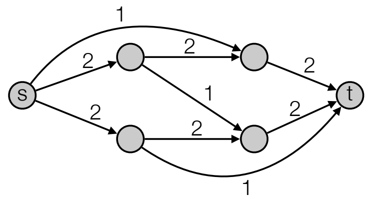
Network Flow II

Inge Li Gørtz

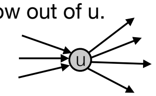
KT 7.3, 7.5, 7.6

Network Flow

- Network flow:
  - graph G=(V,E).
  - Special vertices s (source) and t (sink).
  - Every edge e has a capacity c(e) ≥ 0.
  - Flow:
    - **capacity constraint:** every edge e has a flow 0 ≤ f(e) ≤ c(e).
    - **flow conservation:** for all u ≠ s, t: flow into u equals flow out of u.



$$\sum_{v:(v,u)\in E} f(v,u) = \sum_{v:(u,v)\in E} f(u,v)$$



    - Value of flow f is the sum of flows out of s minus sum of flows into s:

$$v(f) = \sum_{v:(s,v)\in E} f(e) - \sum_{v:(v,s)\in E} f(e) = f^{out}(s) - f^{in}(s)$$

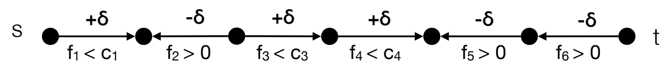
  - **Maximum flow problem:** find s-t flow of maximum value

Today

- Applications
- Finding good augmenting paths. Edmonds-Karp and scaling algorithm.

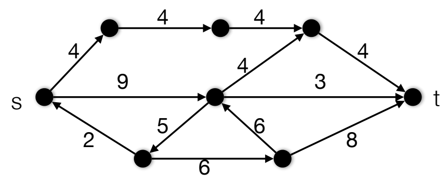
Ford-Fulkerson

- Find (any) augmenting path and use it.
- Augmenting path (definition different than in CLRS): s-t path where
  - forward edges have leftover capacity
  - backwards edges have positive flow



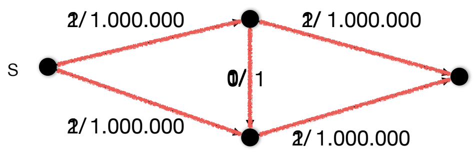
  - Can add extra flow: $\min(c_1 - f_1, f_2, c_3 - f_3, c_4 - f_4, f_5, f_6) = \delta$

- To find augmenting path use DFS or BFS:



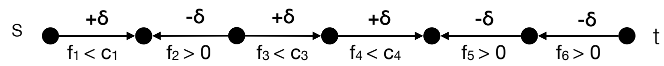
Ford-Fulkerson

- Integral capacities:
  - Each augmenting path increases flow with at least 1.
  - At most v(f) iterations
  - Find augmenting path via DFS/BFS: O(m)
  - Total running time: O(v(f) m)
- Lemma. If all the capacities are integers, then there is a maximum flow where the flow on every edge is an integer.
- Bad example for Ford-Fulkerson:

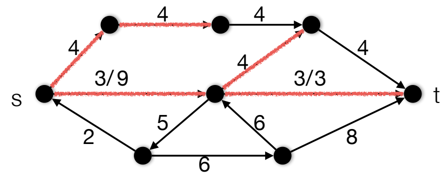


Edmonds-Karp

- Find *shortest* augmenting path and use it.
- Augmenting path (definition different than in CLRS): s-t path where
  - forward edges have leftover capacity
  - backwards edges have positive flow

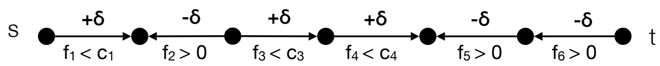


  - Can add extra flow: $\min(c_1 - f_1, f_2, c_3 - f_3, c_4 - f_4, f_5, f_6) = \delta$
- To find augmenting path use *BFS*:

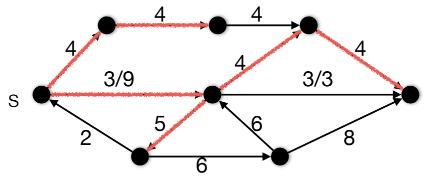


Edmonds-Karp

- Find *shortest* augmenting path and use it.
- Augmenting path (definition different than in CLRS): s-t path where
  - forward edges have leftover capacity
  - backwards edges have positive flow

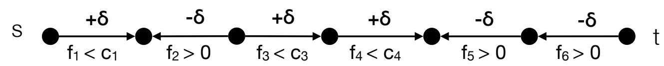


  - Can add extra flow: $\min(c_1 - f_1, f_2, c_3 - f_3, c_4 - f_4, f_5, f_6) = \delta$
- To find augmenting path use *BFS*:

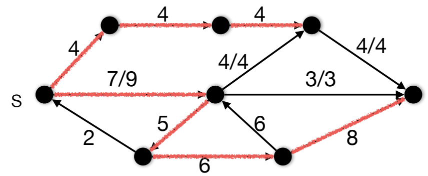


Edmonds-Karp

- Find *shortest* augmenting path and use it.
- Augmenting path (definition different than in CLRS): s-t path where
  - forward edges have leftover capacity
  - backwards edges have positive flow

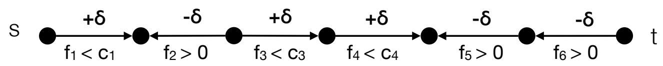


  - Can add extra flow: $\min(c_1 - f_1, f_2, c_3 - f_3, c_4 - f_4, f_5, f_6) = \delta$
- To find augmenting path use *BFS*:



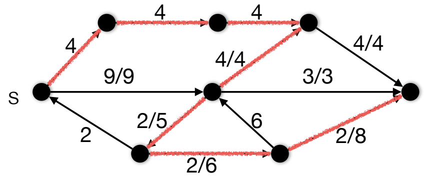
Edmonds-Karp

- Find *shortest* augmenting path and use it.
- Augmenting path (definition different than in CLRS): s-t path where
  - forward edges have leftover capacity
  - backwards edges have positive flow



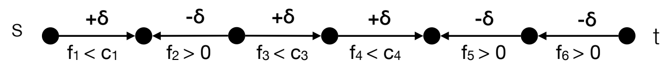
  - Can add extra flow: $\min(c_1 - f_1, f_2, c_3 - f_3, c_4 - f_4, f_5, f_6) = \delta$

- To find augmenting path use *BFS*:



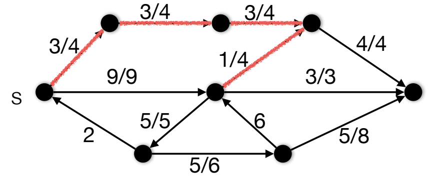
Edmonds-Karp

- Find *shortest* augmenting path and use it.
- Augmenting path (definition different than in CLRS): s-t path where
  - forward edges have leftover capacity
  - backwards edges have positive flow



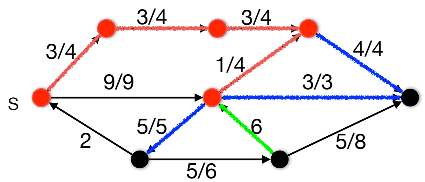
  - Can add extra flow: $\min(c_1 - f_1, f_2, c_3 - f_3, c_4 - f_4, f_5, f_6) = \delta$

- To find augmenting path use *BFS*:



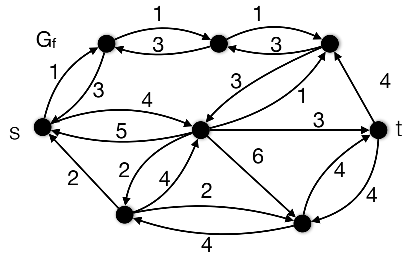
Find a minimum cut

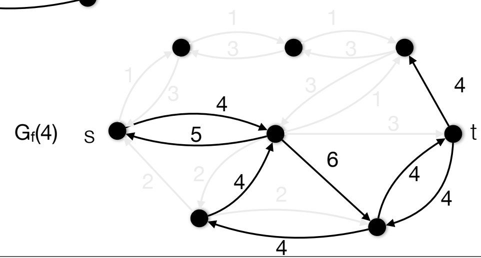
- When there are no more augmenting s-t paths:
- Find all augmenting paths from s.
- The nodes S that can be reached by these augmenting paths form the left side of a minimum cut.
  - edges out of S have $c_e = f_e$.
  - edges into S have $f_e = 0$.
  - Capacity of the cut equals the flow.



Scaling algorithm

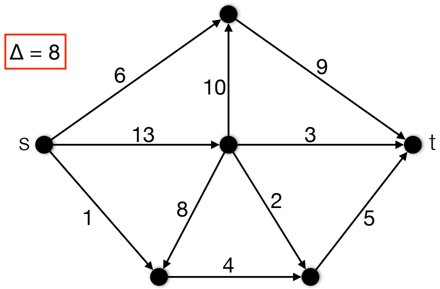
- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Example: Δ = 4





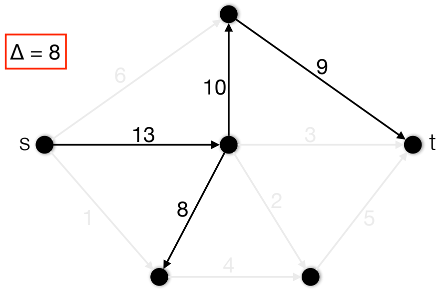
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"



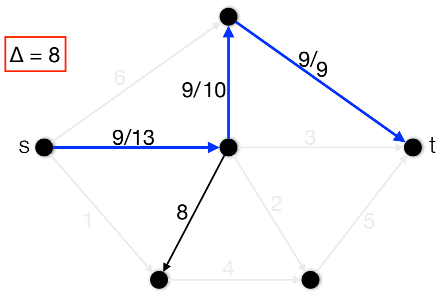
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"



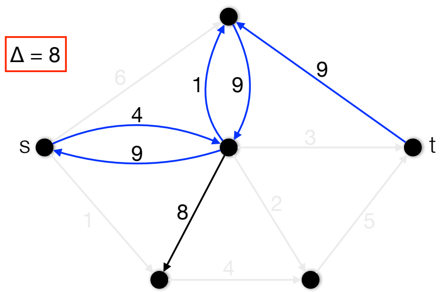
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"



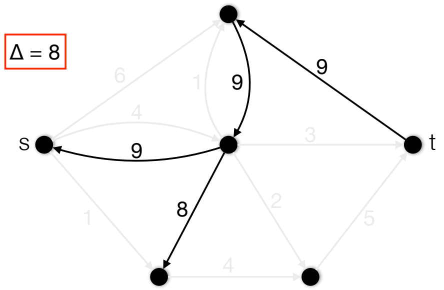
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"



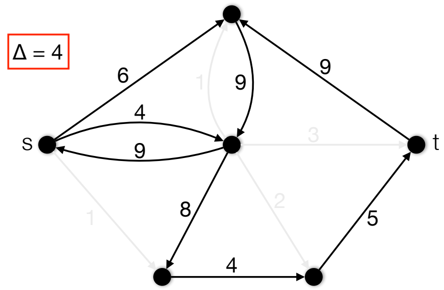
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



