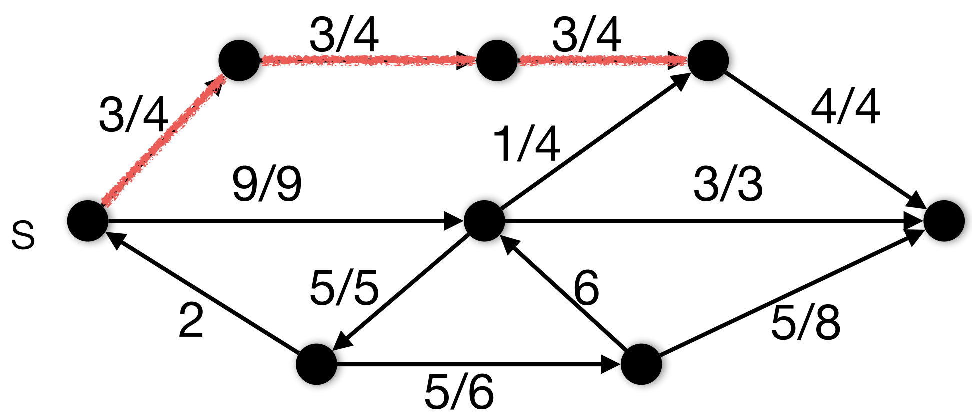

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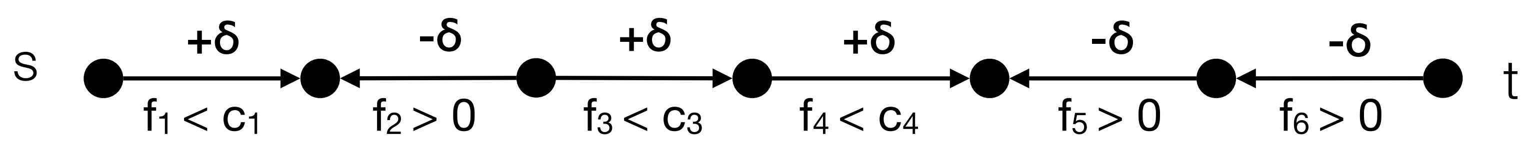

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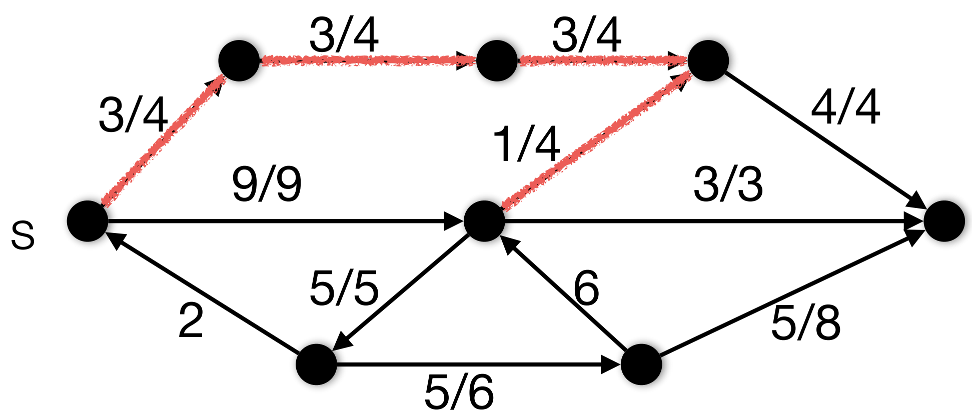

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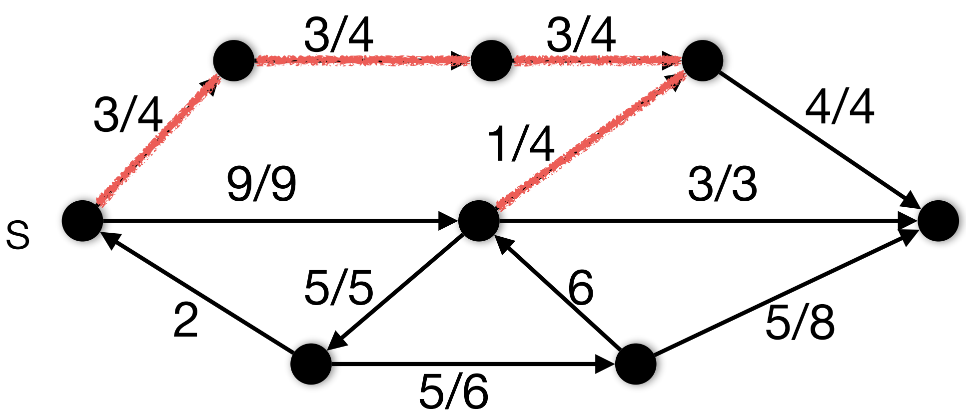

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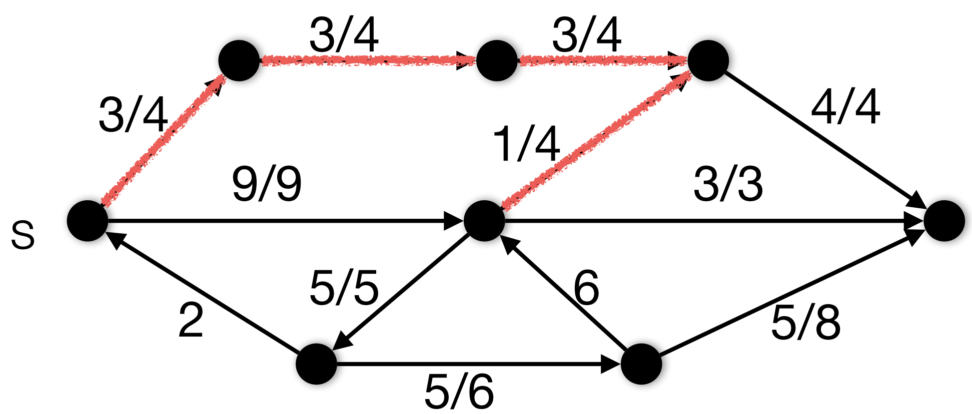

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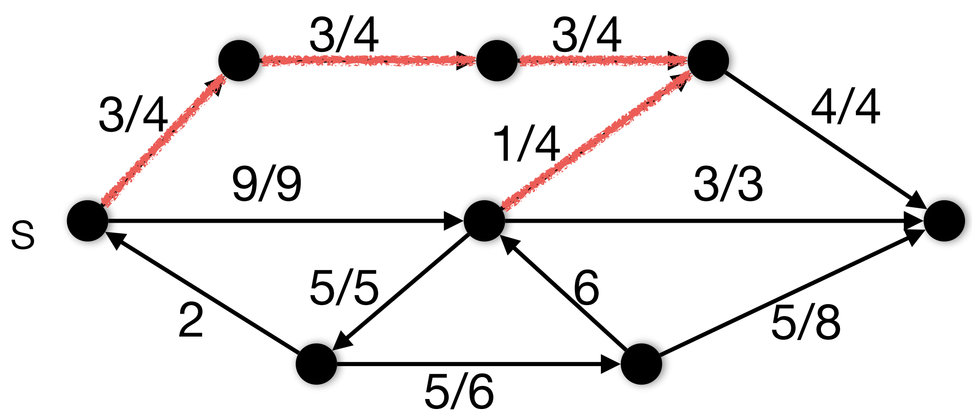

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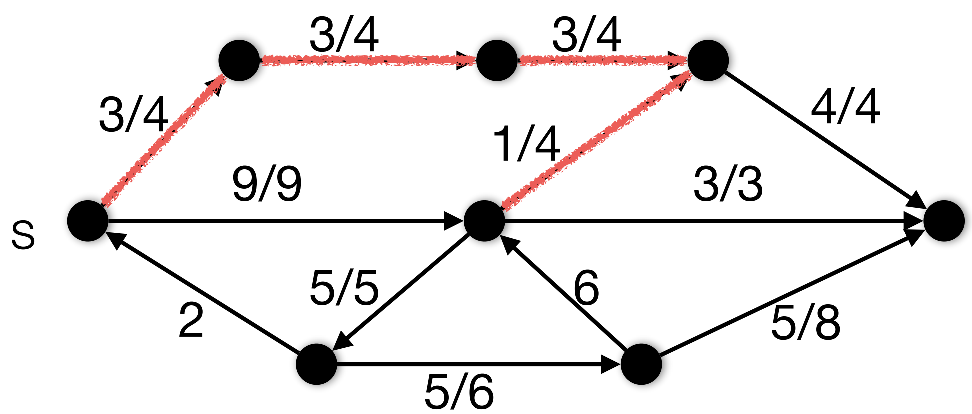

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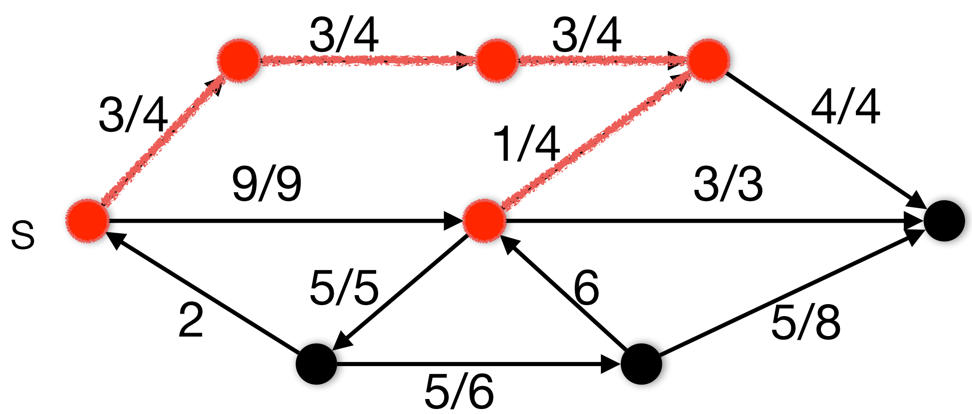

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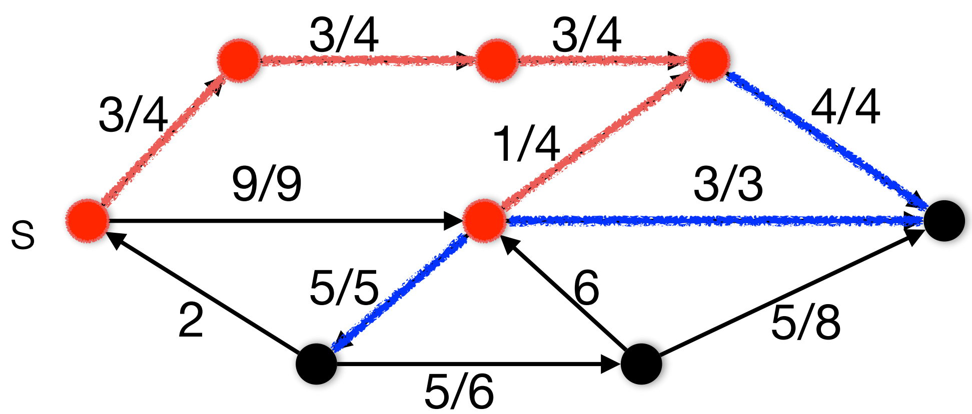

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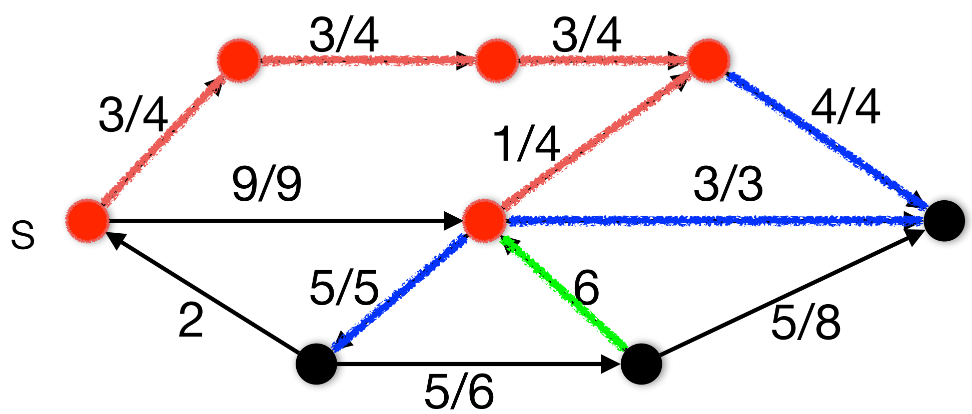

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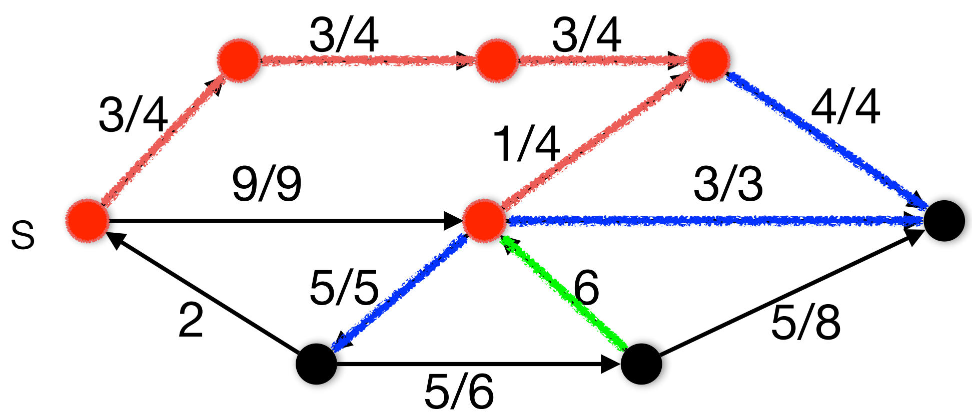

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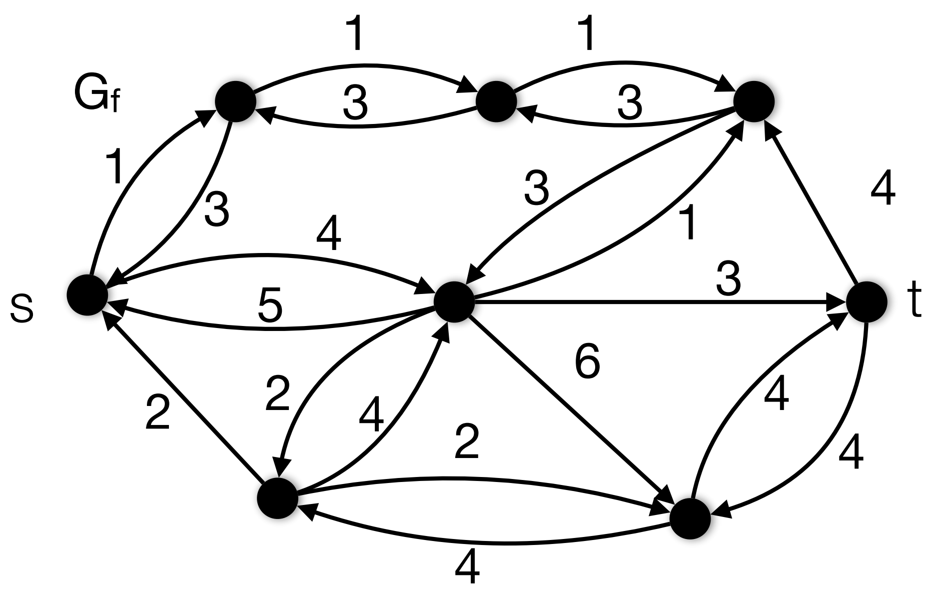

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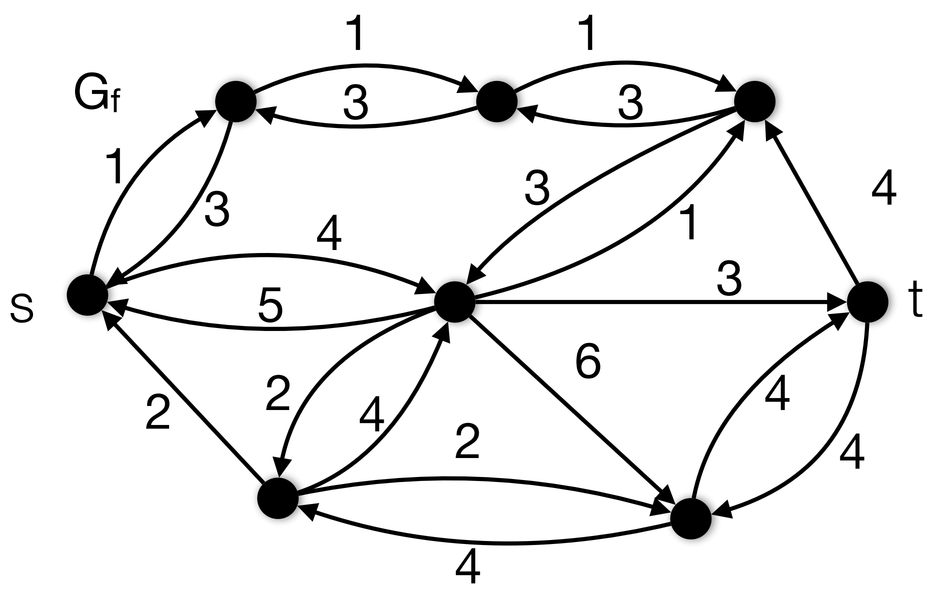

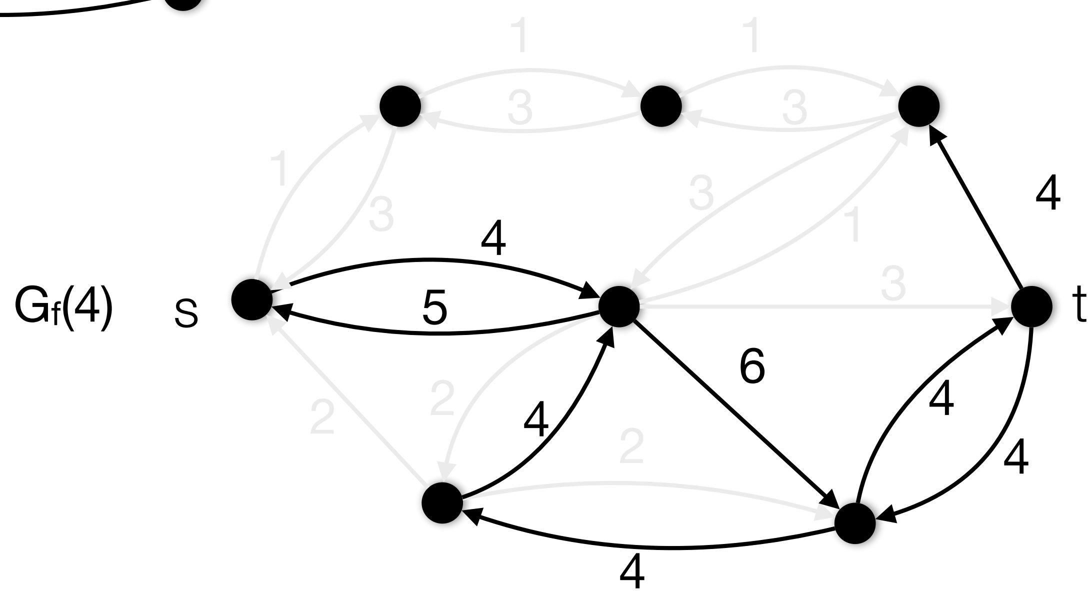

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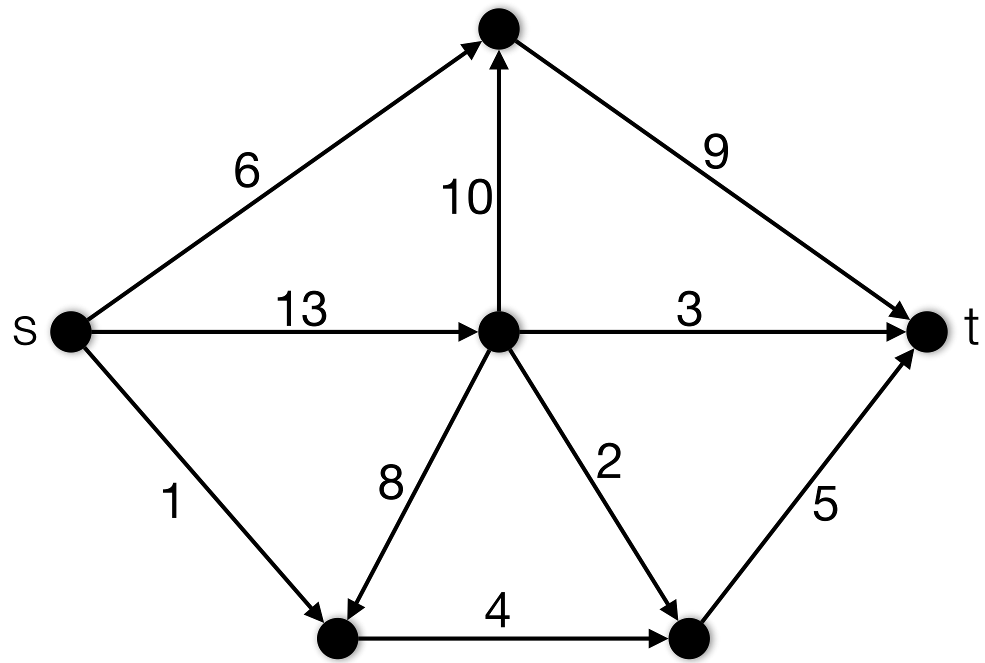

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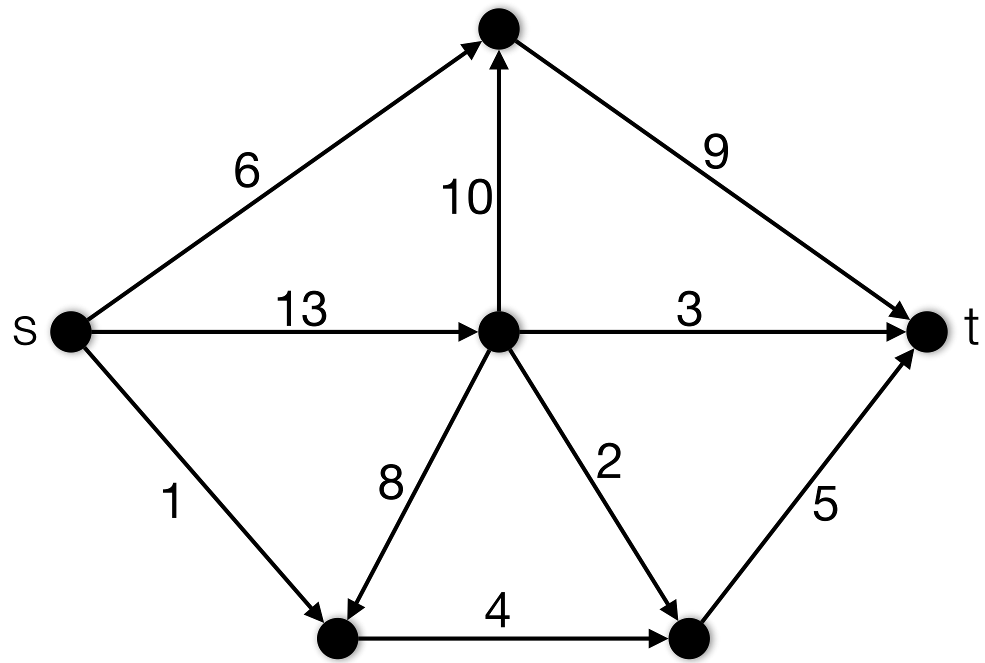

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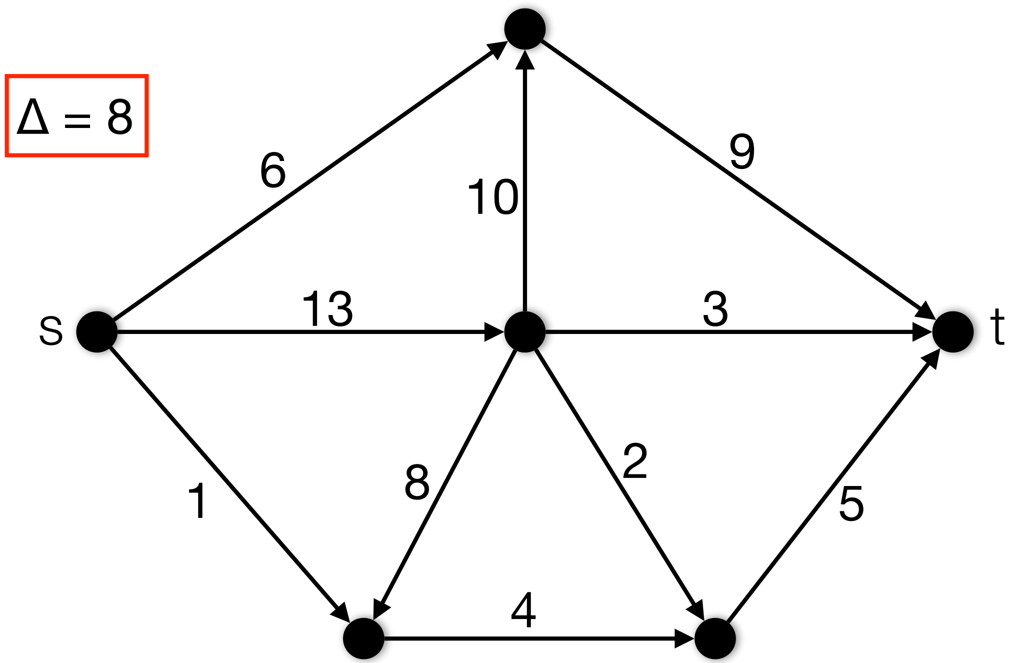

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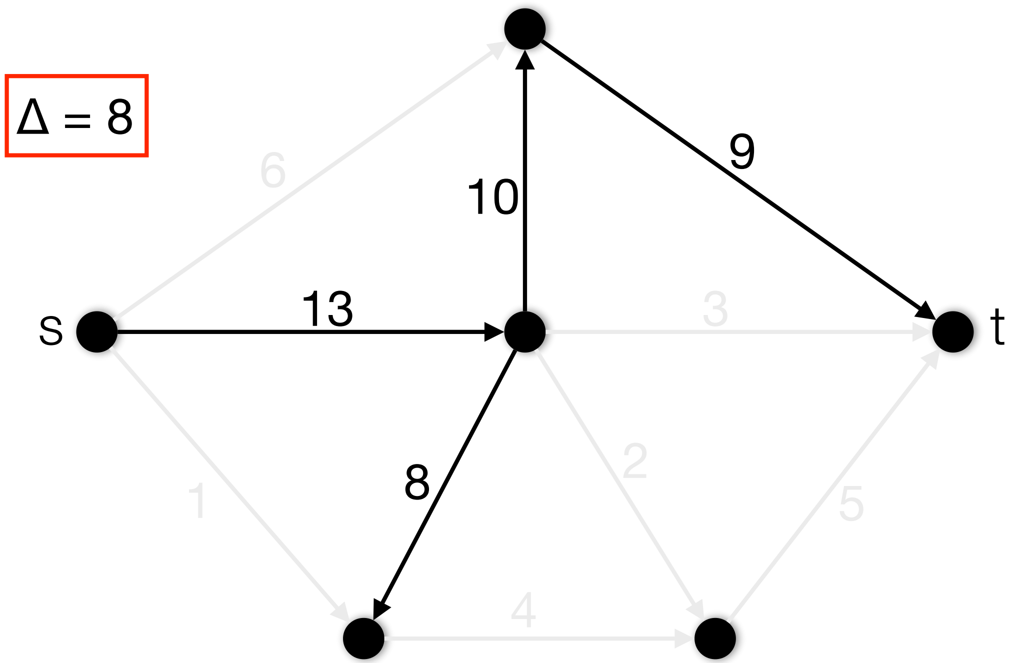

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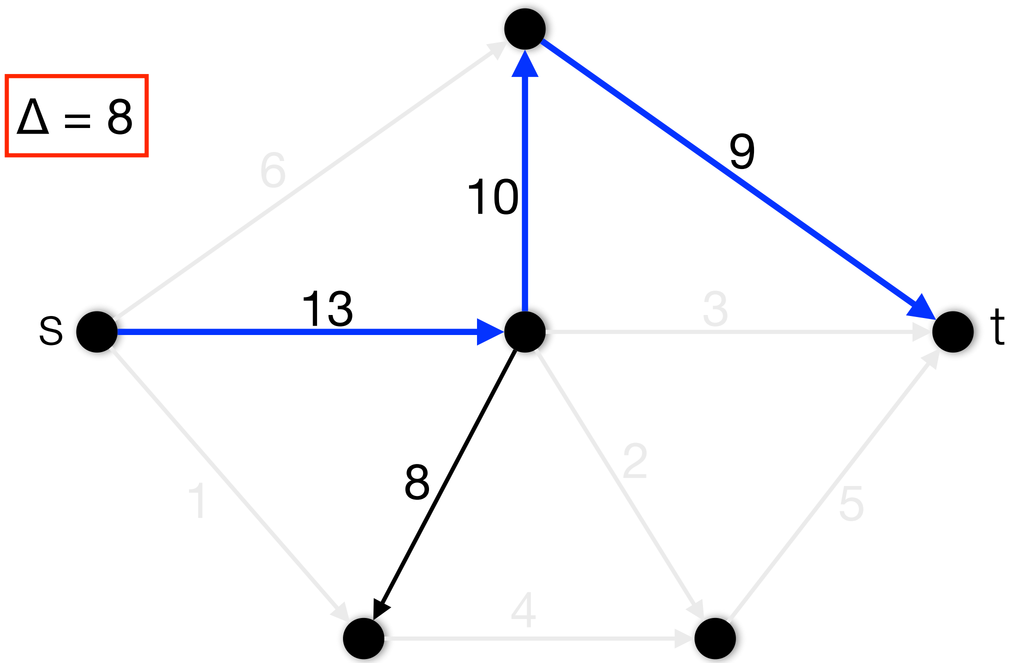

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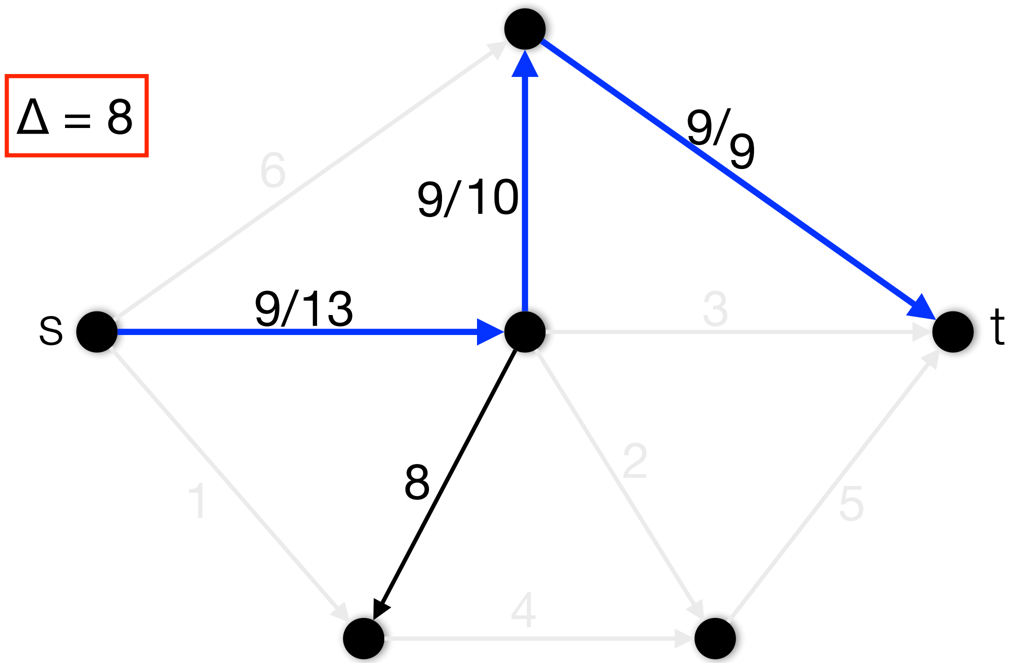

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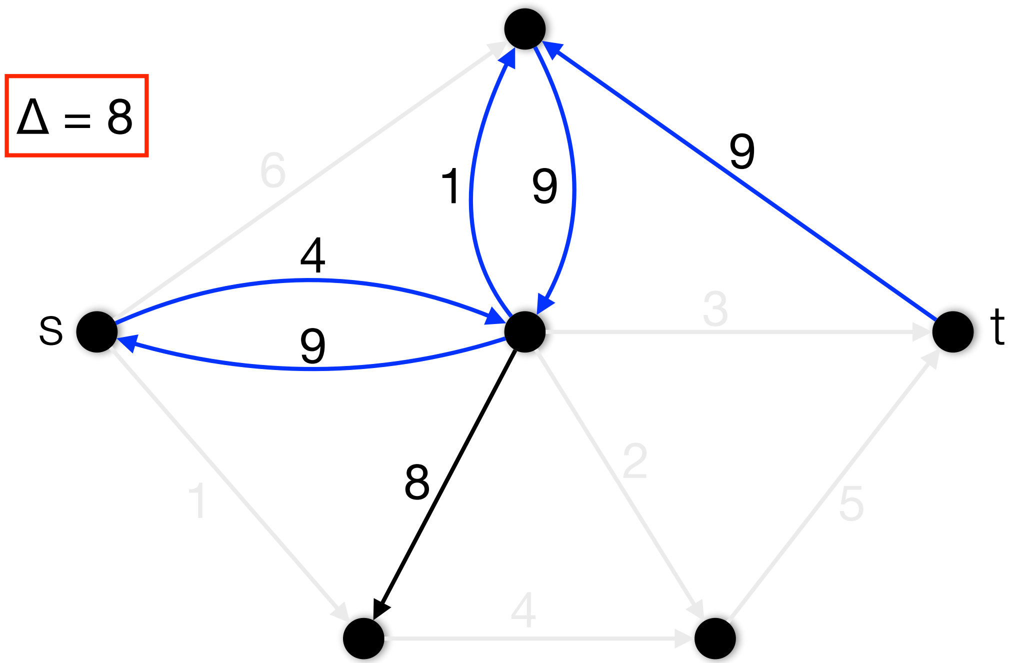

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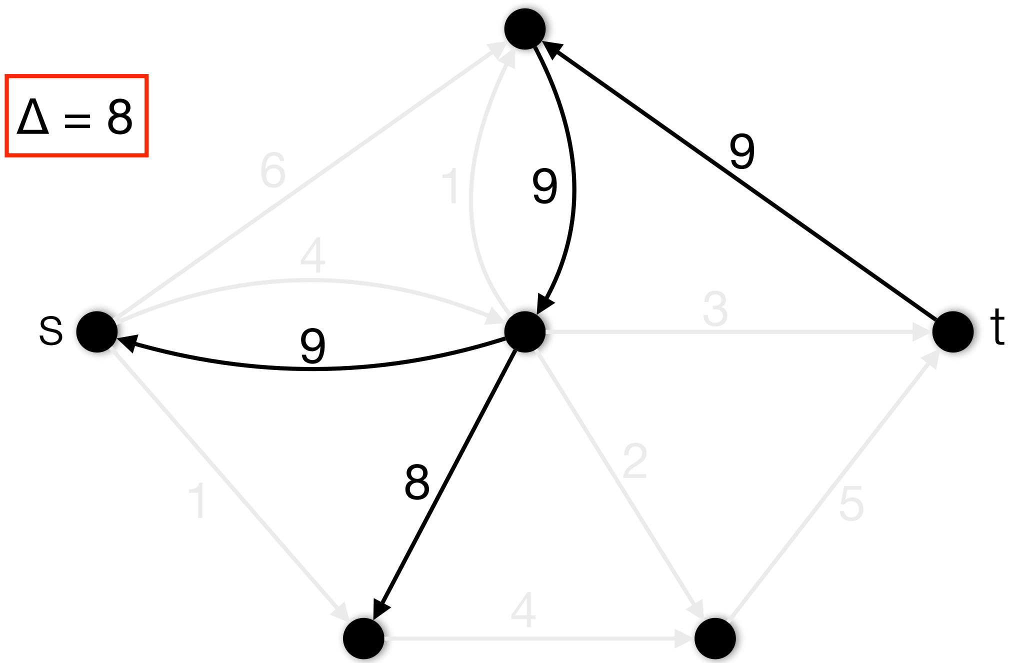

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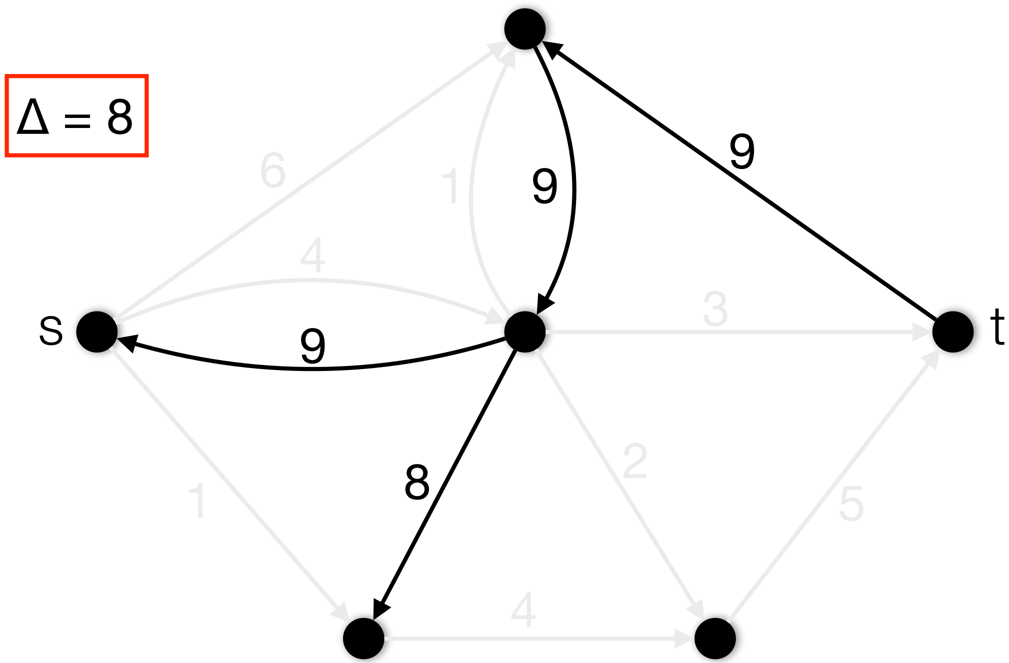

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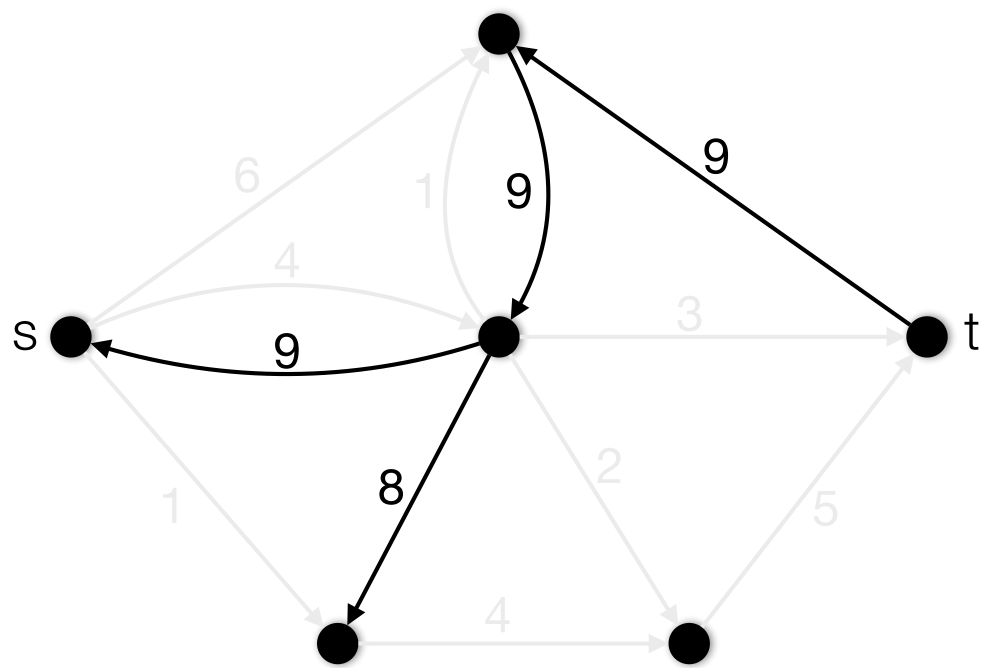

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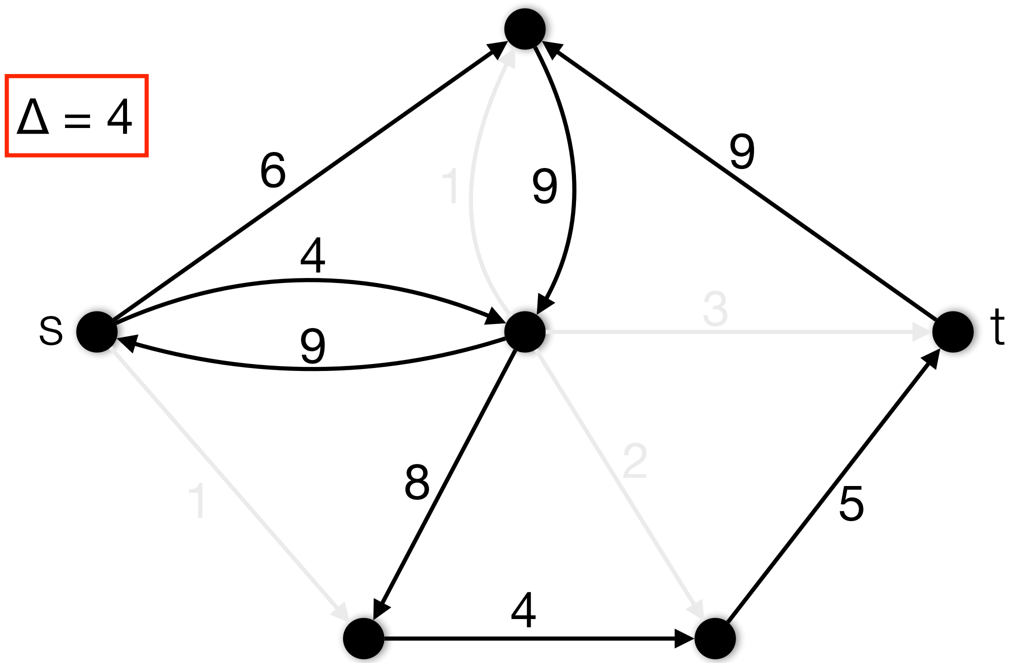

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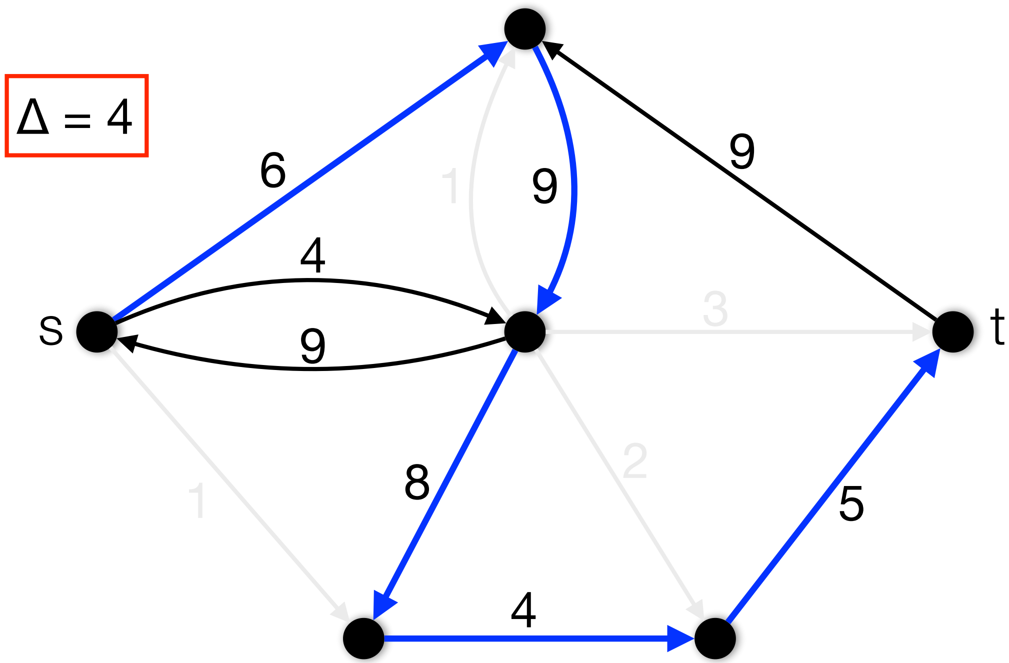

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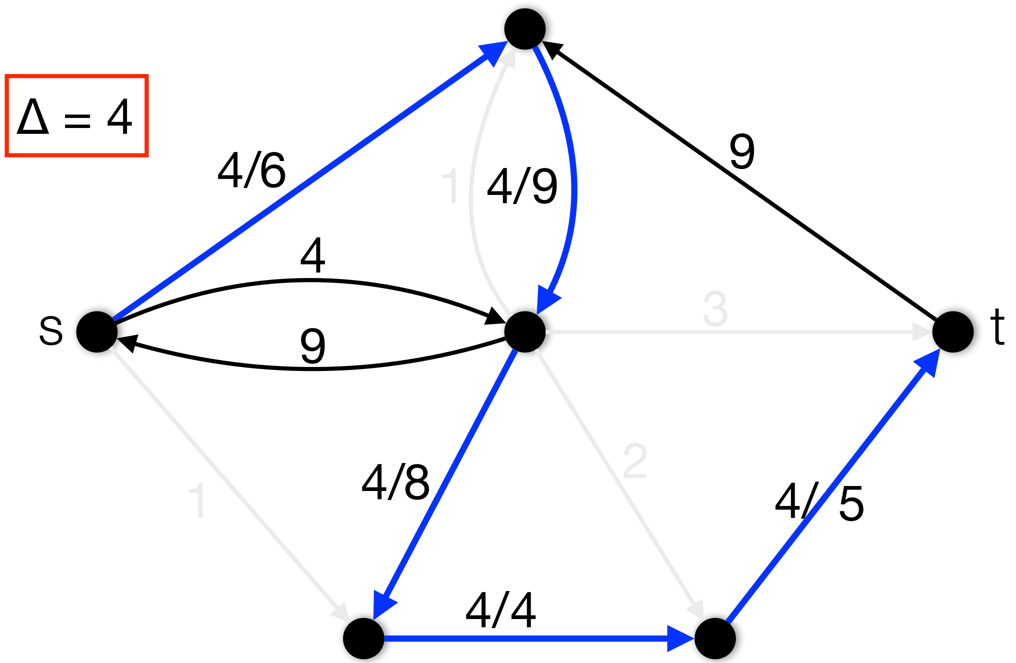

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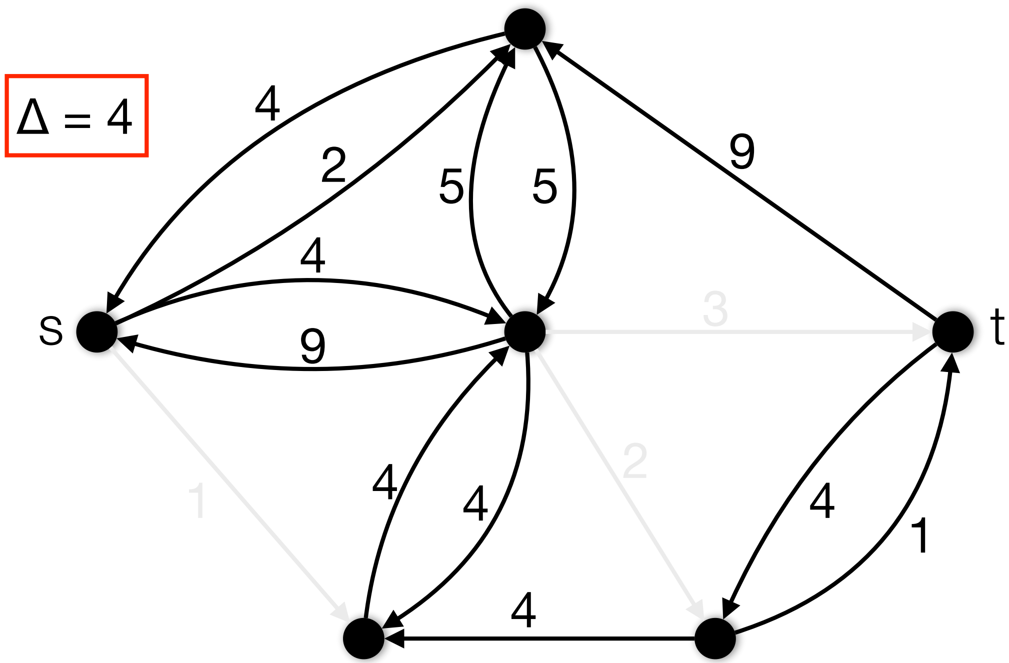

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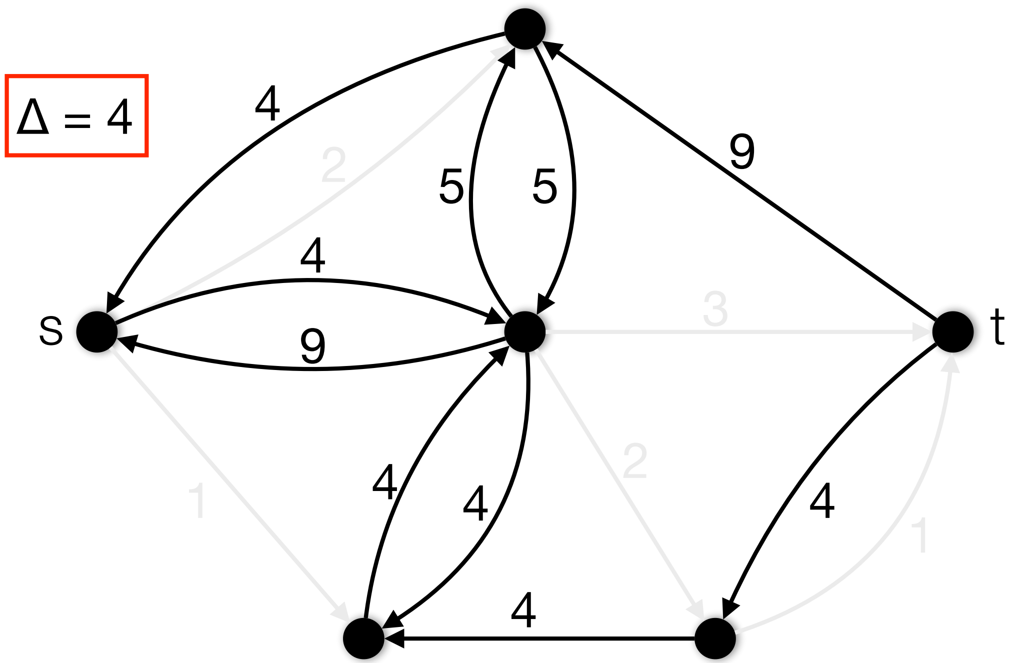