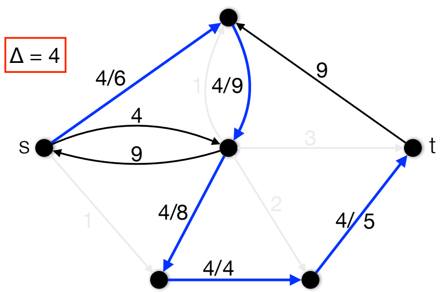
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



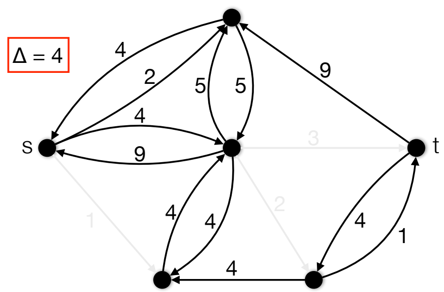
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



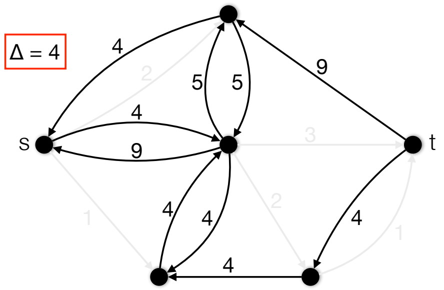
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



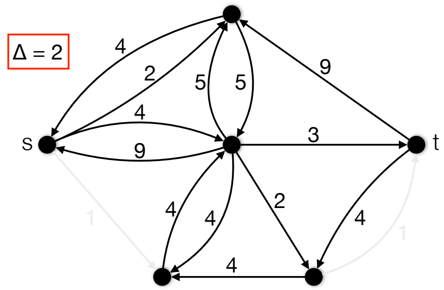
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = “highest power of 2 ≤ largest capacity out of s”
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



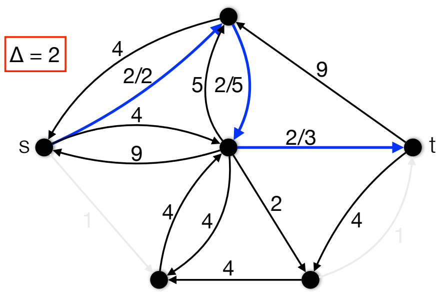
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = “highest power of 2 ≤ largest capacity out of s”
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



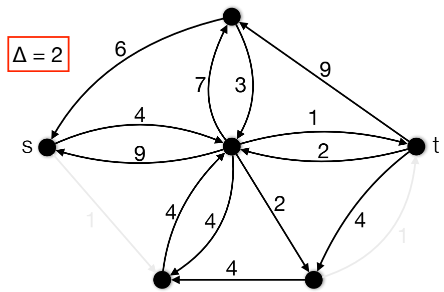
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = “highest power of 2 ≤ largest capacity out of s”
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



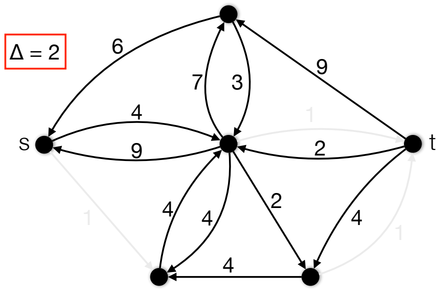
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = “highest power of 2 ≤ largest capacity out of s”
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



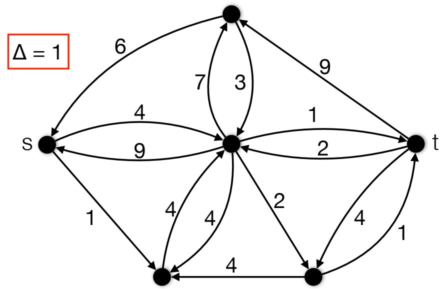
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