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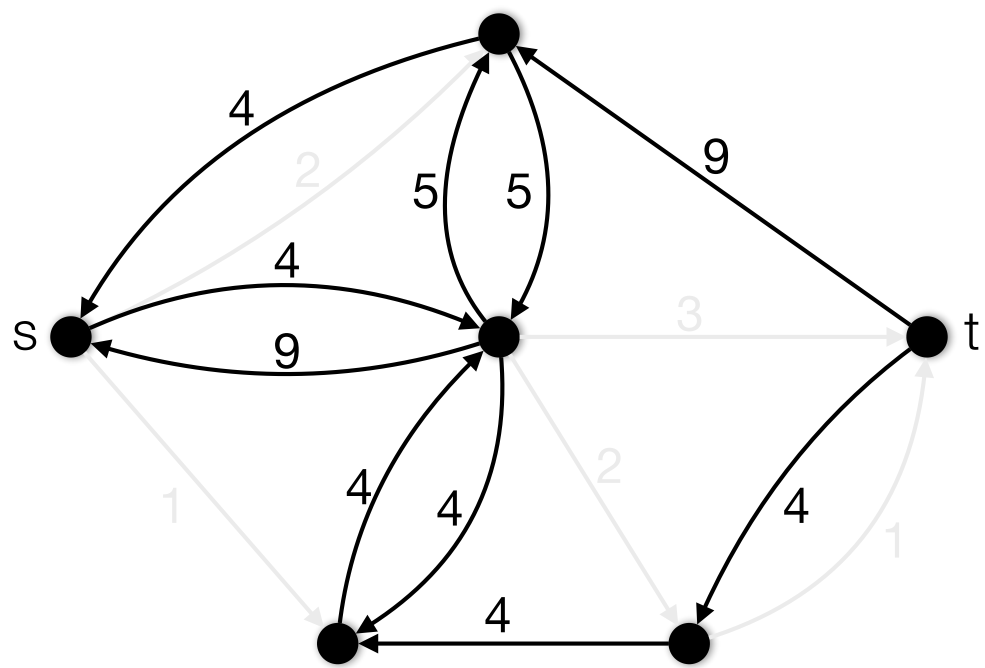

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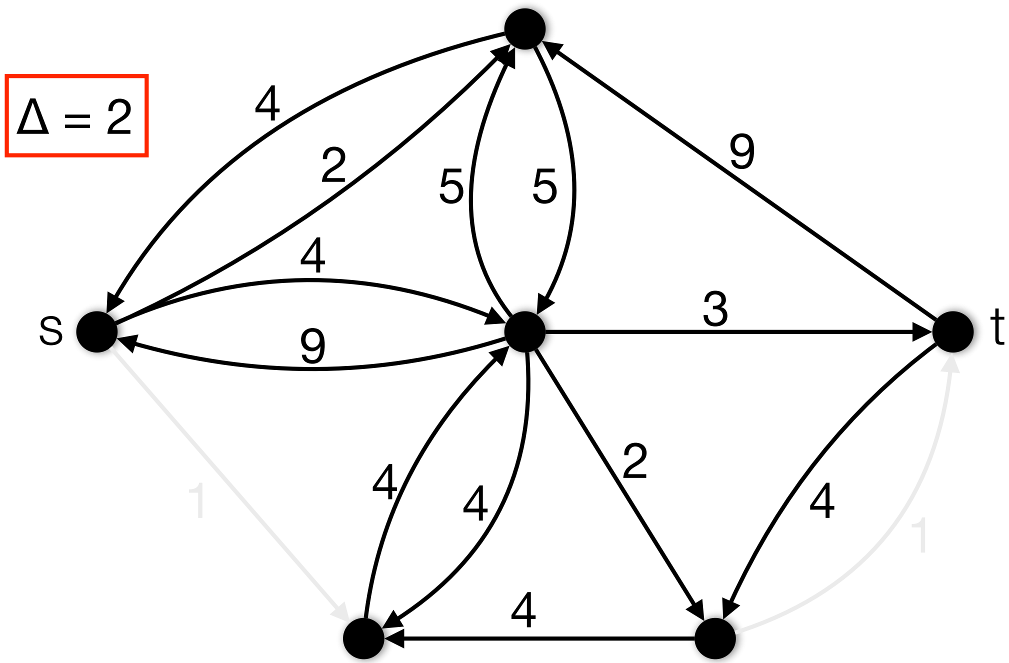

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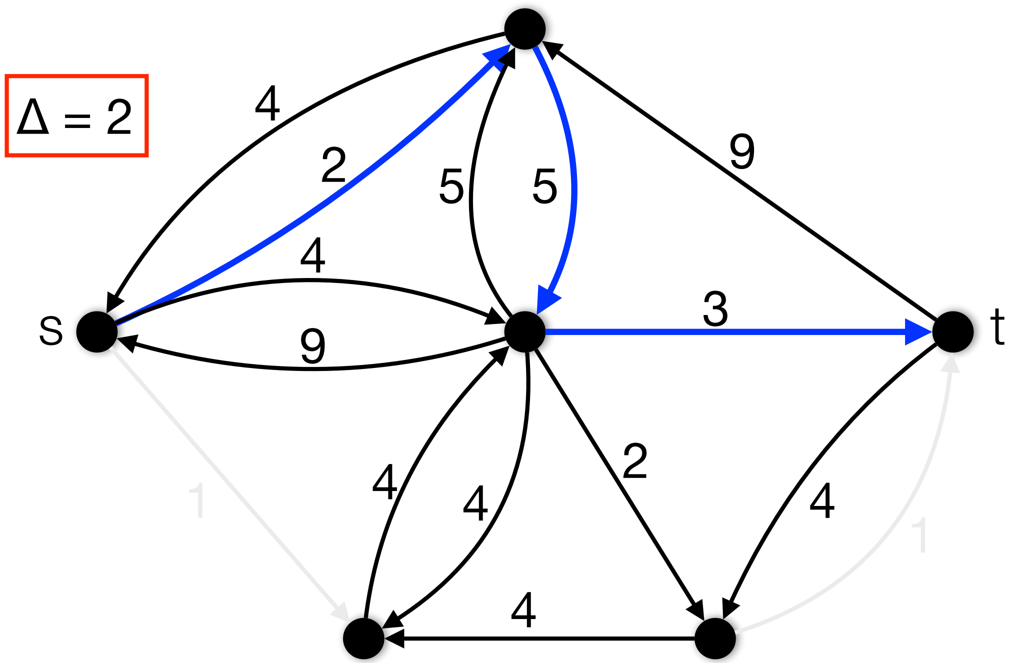

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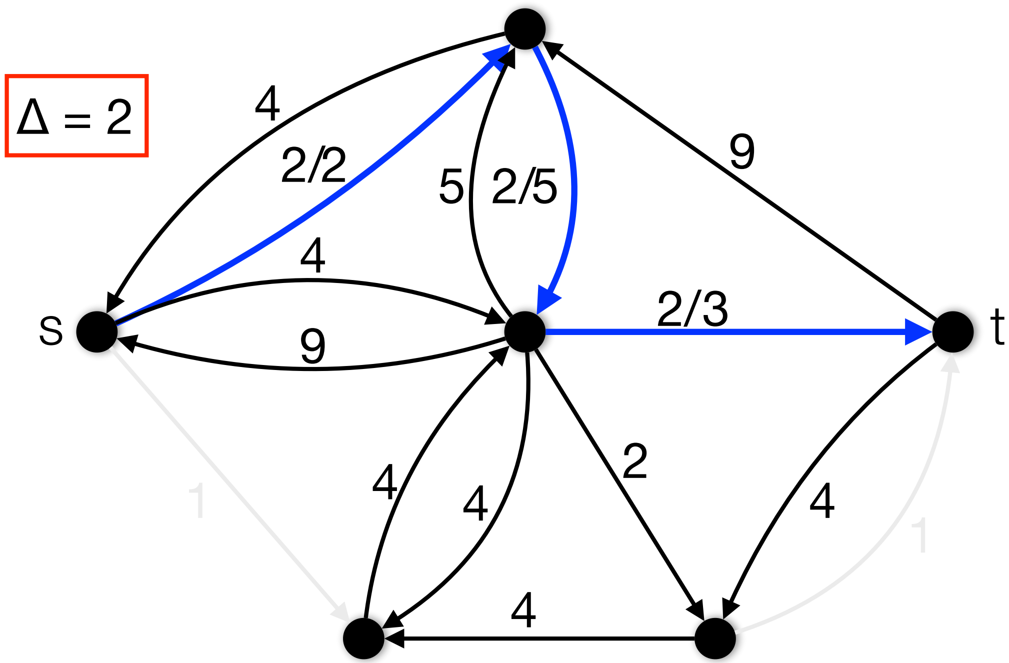

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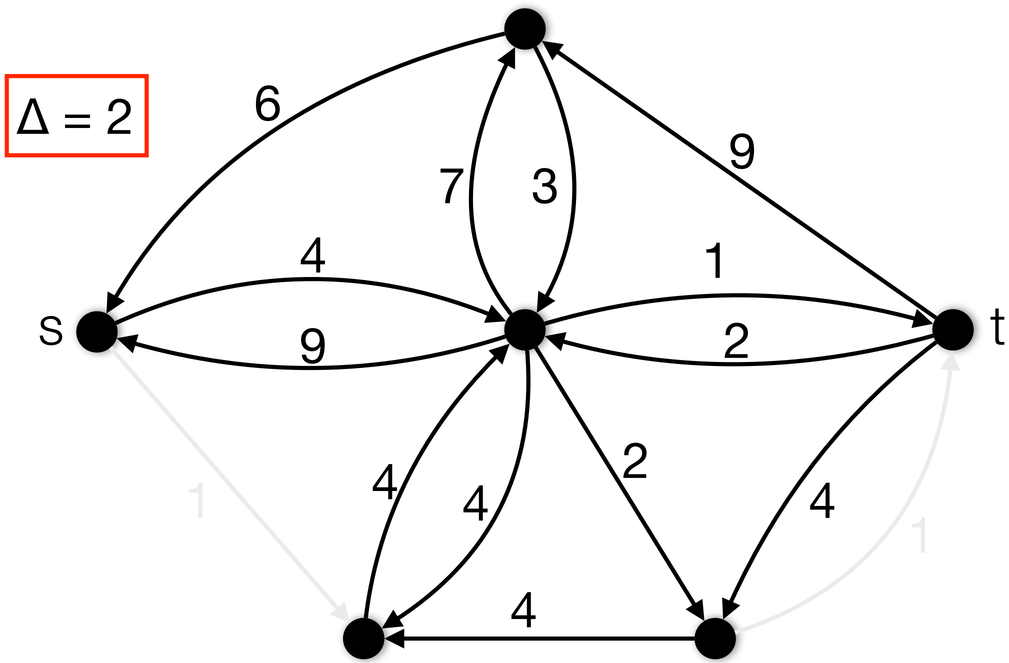

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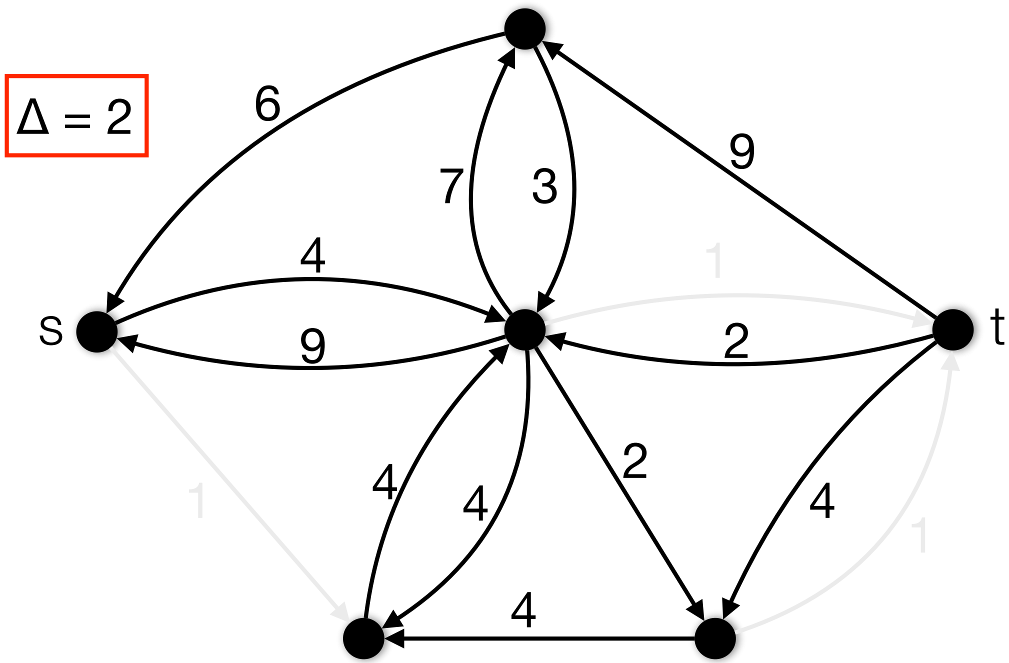

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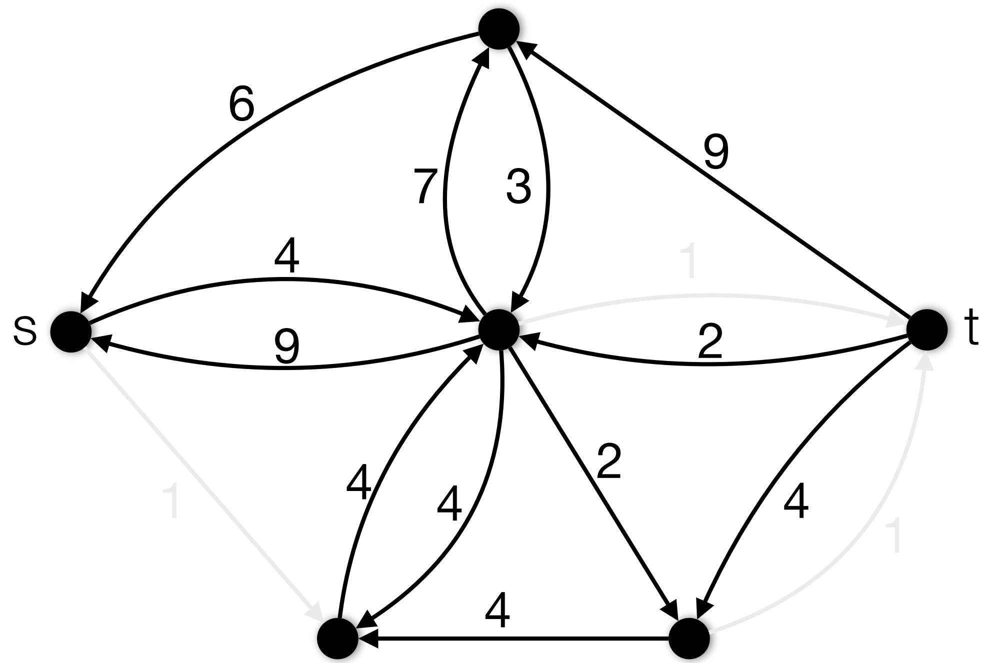

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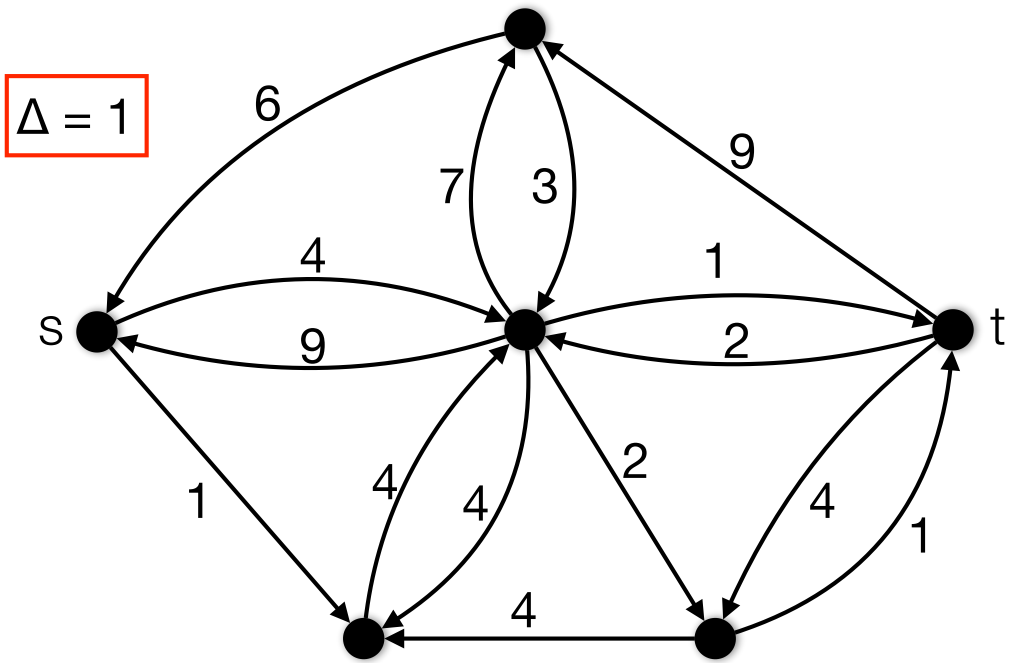

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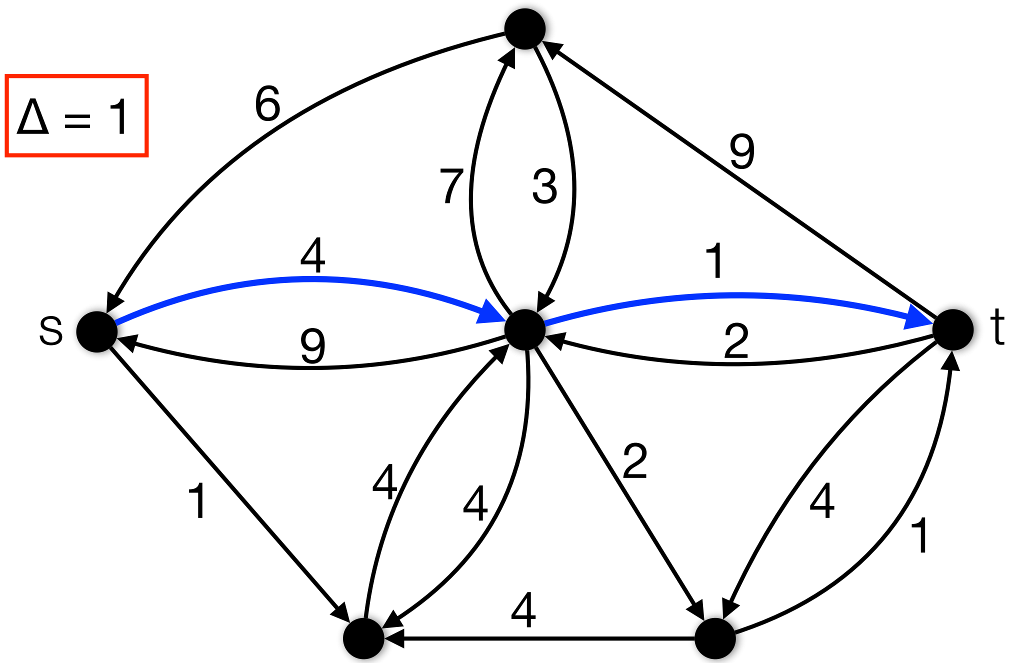

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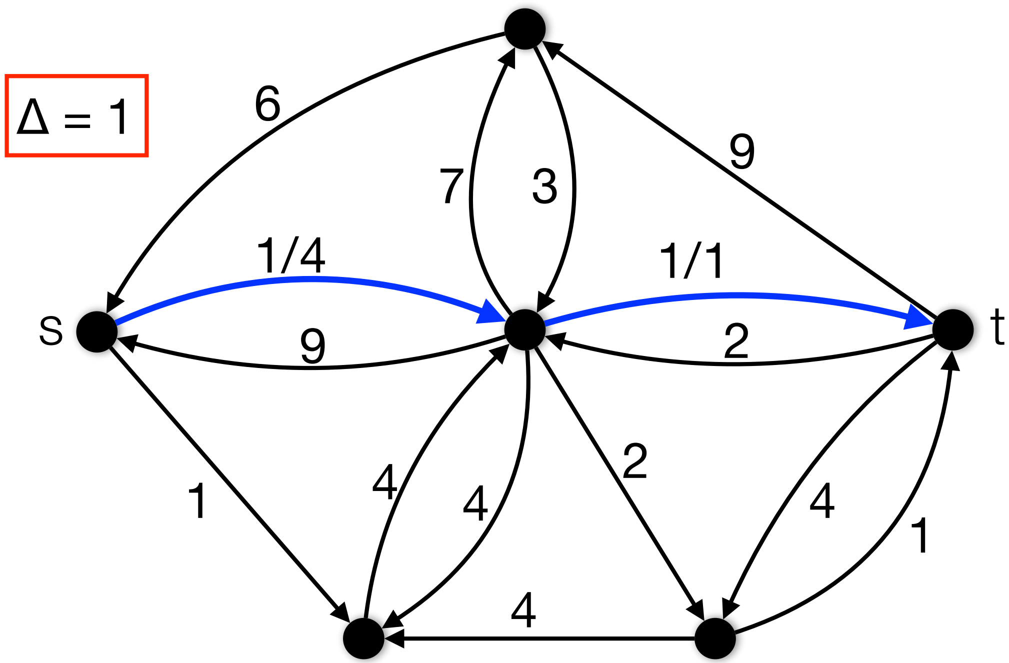

Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
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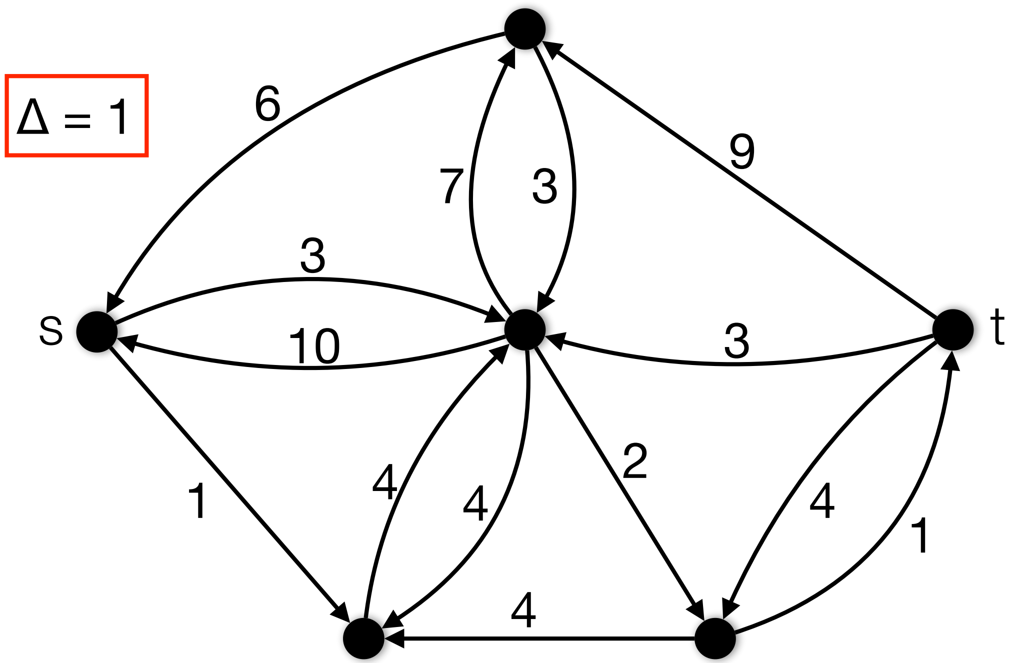

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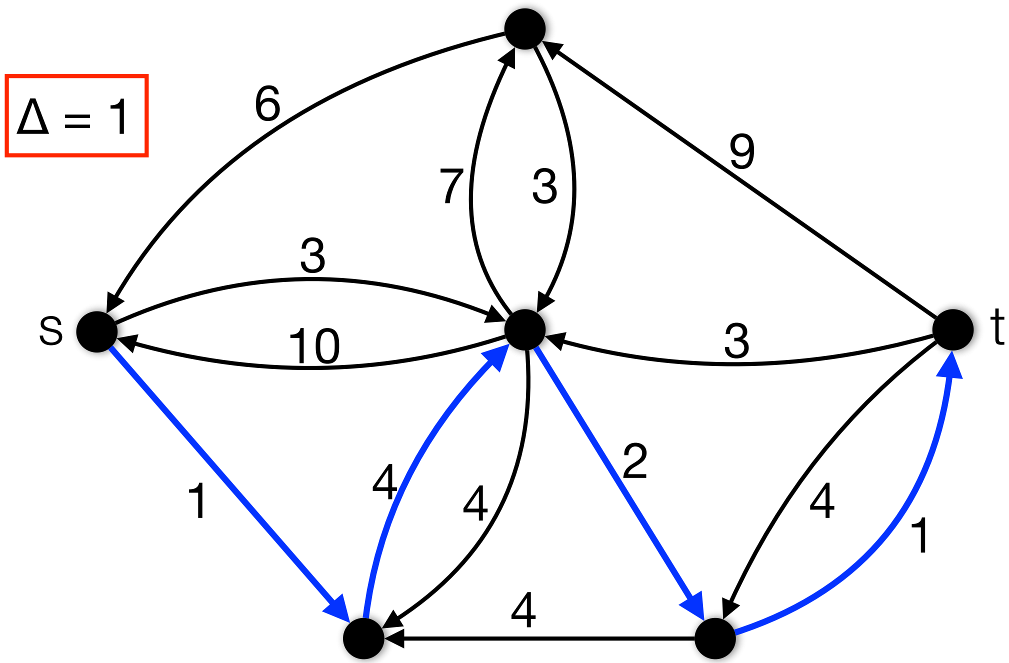