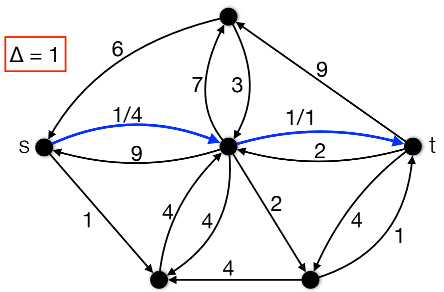
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



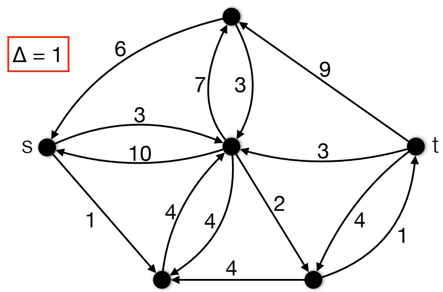
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



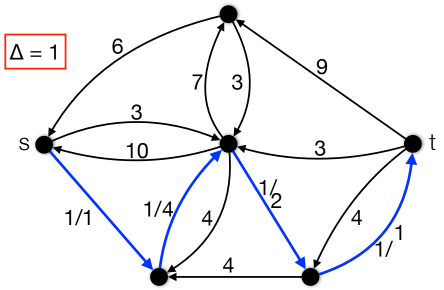
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



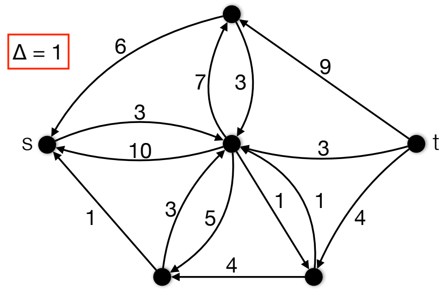
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).



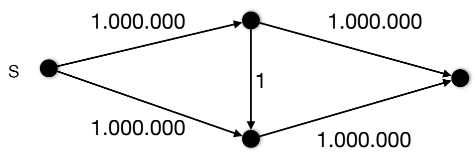
Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Start with Δ = "highest power of 2 ≤ largest capacity out of s"
- When no more augmenting paths in $G_f(\Delta)$: Δ = Δ/2 (new phase).

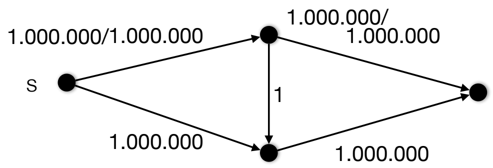


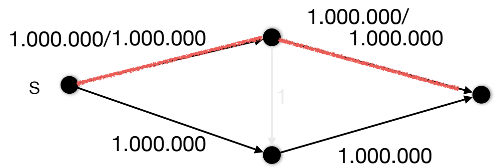
- Stop when no more augmenting paths in $G_f(1)$.