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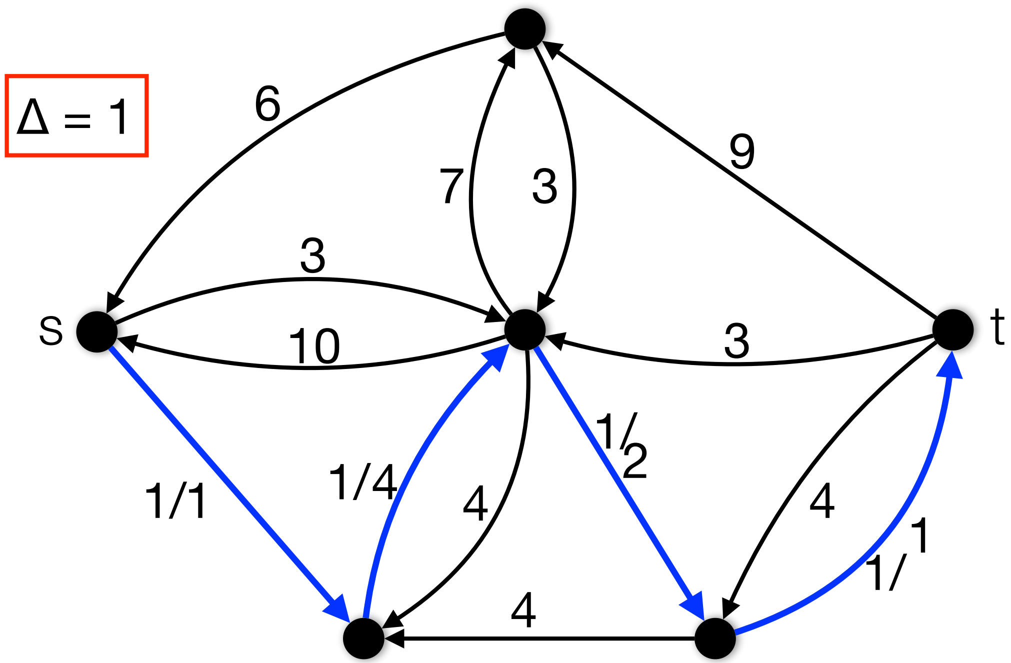

Scaling algorithm

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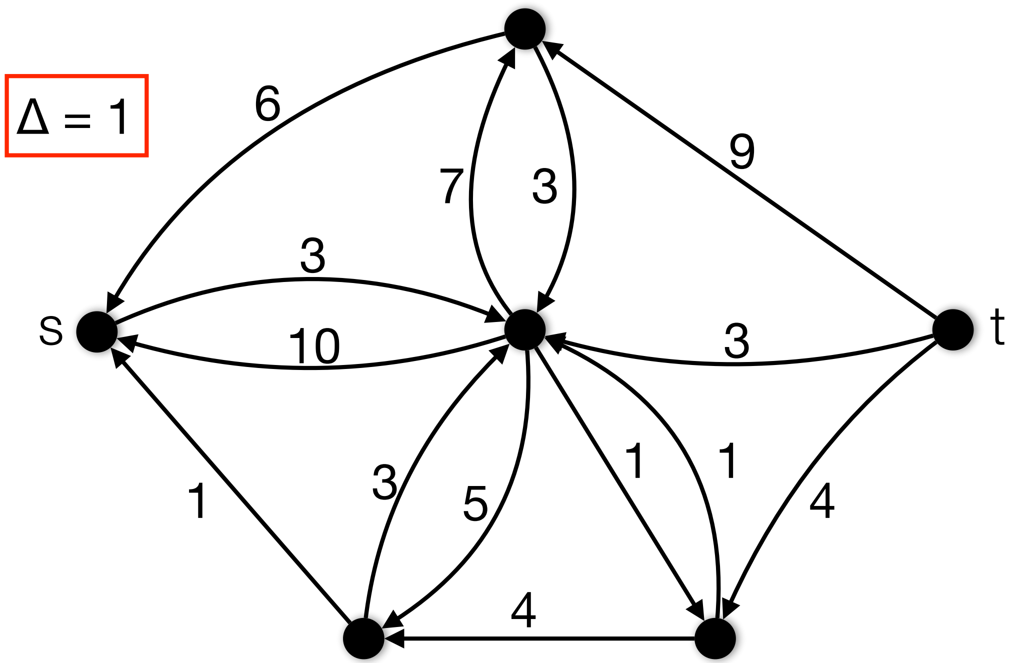

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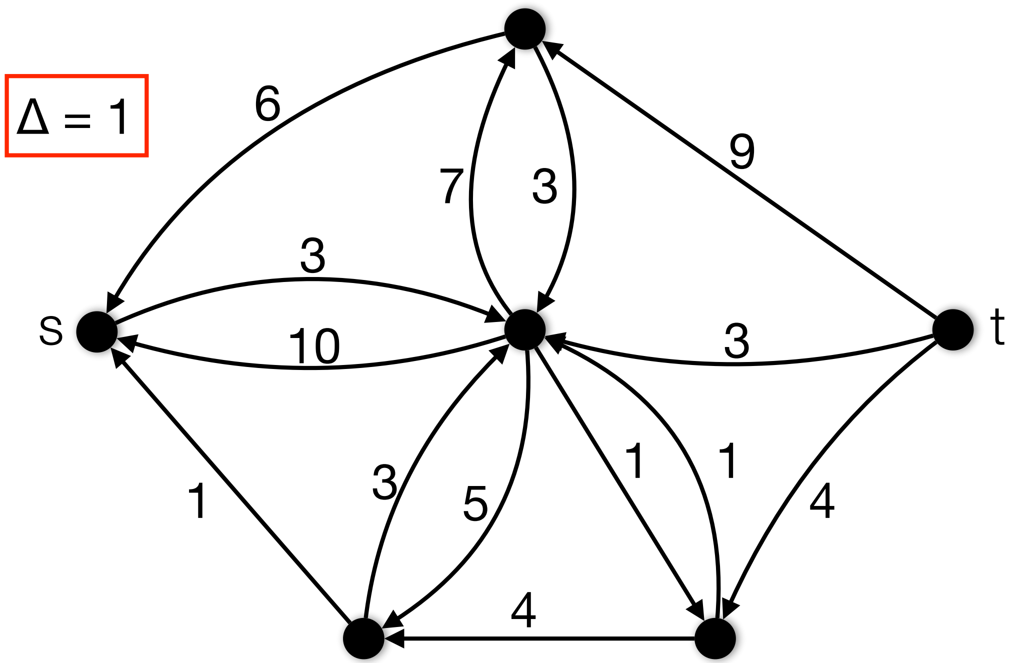
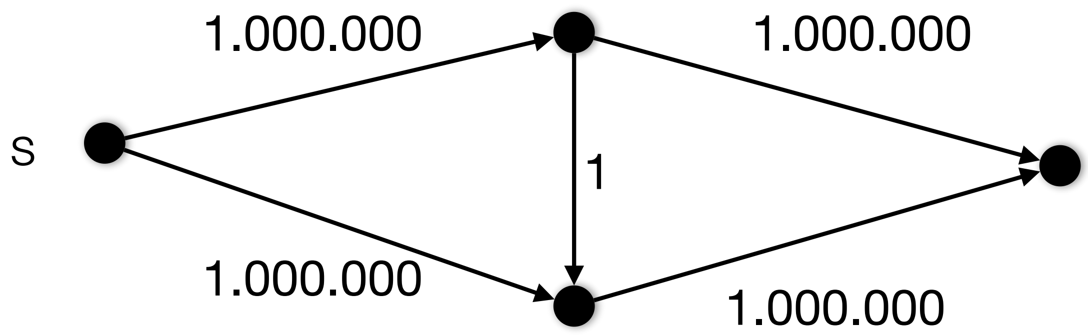


- Stop when no more augmenting paths in $G_f(1)$.

Scaling algorithm



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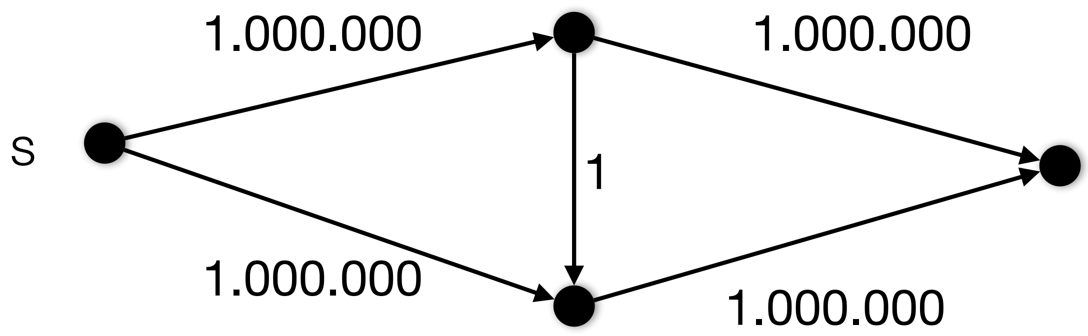

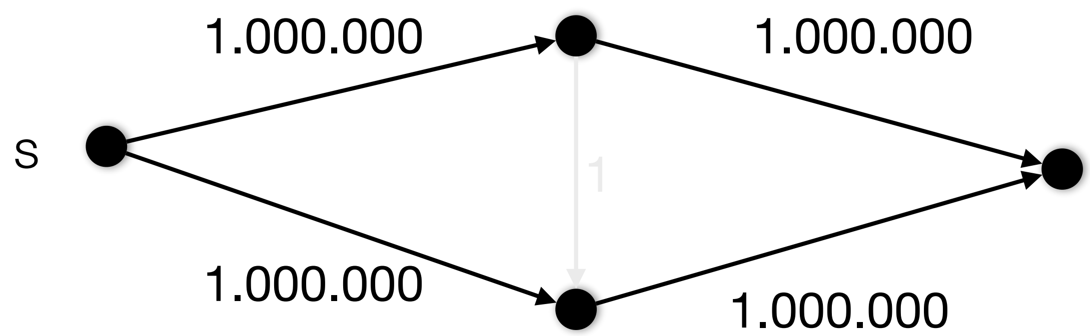

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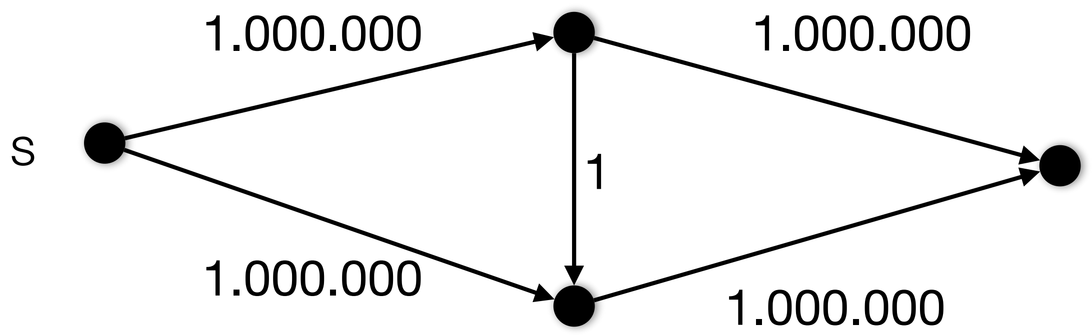

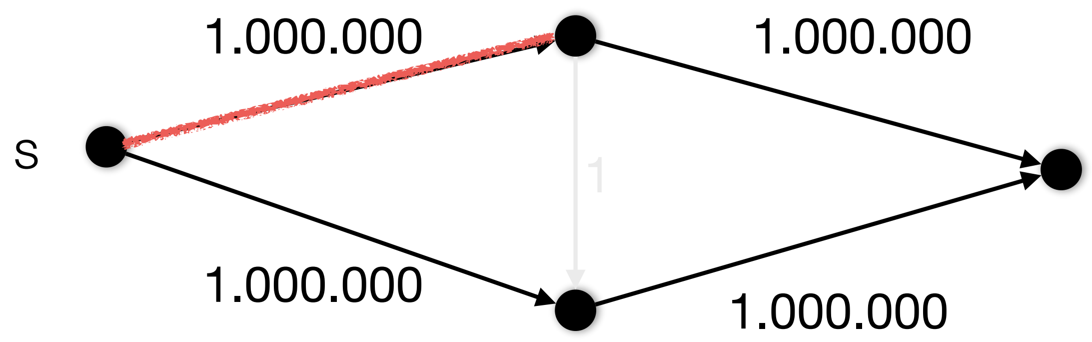

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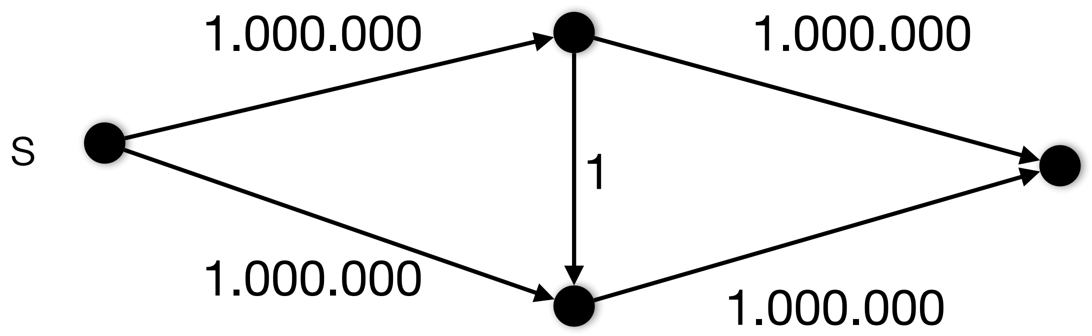

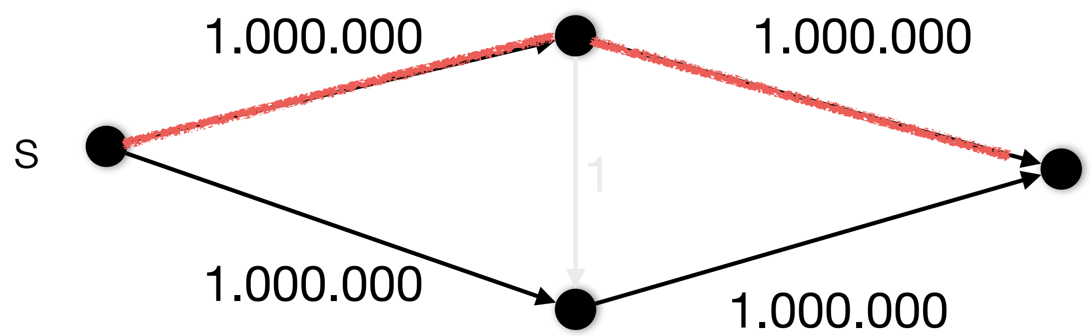

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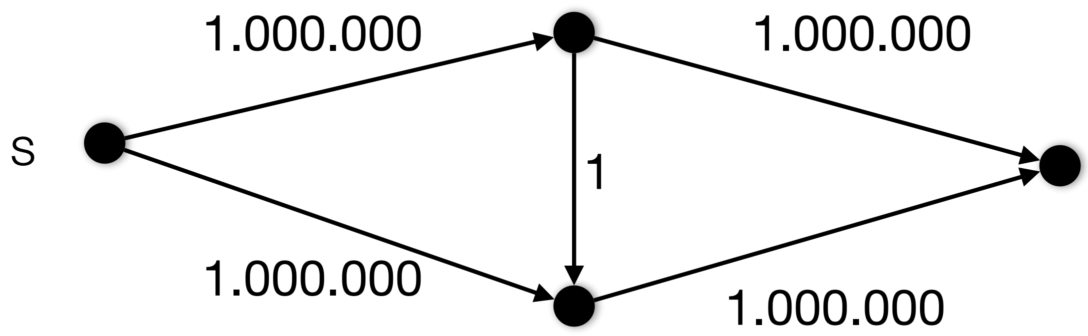

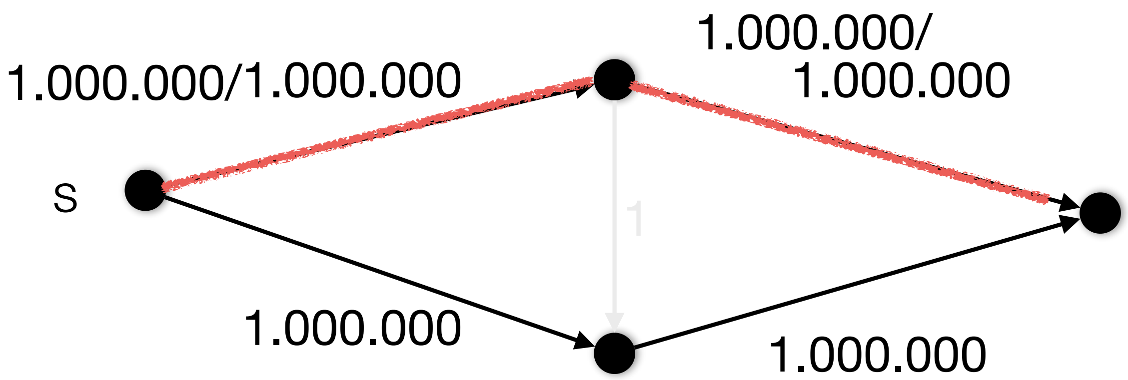

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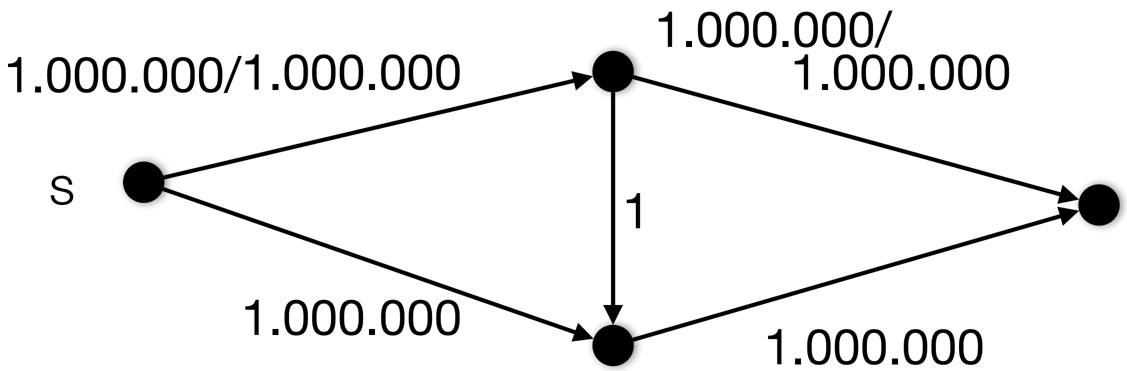

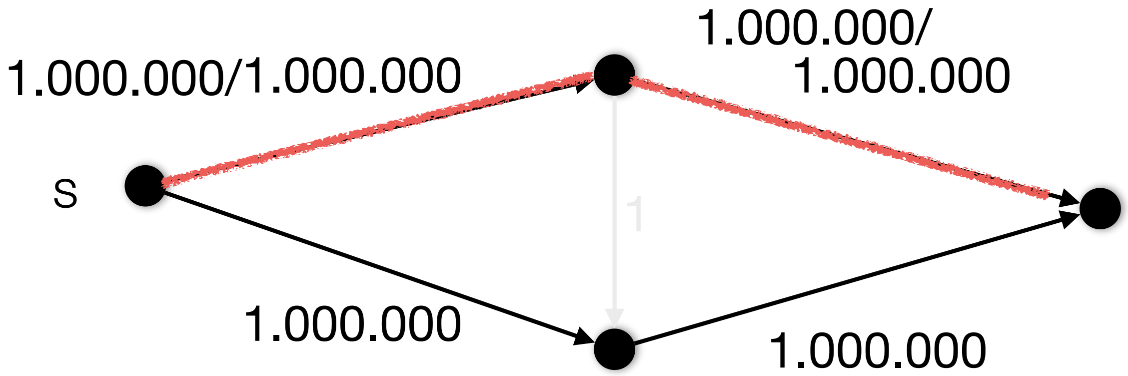

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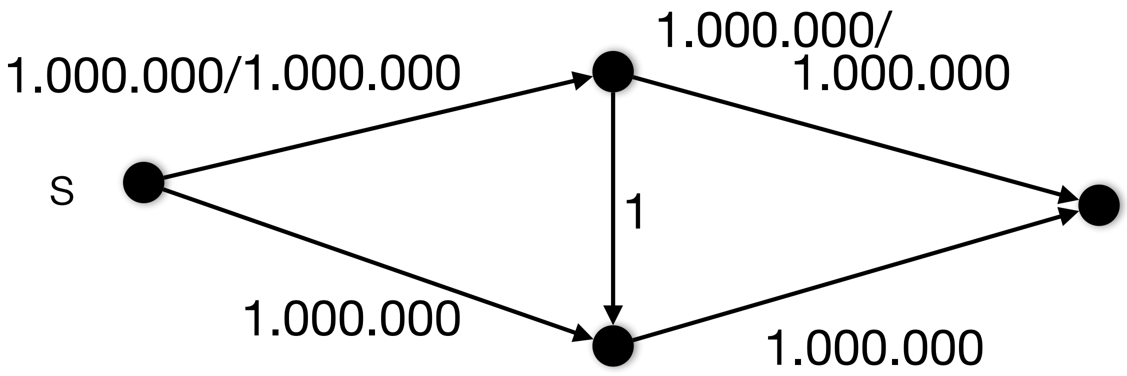

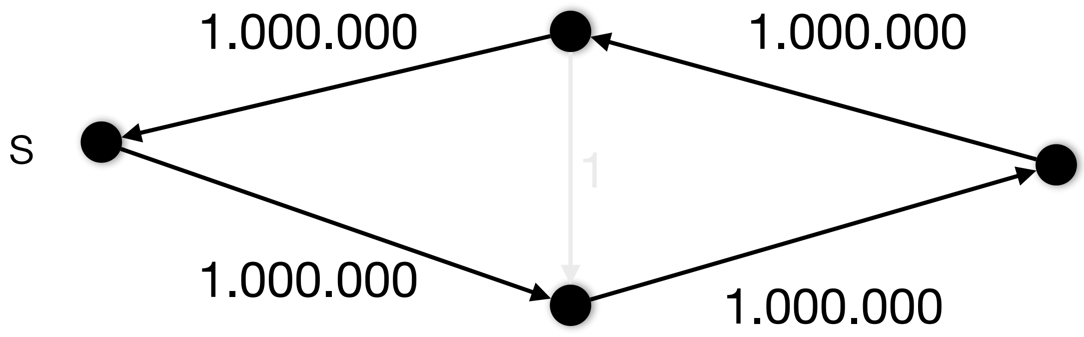

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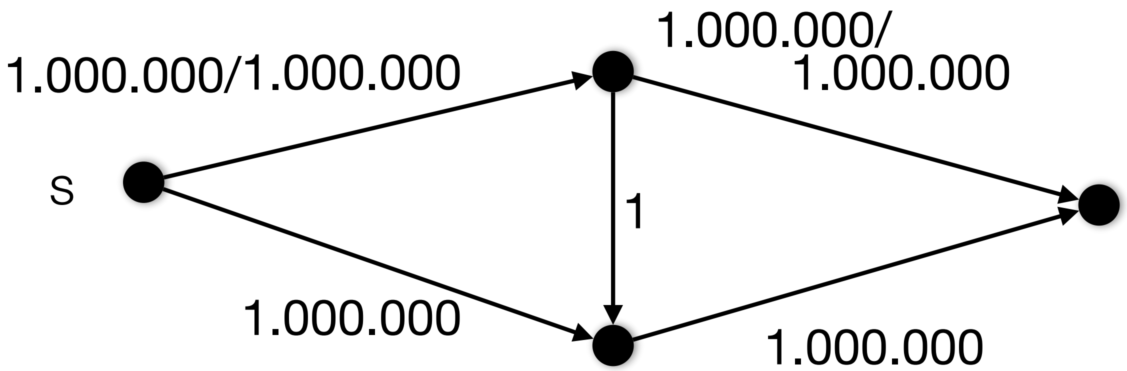

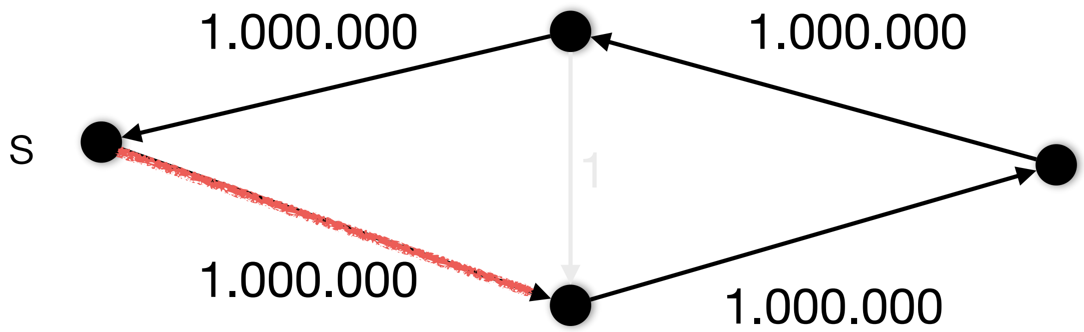

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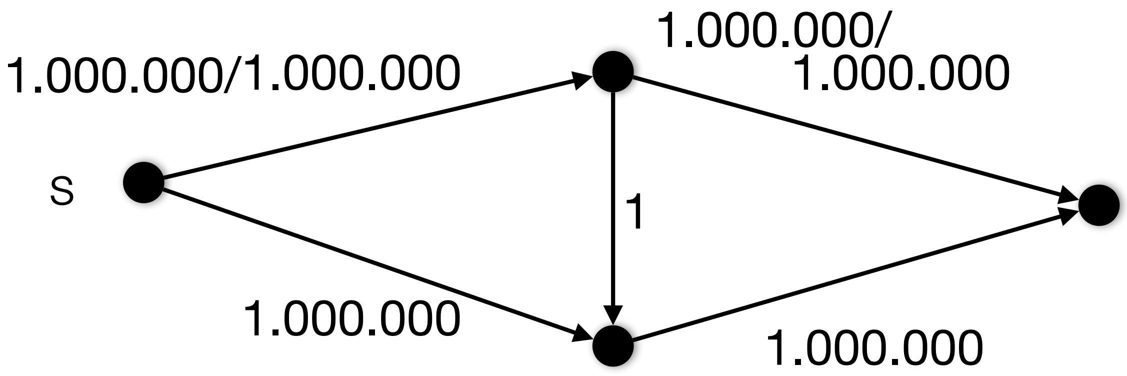

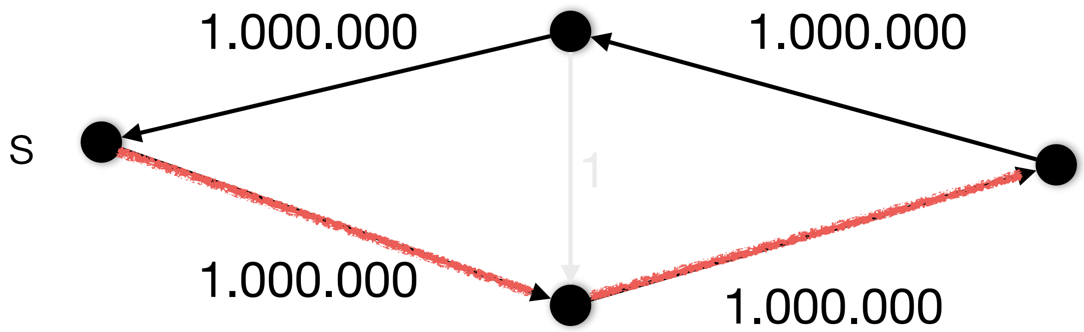

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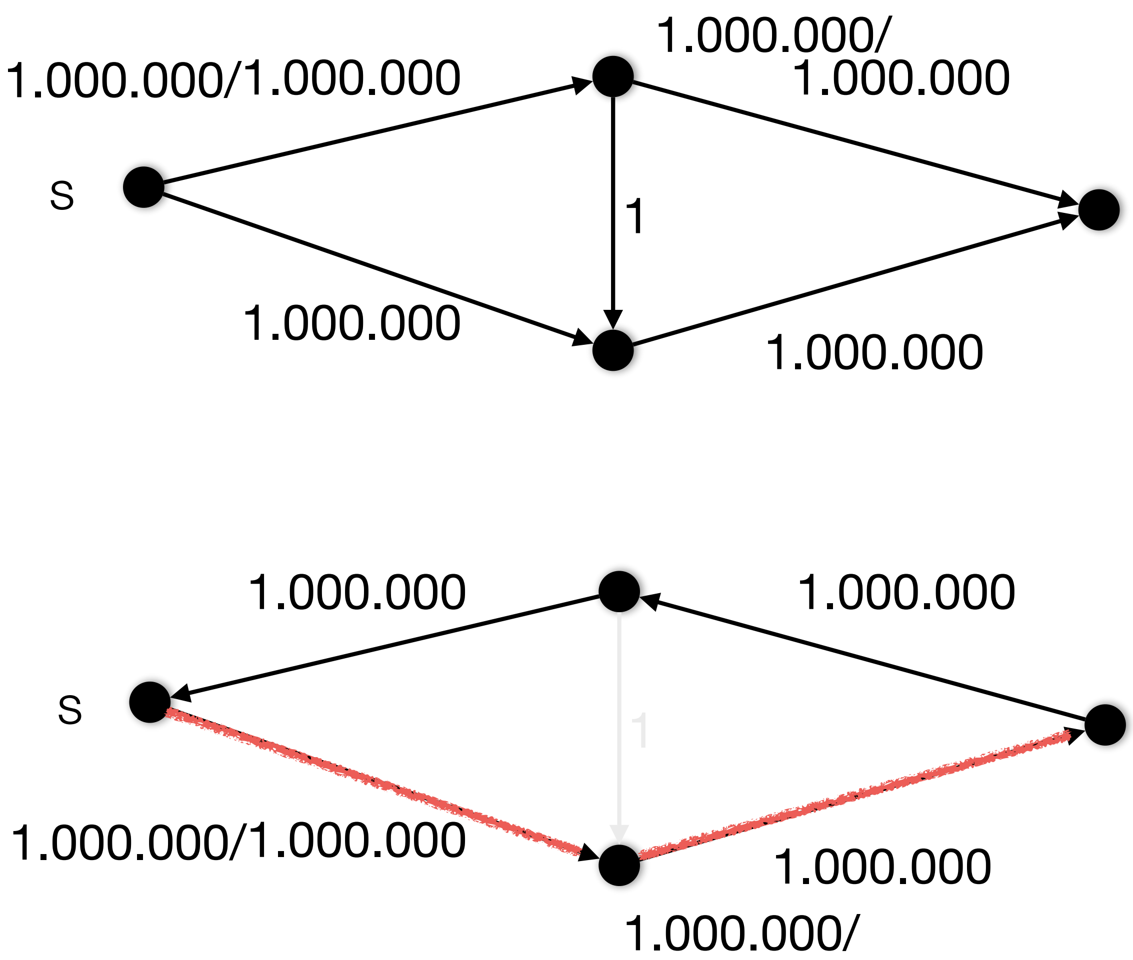

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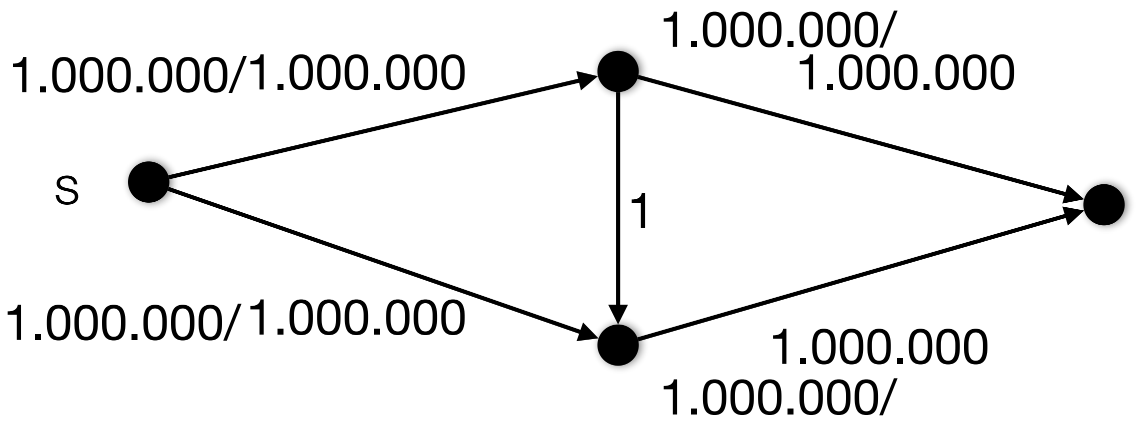

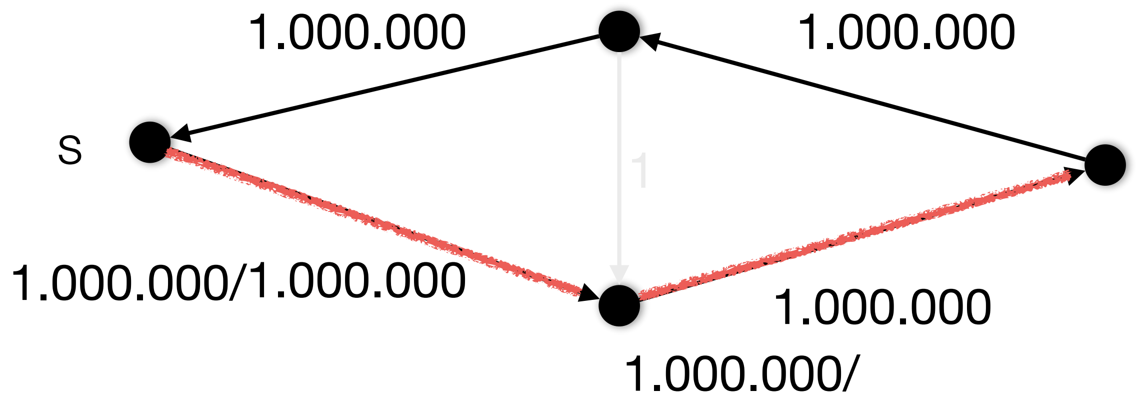

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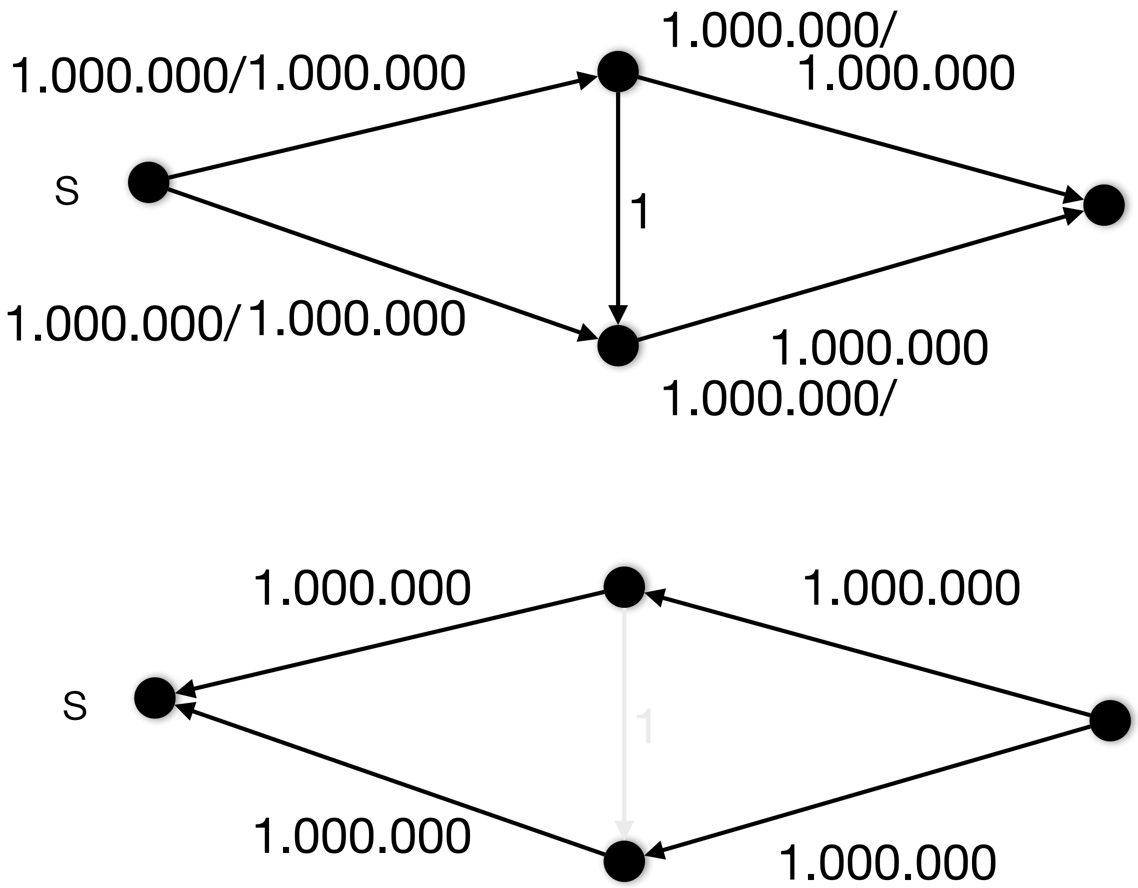

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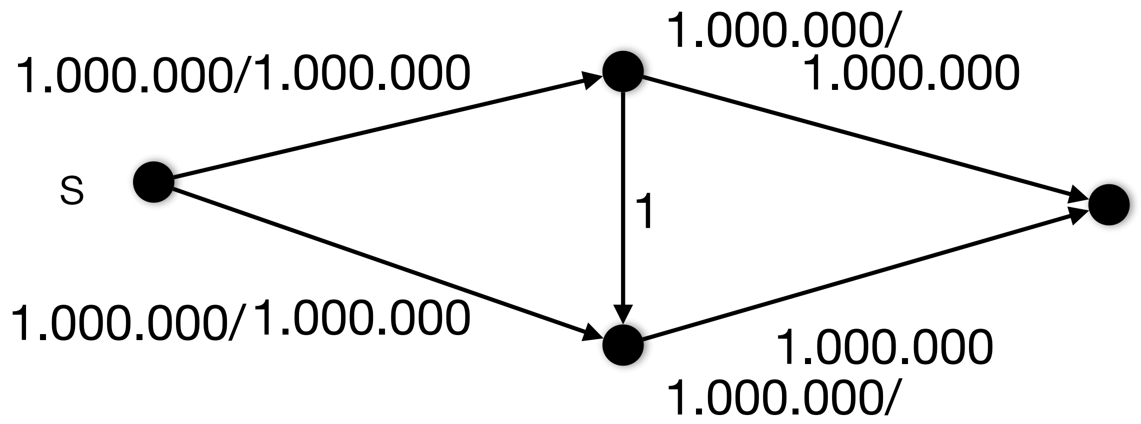
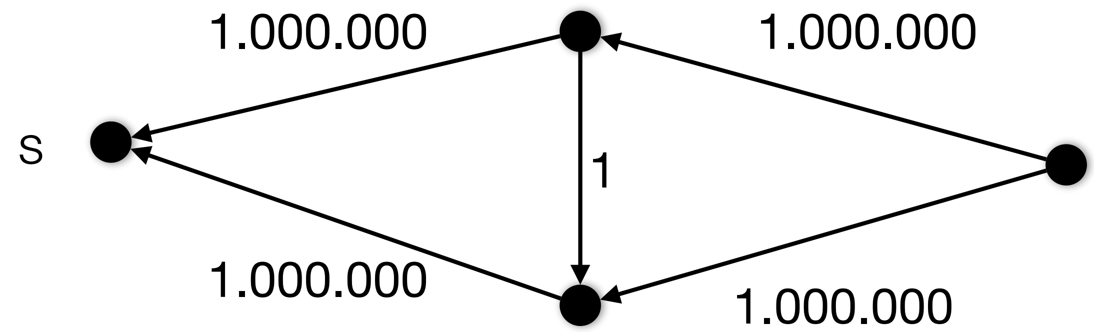




Scaling algorithm

- Running time: $O(m^2 \log C)$, where C is the largest capacity out of s.

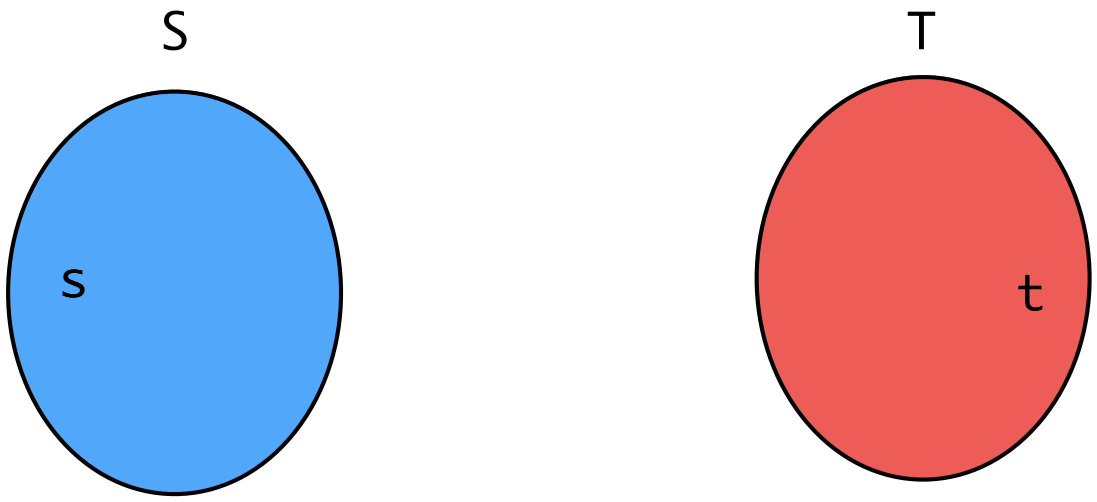
- Lemma 1. *Number of scaling phases:* $1 + \lceil \log C \rceil$

- Lemma 2. *Let f the flow when Δ-scaling phase ends, and let f*be the maximum flow. Then* $v(f^*) \leq v(f) + m\Delta$.

- Lemma 3. *The number of augmentations in a scaling phase is at most 2m.*
  - First phase: can use each edge out of s in at most one augmenting path.
  - f flow at the end of previous phase.
  - Used $\Delta' = 2\Delta$ in last round.
  - Lemma 2: $v(f^*) \leq v(f) + m\Delta' = v(f) + 2m\Delta$.
  - "Leftover flow" to be found $\leq 2m\Delta$.
  - Each agumentation in a Δ-scaling phase augments flow with at least Δ.

Scaling algorithm

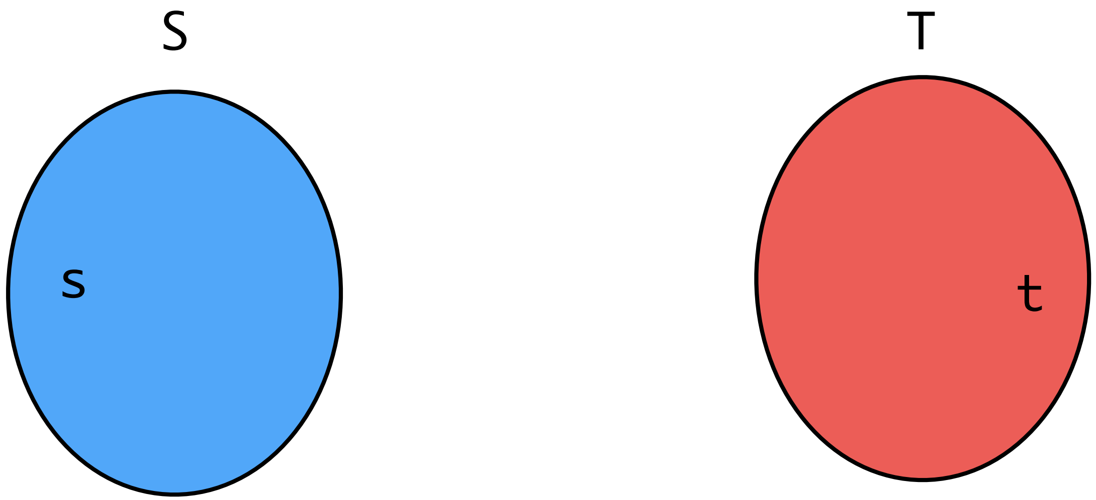
- Lemma 2. *Let f the flow when Δ-scaling phase ends, and let f*be the maximum flow. Then $v(f^*) \leq v(f) + m\Delta$.*



Scaling algorithm

- Lemma 2. *Let f the flow when Δ-scaling phase ends, and let f*be the maximum flow. Then v(f*) ≤ v(f) + mΔ.*

- By the end of the phase there is a cut c(S,T) ≤ v(f) + mΔ.



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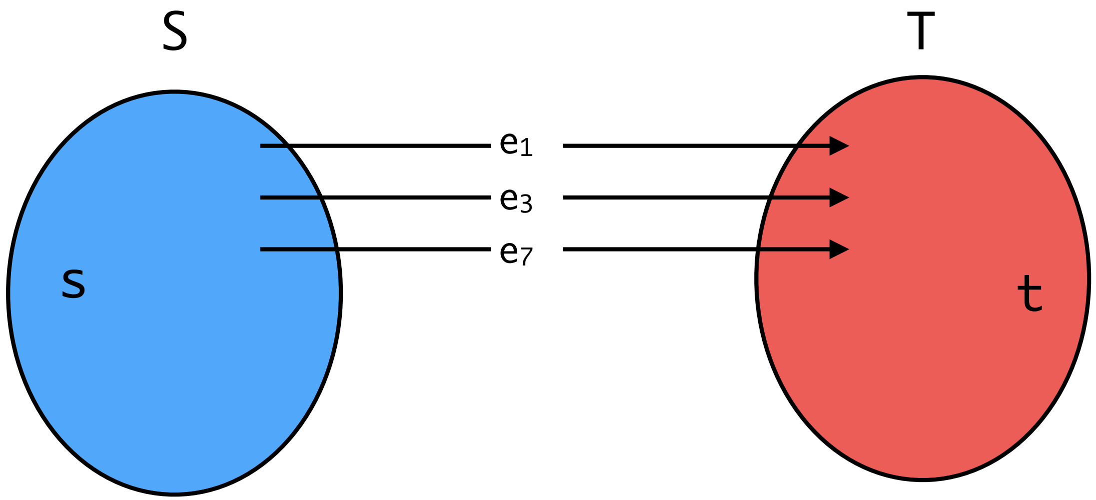

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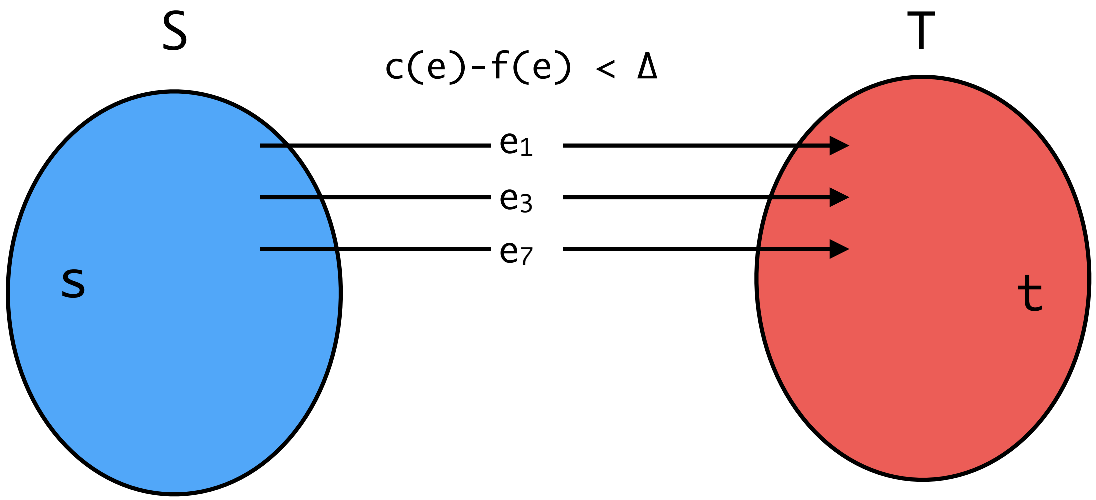

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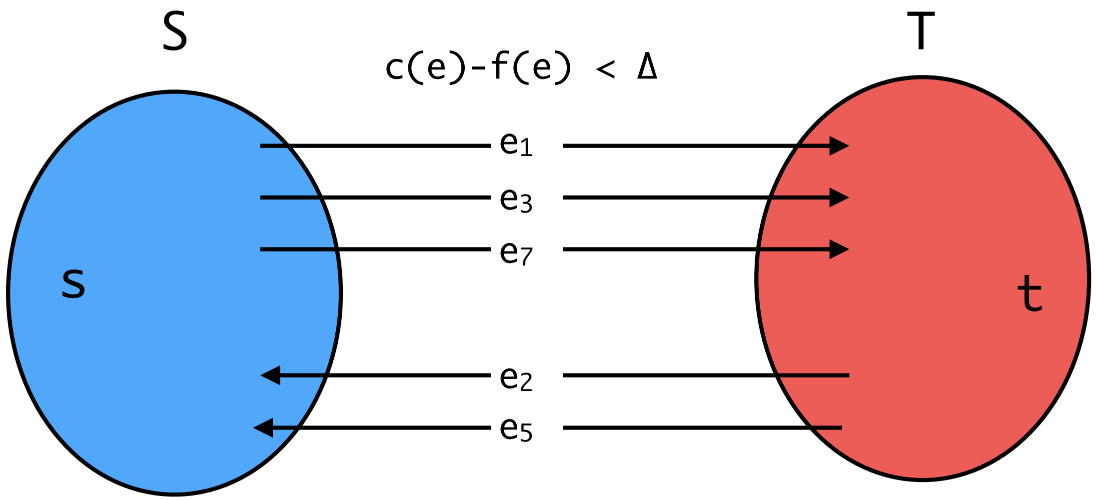

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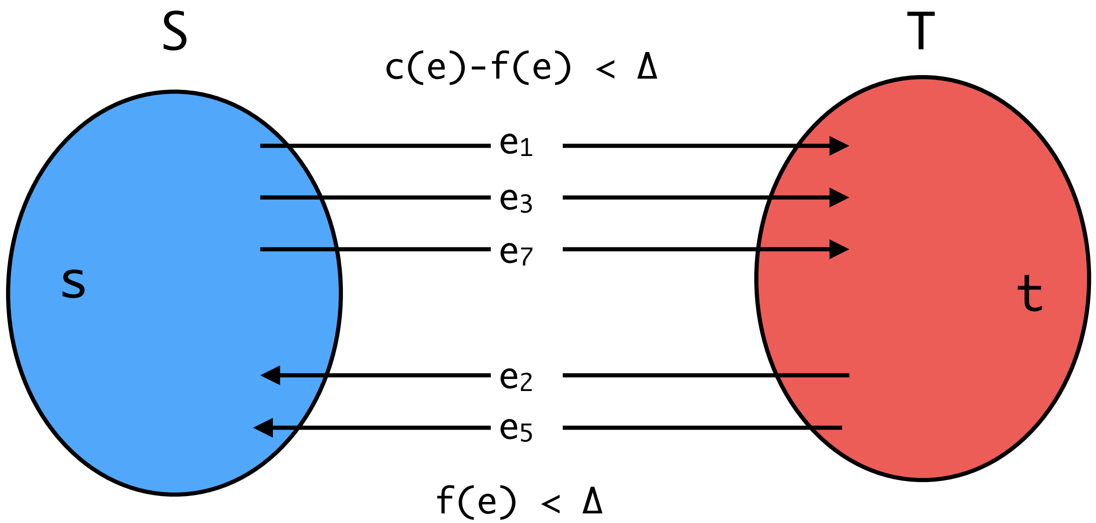

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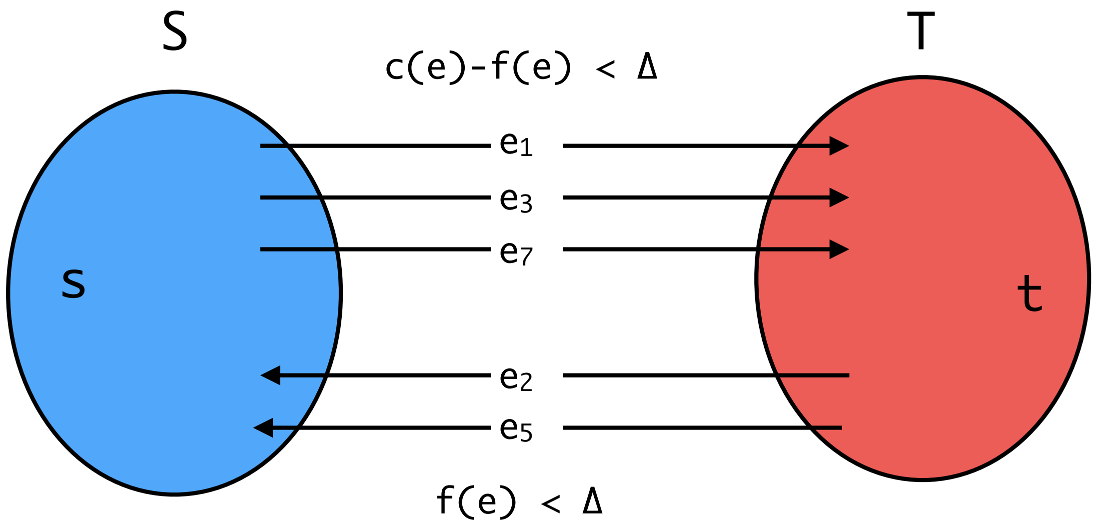

$$c(S, T) = c(e_1) + c(e_3) + c(e_7)$$

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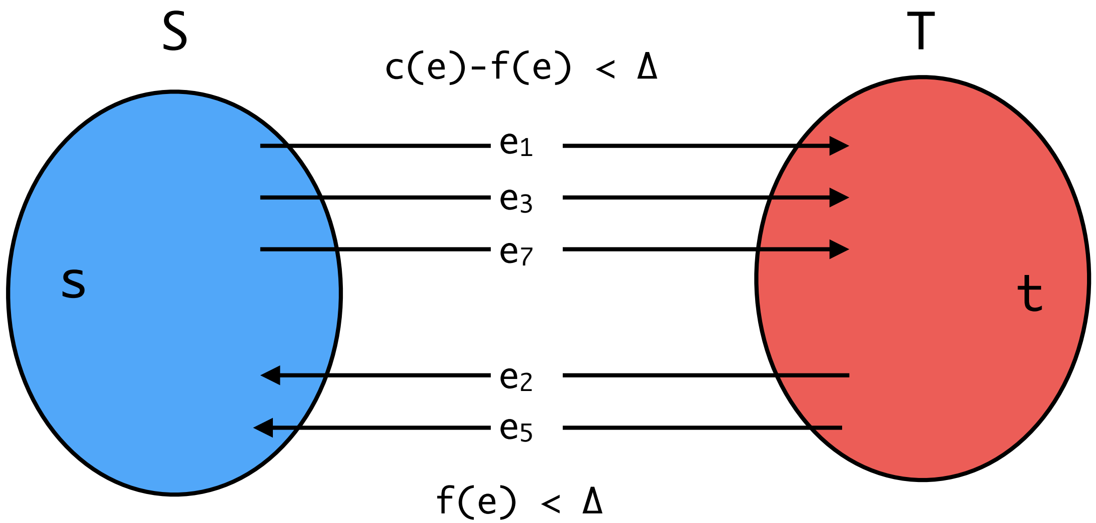

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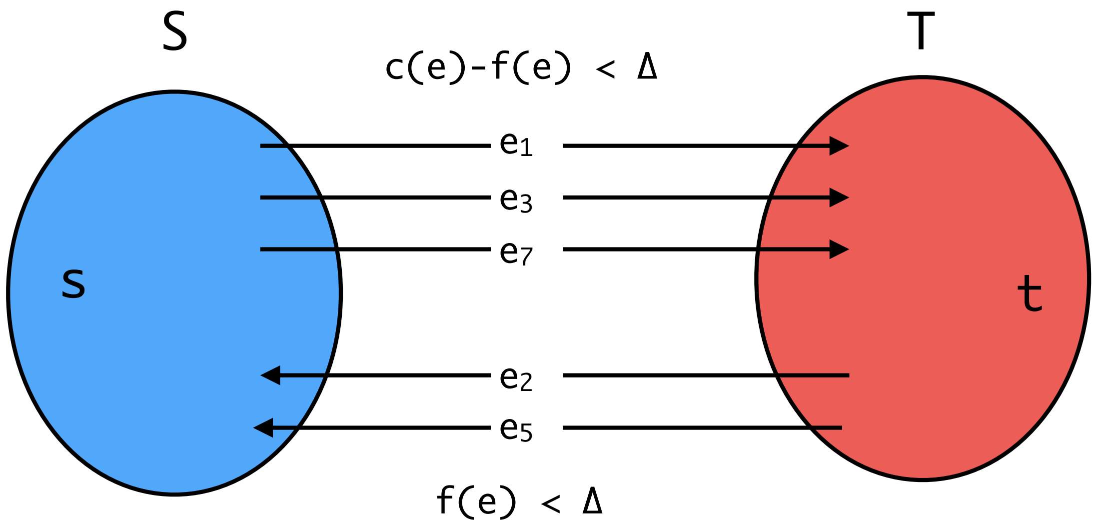

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Scaling algorithm

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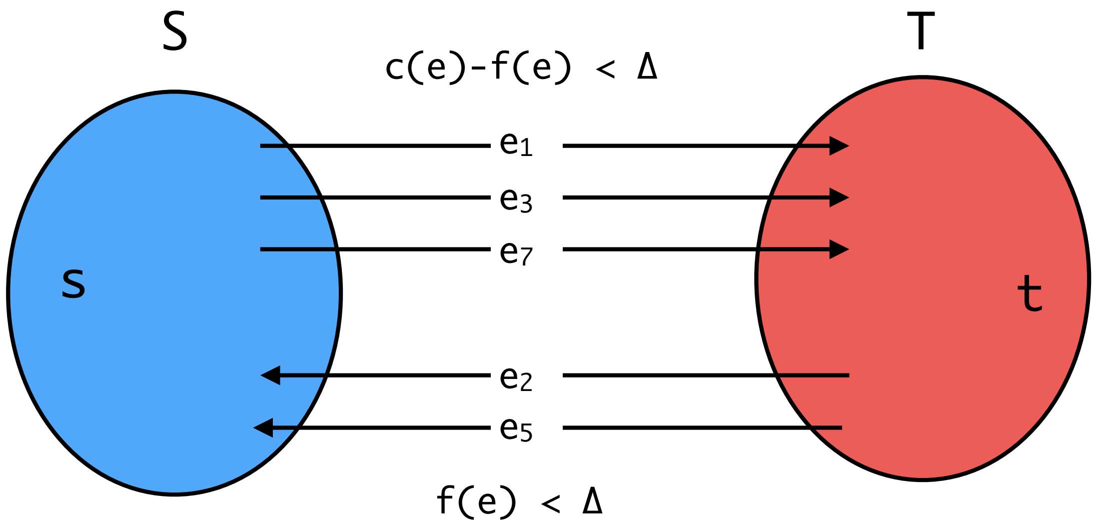

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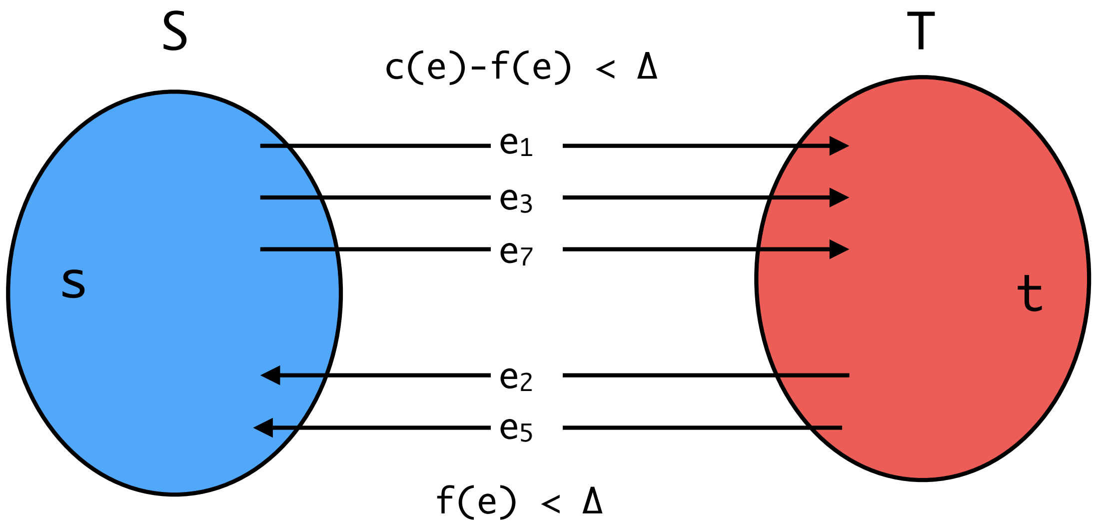

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$$\leq \Delta + \Delta + \Delta + \Delta + \Delta + \Delta = 5\Delta$$

Maximum flow algorithms

- Edmonds-Karp: $O(m^2 n)$
- Scaling: $O(m^2 \log C)$
- Ford-Fulkerson $O(m\, v(f))$.
- Preflow-push $O(n^3)$
- Other algorithms: $O(mn \log n)$ or $O(\min(n^{2/3}, m^{1/2}) m \log n \log U)$.

Maximum Bipartite Matching

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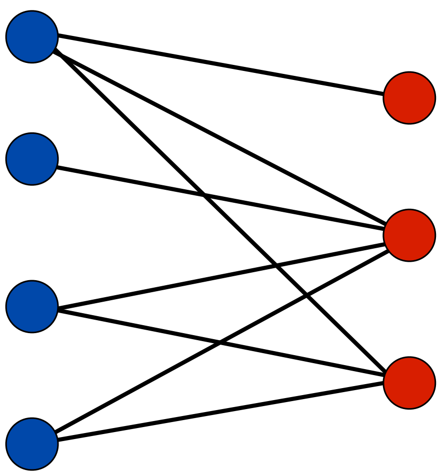

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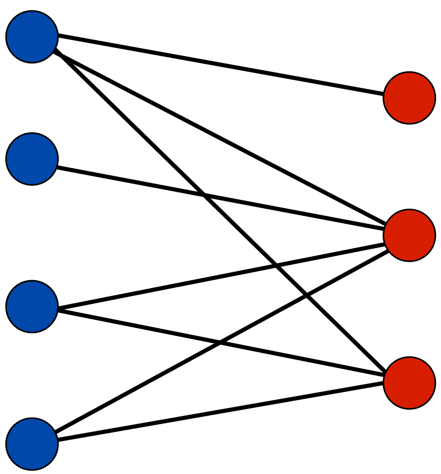

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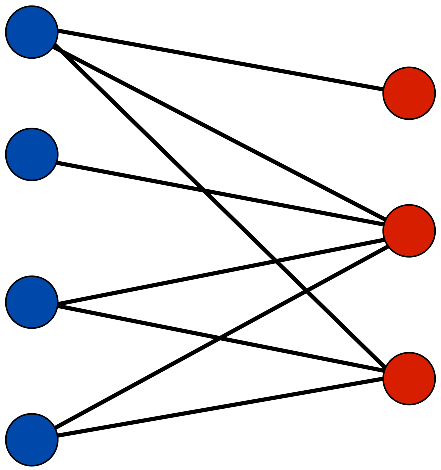
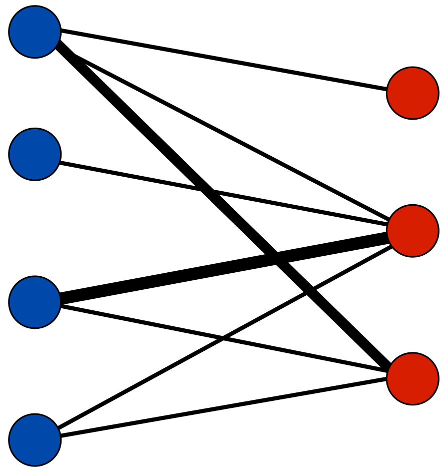


Matching



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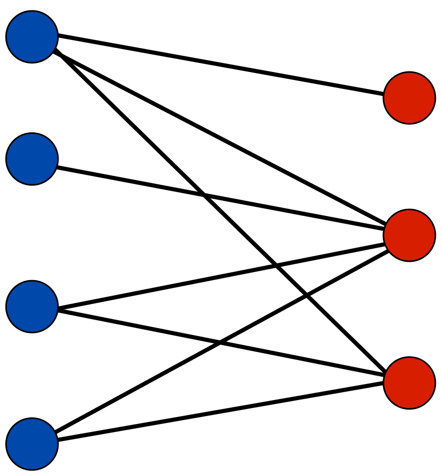

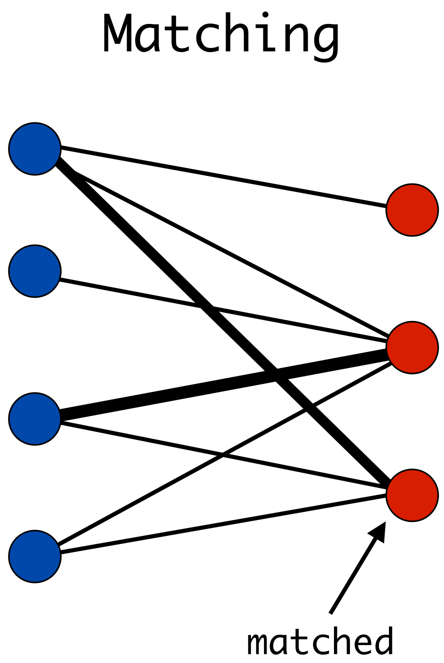

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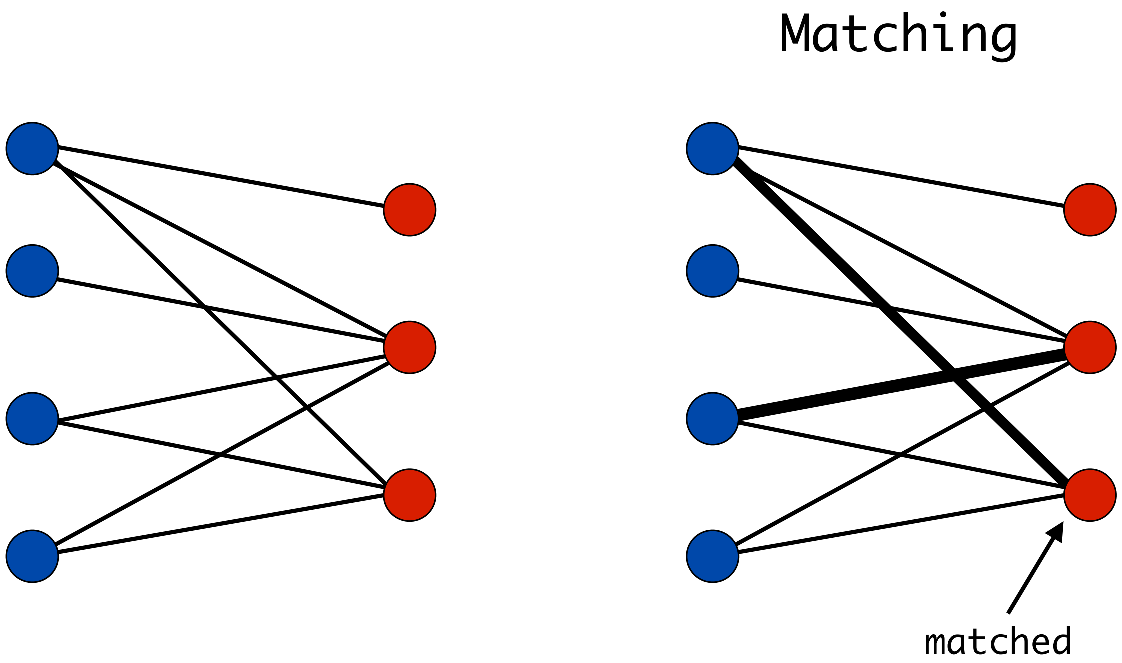

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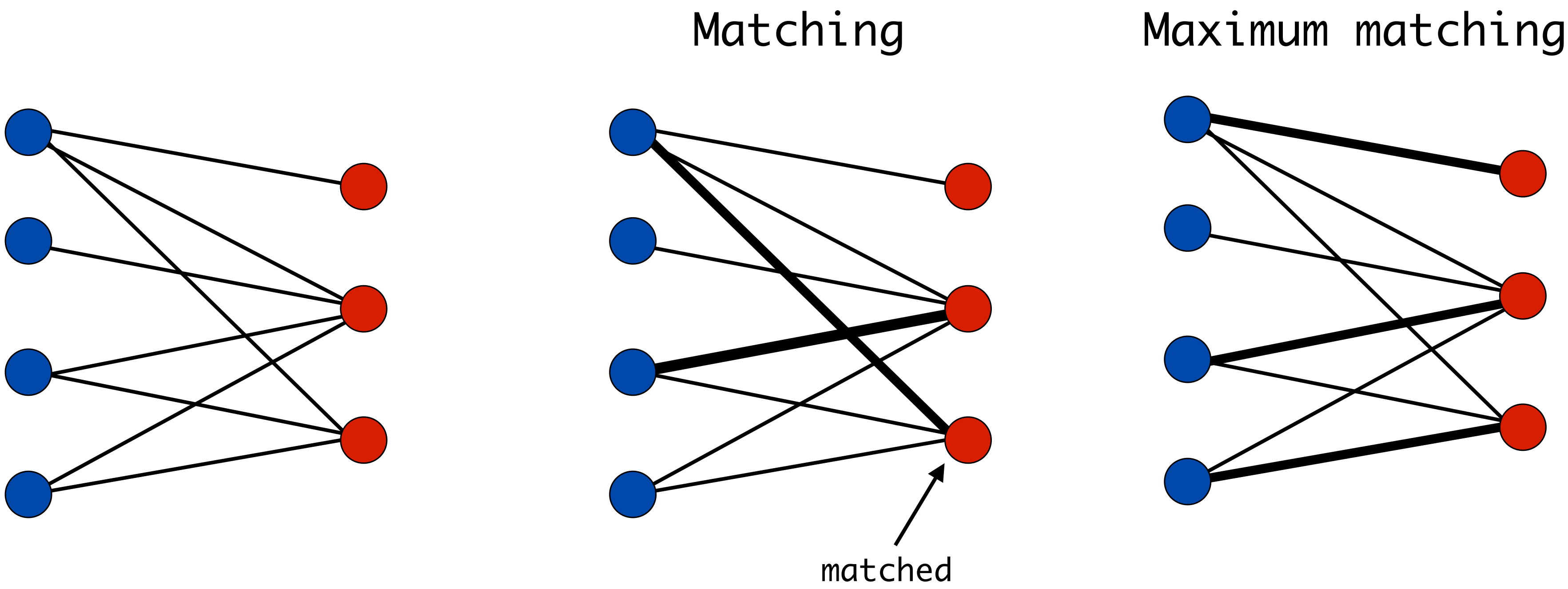

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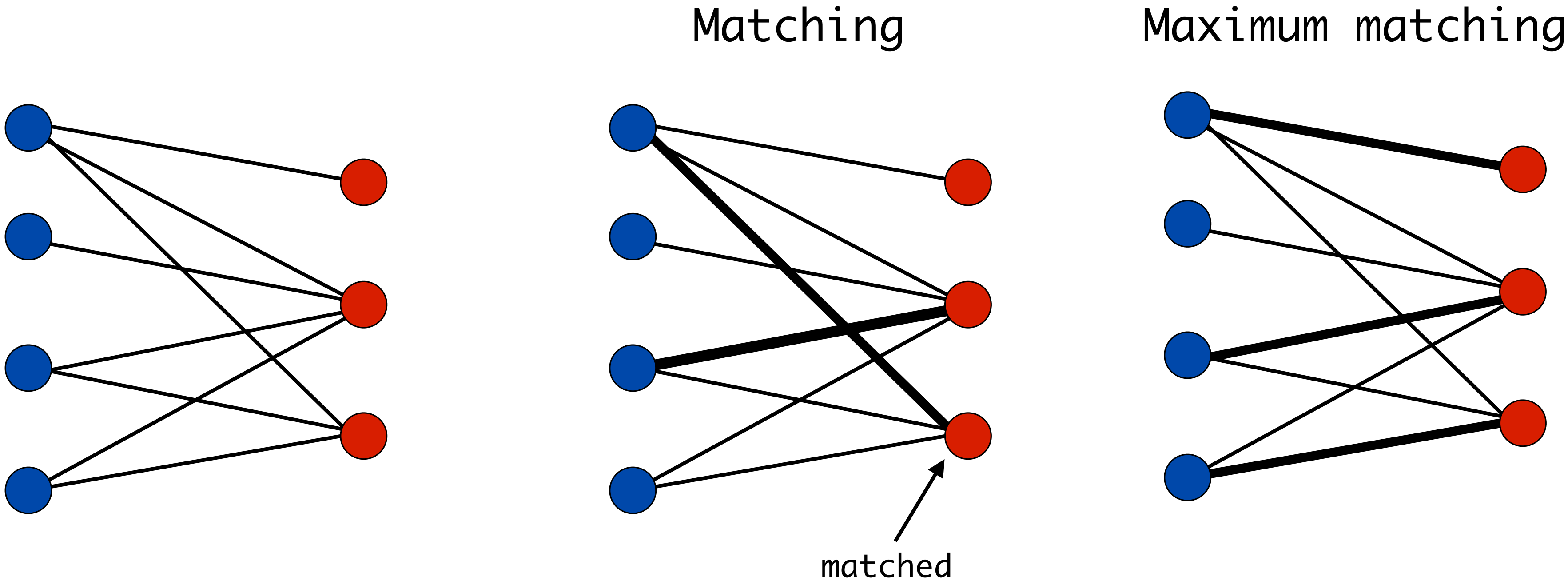
- Applications:
  - planes to routes
  - jobs to workers/machines



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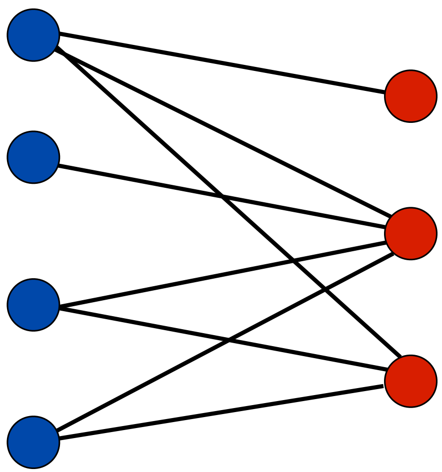

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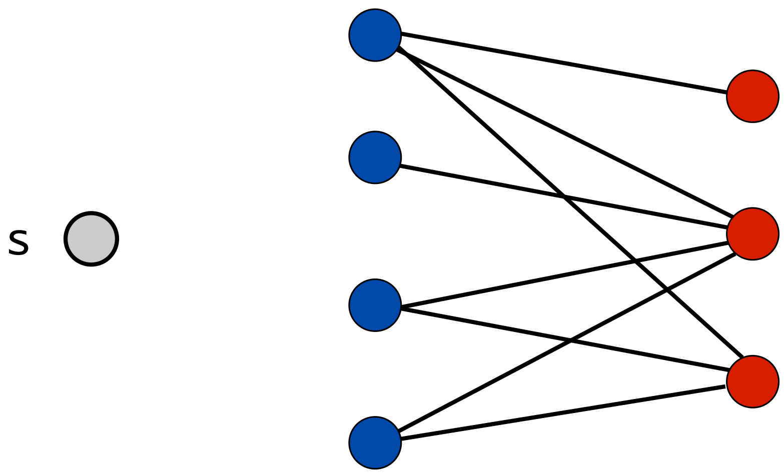

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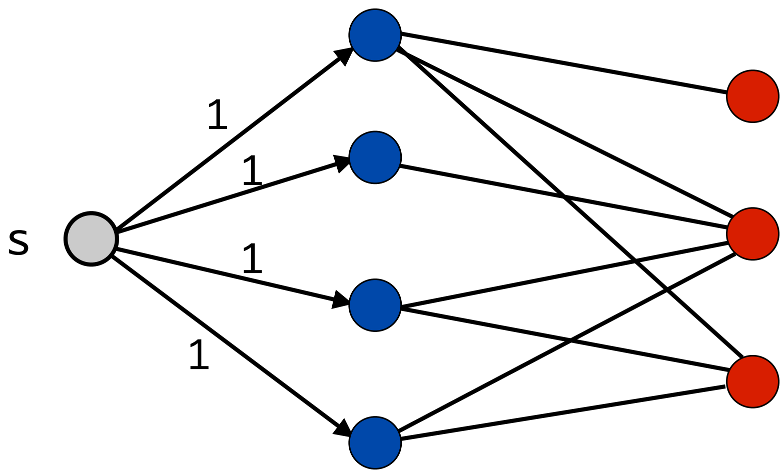

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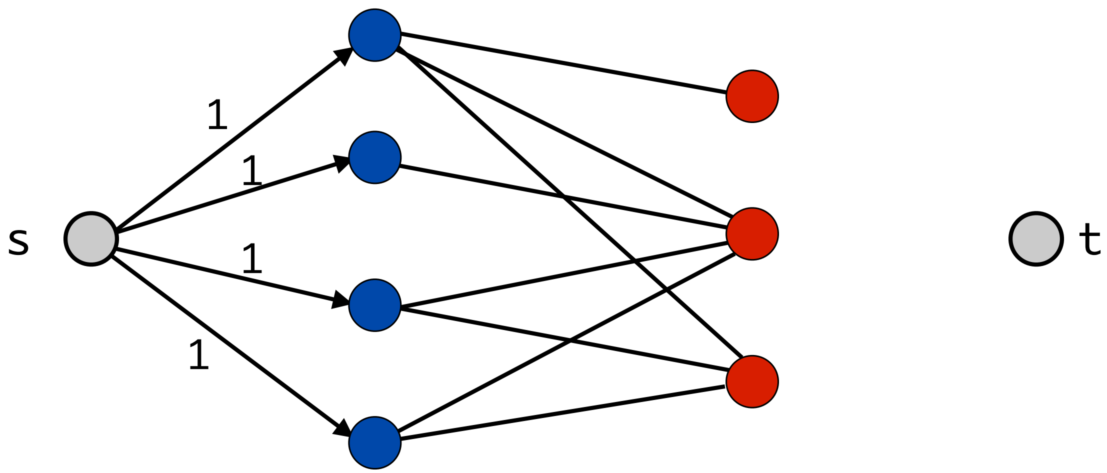

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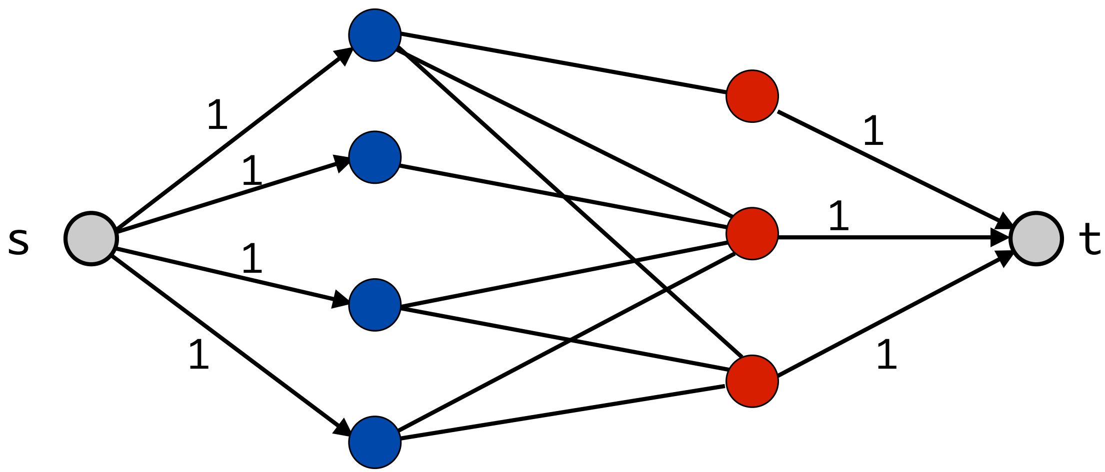

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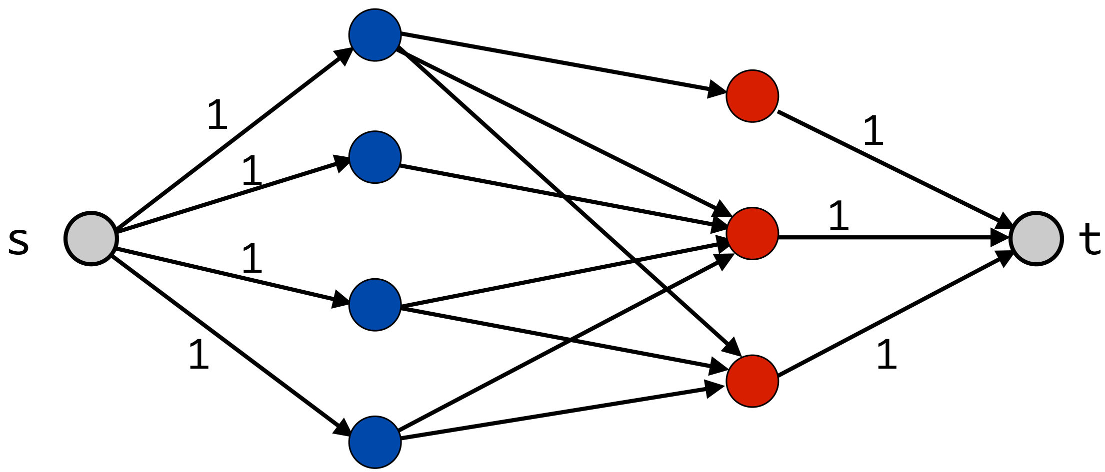

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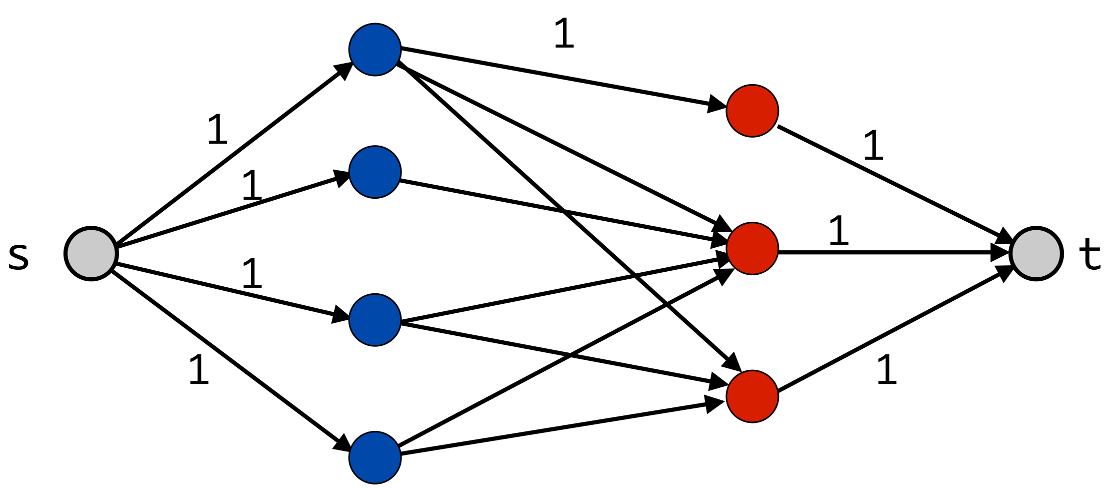

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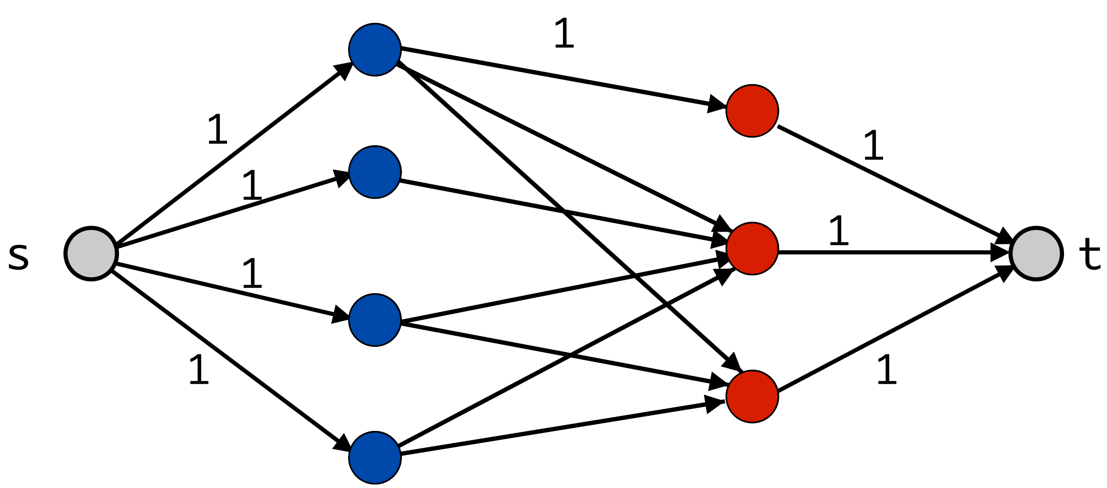

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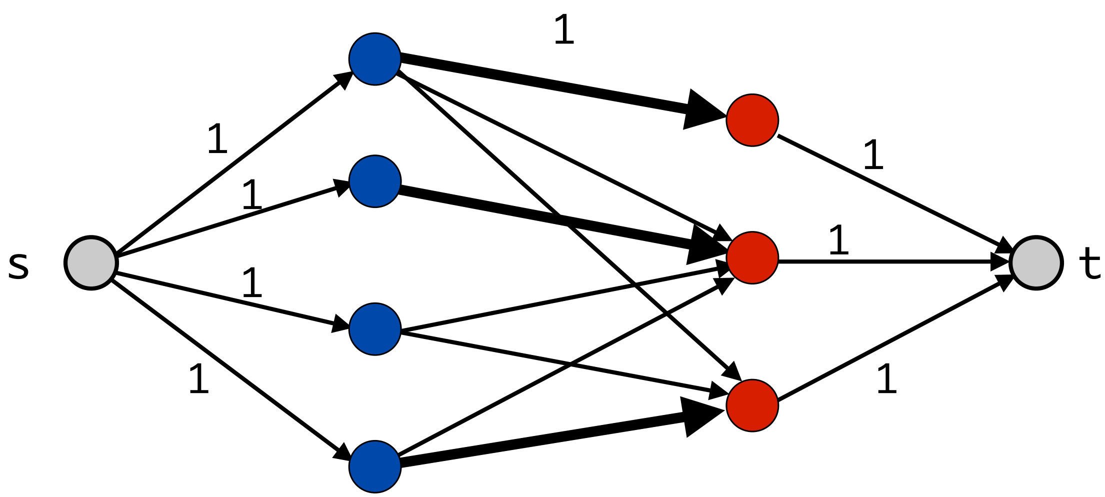

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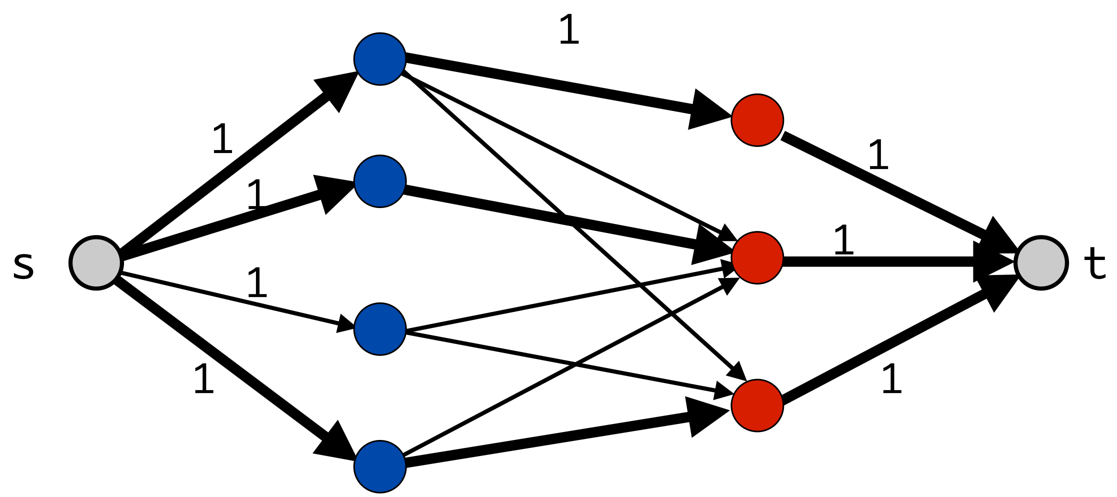

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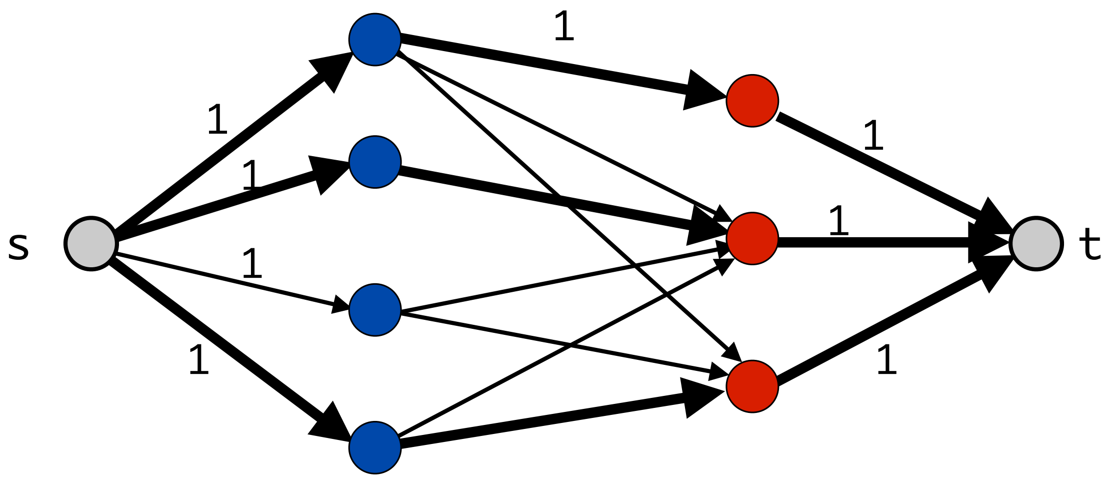
  - Flow of value v(f) => matching of size v(f)



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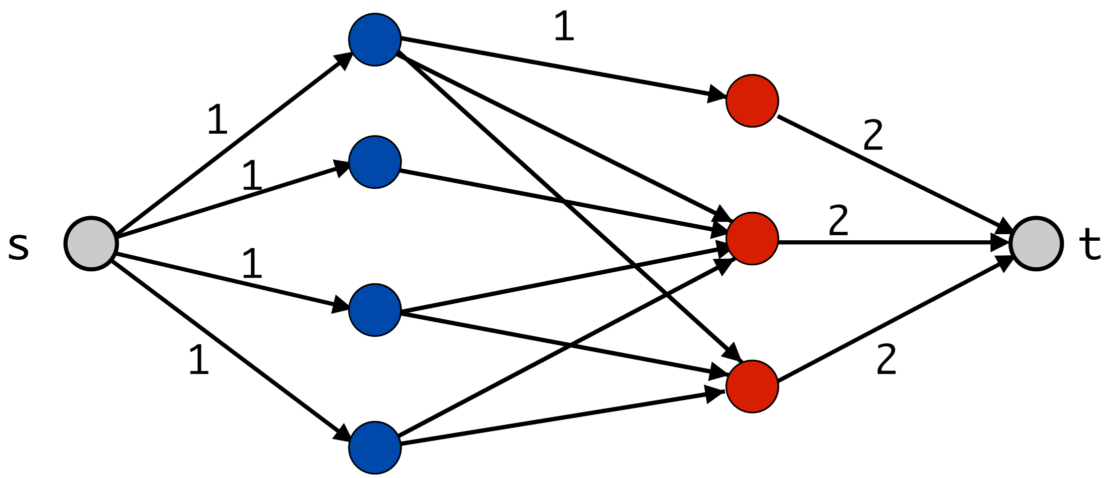
- Solve via flow:
- Can generalize to general matchings



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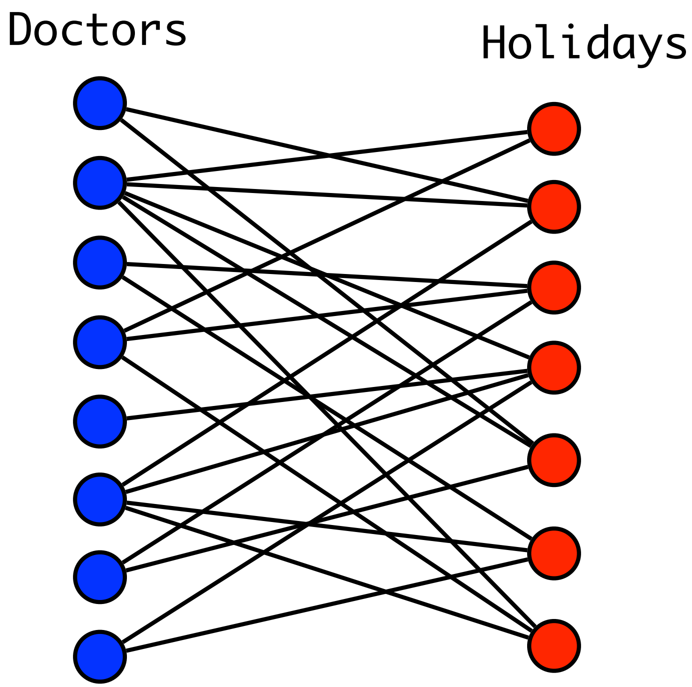

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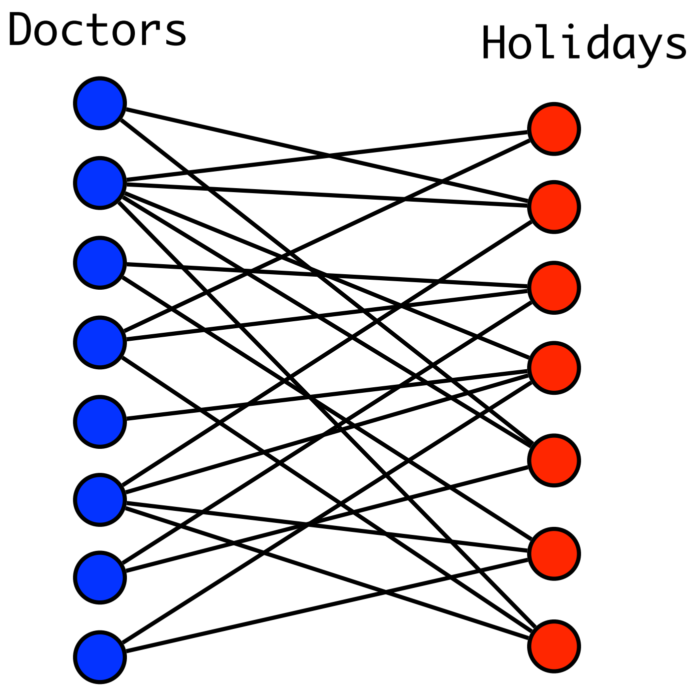

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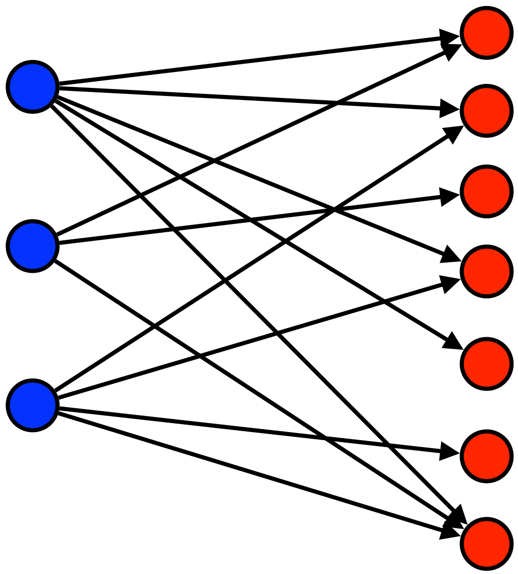

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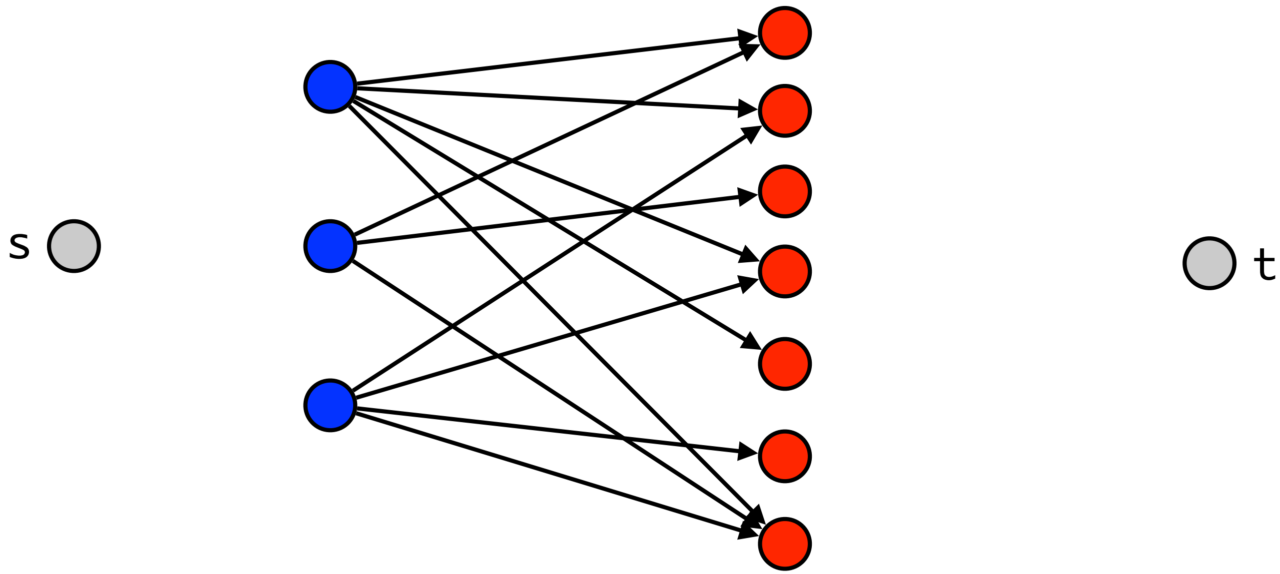

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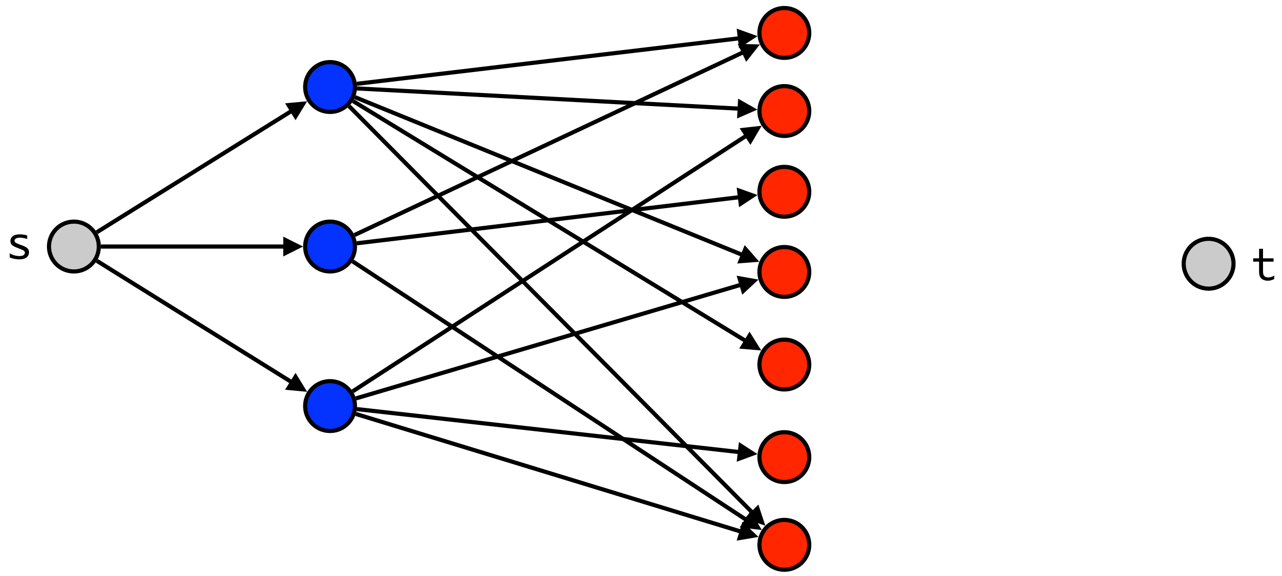

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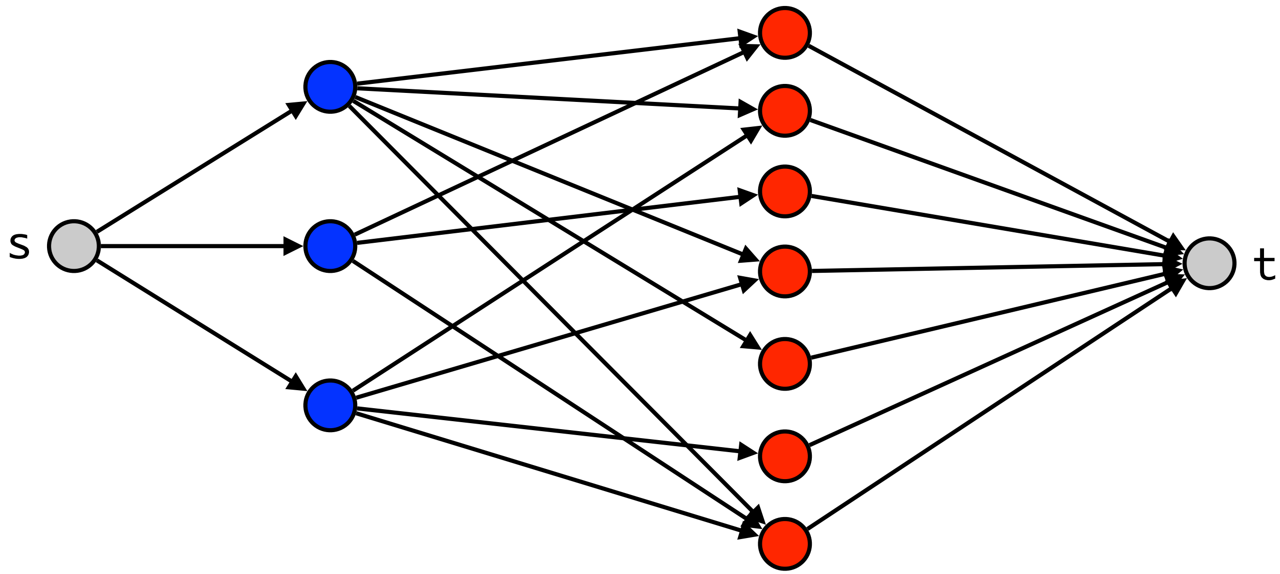

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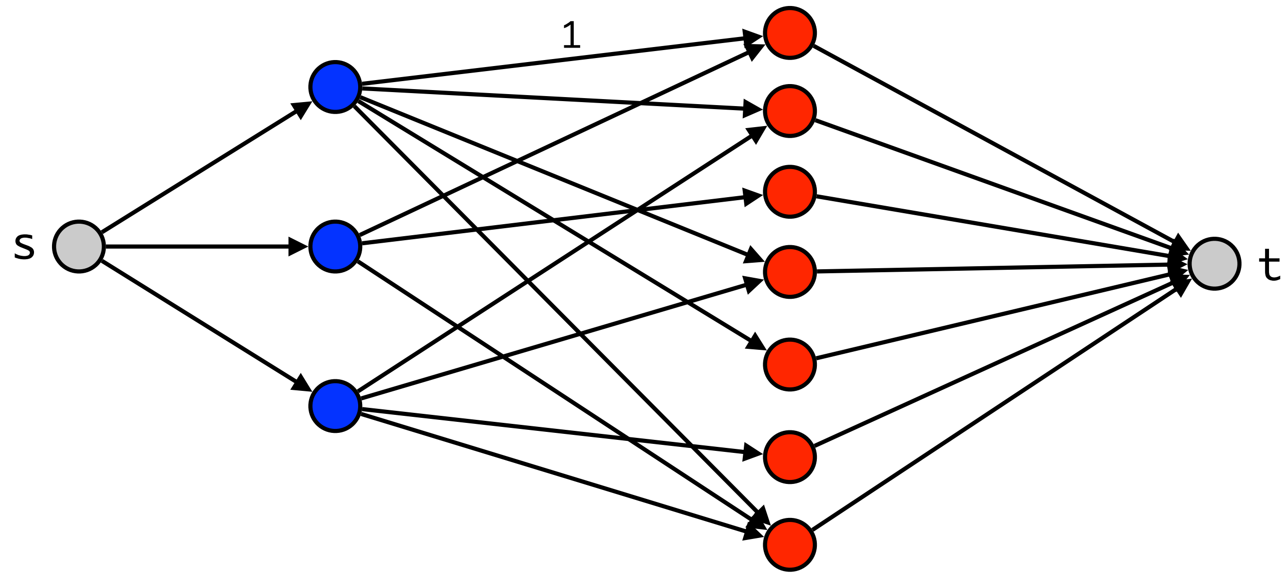

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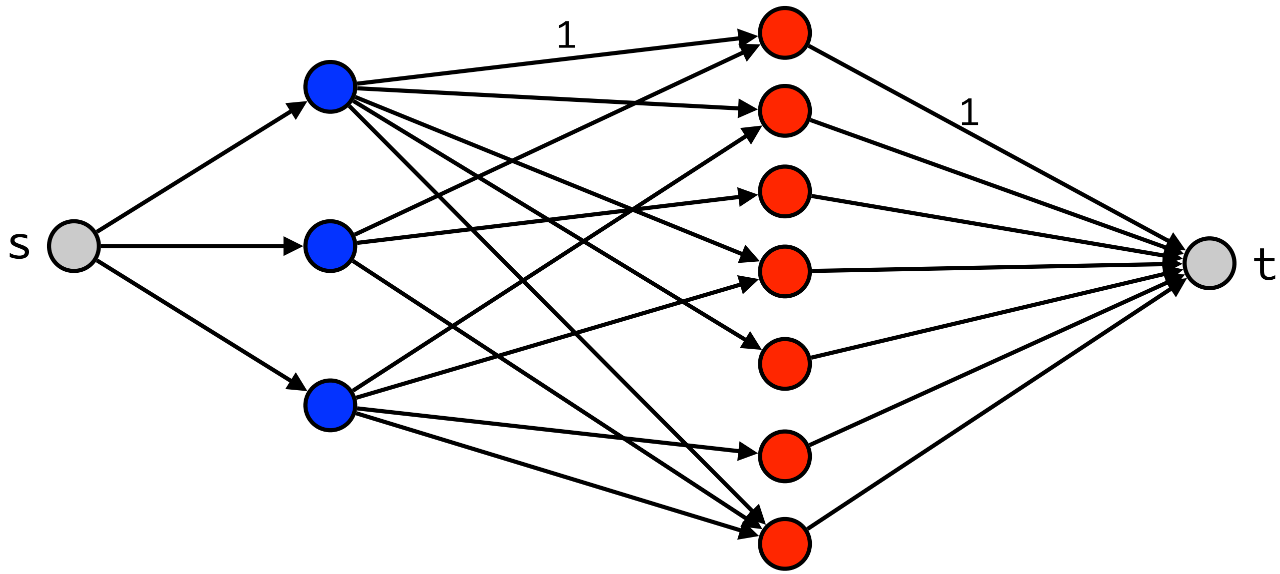

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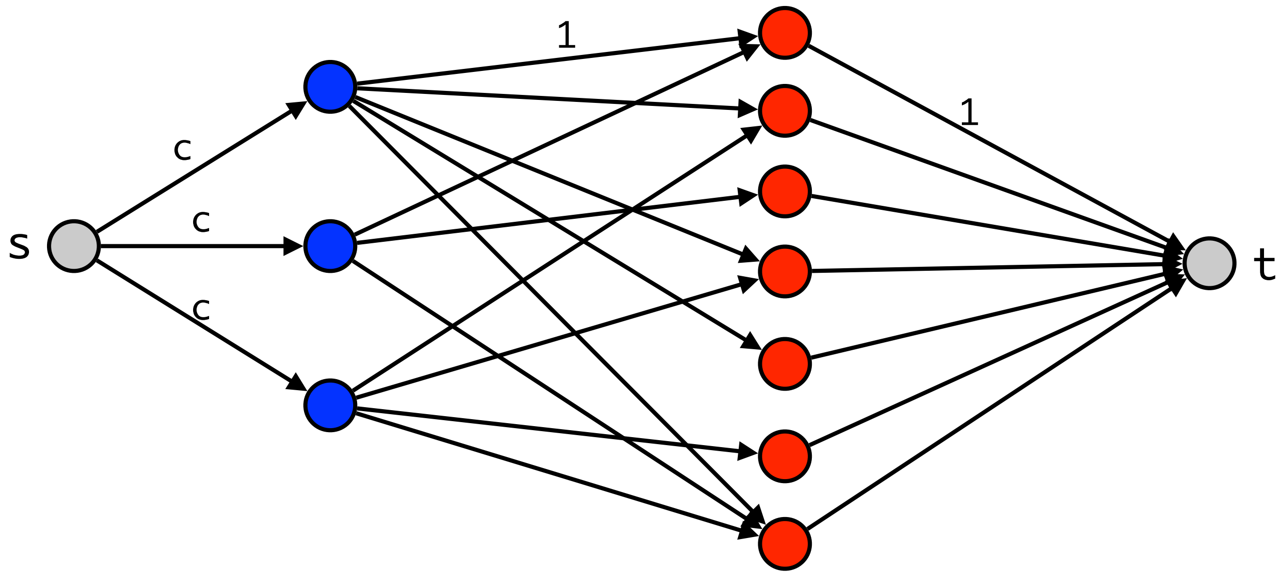

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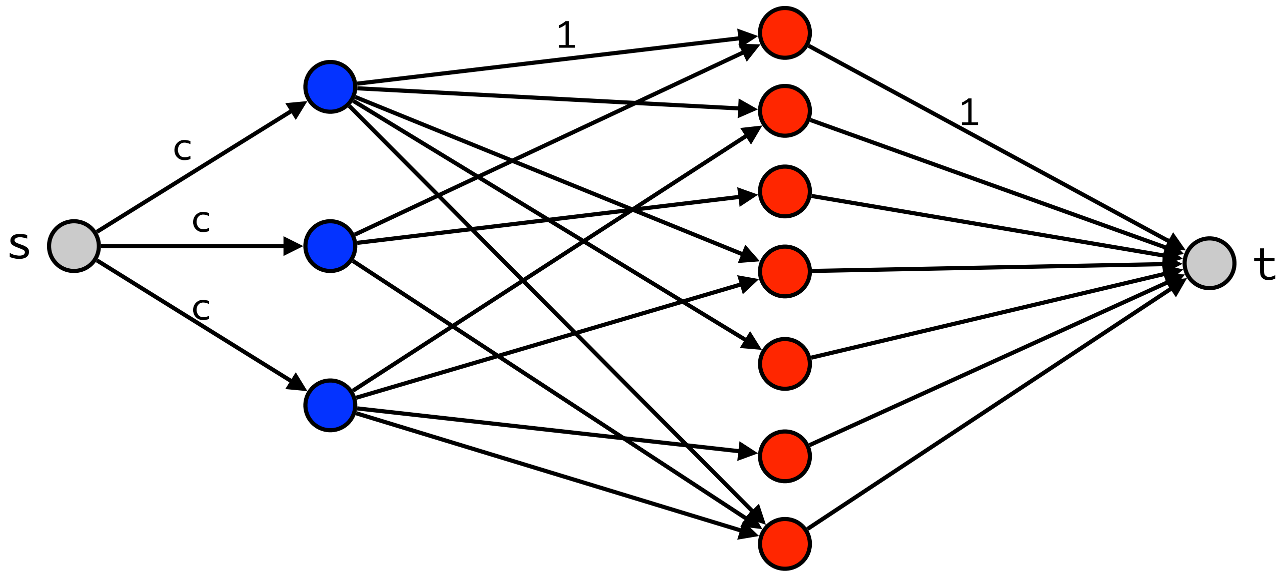

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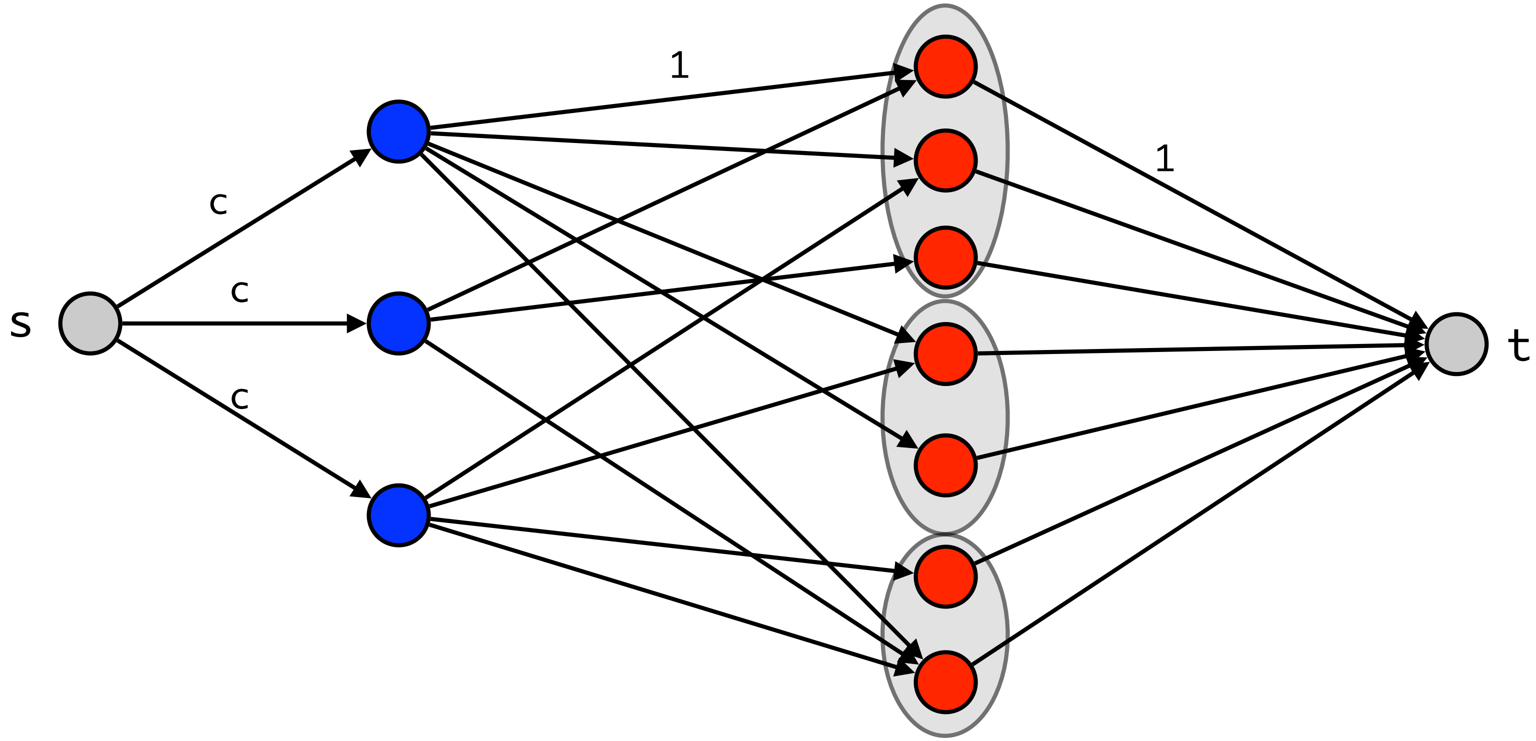

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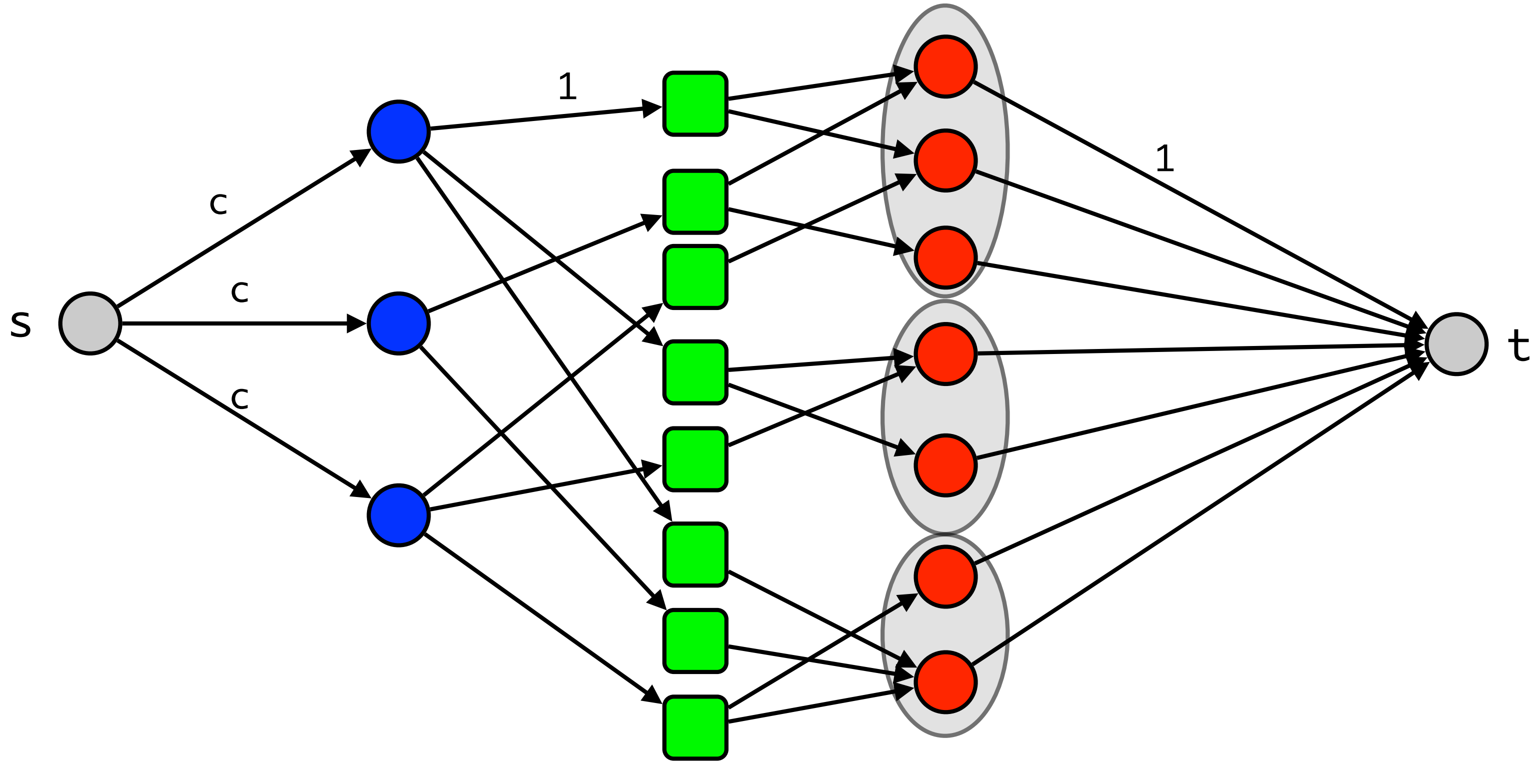
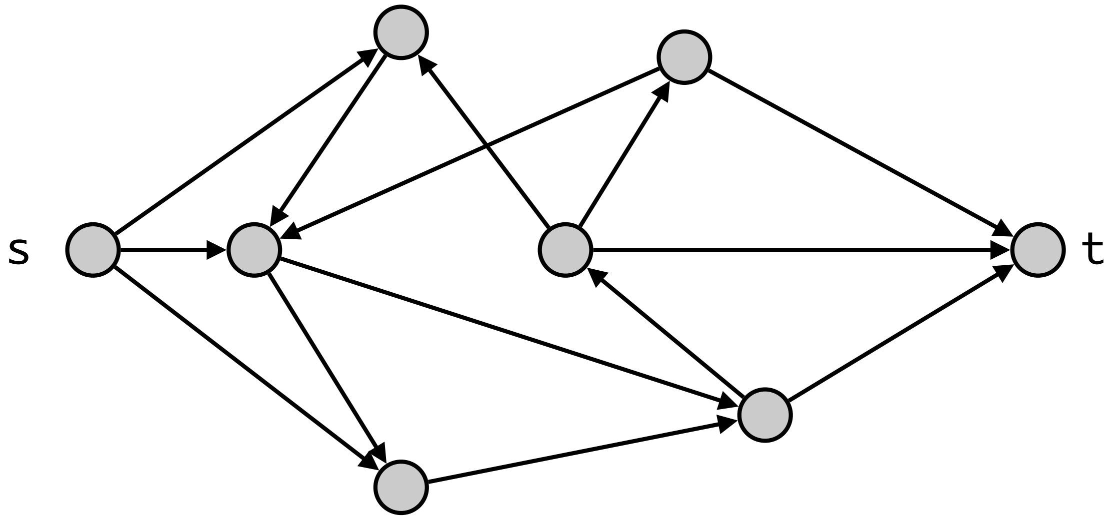


Edge Disjoint paths

- Problem: Find maximum number of edge-disjoint paths from s to t.
- Two paths are edge-disjoint if they have no edge in common.



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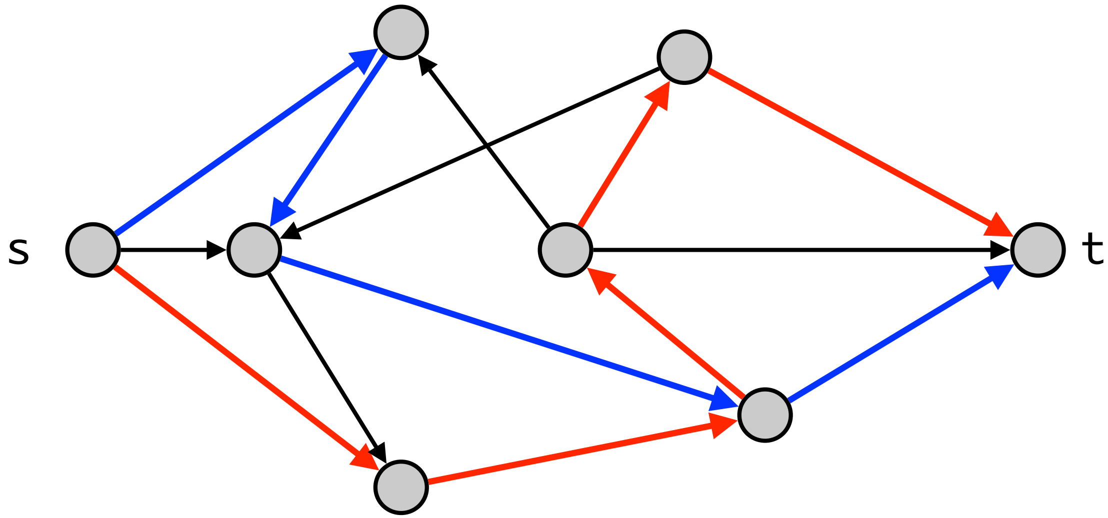
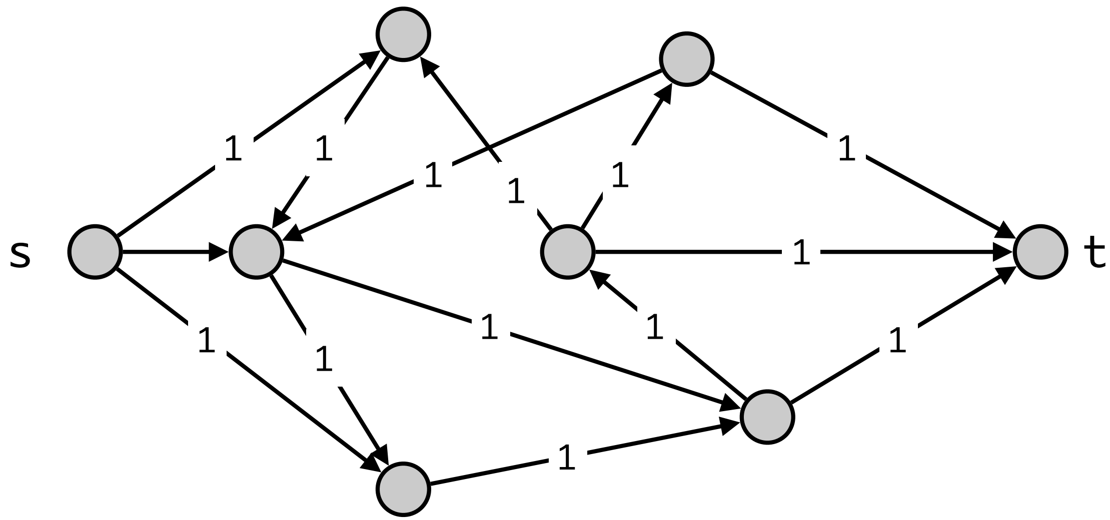


Edge Disjoint Paths

- Reduction to max flow: assign capacity 1 to each edge.



- Thm. Max number of edge-disjoint s-t paths is equal to the value of a maximum flow.
  - Suppose there are k edge-disjoint paths: then there is a flow of k (let all edges on the paths have flow 1).
  - Other way (graph theory course).

- Ford-Fulkerson: $v(f) \leq n$ (no multiple edges and therefore at most n edges out of s) => running time $O(nm)$.

Network Connectivity

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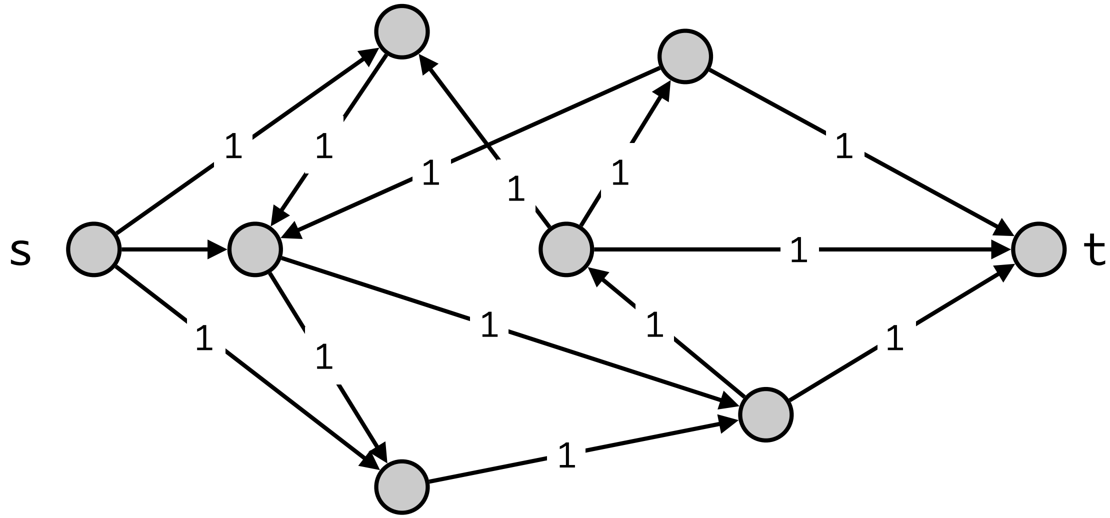

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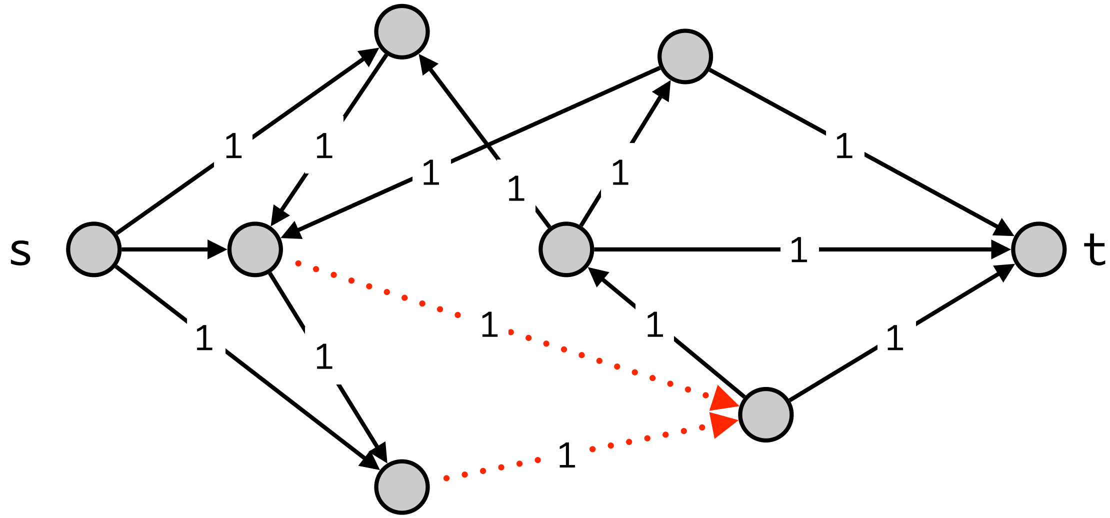

- Set all capacities to 1 and find minimum cut.
- Thm. (Menger) The maximum number of edge-disjoint s-t paths is equal to the minimum number of edges whose removal disconnects t from s.

Node capacities

- Capacities on nodes.