Scaling algorithm

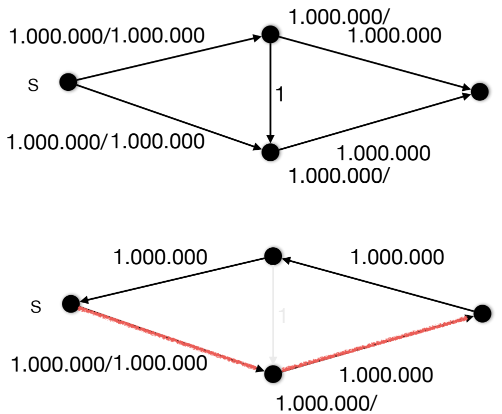


Scaling algorithm

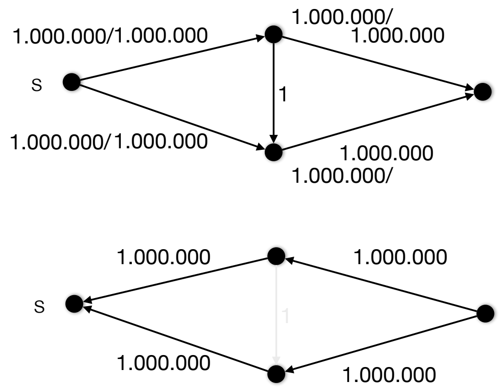




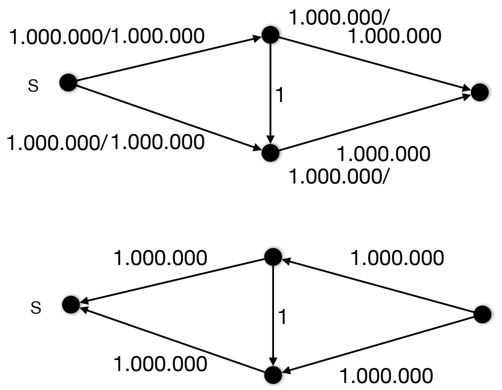
Scaling algorithm



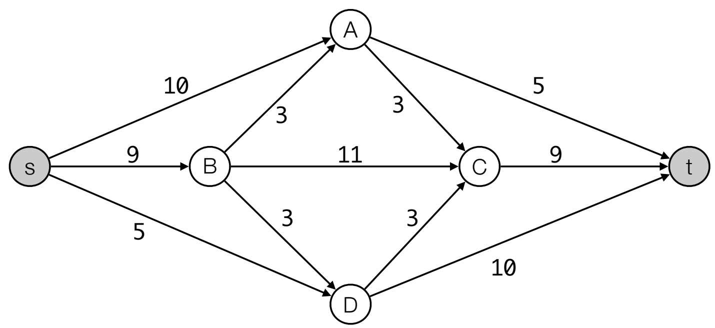
Scaling algorithm



Scaling algorithm



Exercise



Scaling algorithm

- Running time: $O(m^2 \log C)$, where C is the largest capacity out of s.

- Lemma 1. *Number of scaling phases: $1 + \lceil \lg C \rceil$*

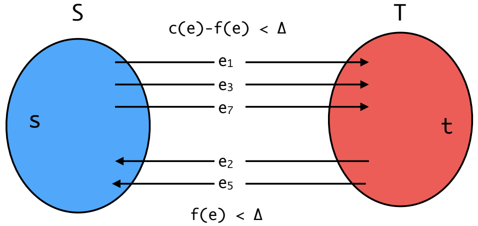
- Lemma 2. *Let f the flow when Δ-scaling phase ends, and let f*be the maximum flow. Then $v(f^*) \le v(f) + m\Delta$.*

- Lemma 3. *The number of augmentations in a scaling phase is at most 2m.*
  - First phase: can use each edge out of s in at most one augmenting path.
  - f flow at the end of previous phase.
  - Used Δ' = 2Δ in last round.
  - Lemma 2: $v(f^*) \le v(f) + m\Delta' = v(f) + 2m\Delta$.
  - "Leftover flow" to be found $\le 2m\Delta$.
  - Each agumentation in a Δ-scaling phase augments flow with at least Δ.

Scaling algorithm

- Lemma 2. *Let f the flow when Δ-scaling phase ends, and let f*be the maximum flow. Then $v(f^*) \le v(f) + m\Delta$.*

- By the end of the phase there is a cut $c(S,T) \le v(f) + m\Delta$.



$$c(S,T) = c(e_1) + c(e_3) + c(e_7)$$

$$v(f) = f(e_1) + f(e_3) + f(e_7) - f(e_2) - f(e_5)$$

$$c(S,T) - v(f) = c(e_1) + c(e_3) + c(e_7) - f(e_1) - f(e_3) - f(e_7) + f(e_2) + f(e_5)$$
$$= c(e_1) - f(e_1) + c(e_3) - f(e_3) + c(e_7) - f(e_7) + f(e_2) + f(e_5)$$
$$< \Delta + \Delta + \Delta + \Delta + \Delta = 5\Delta$$

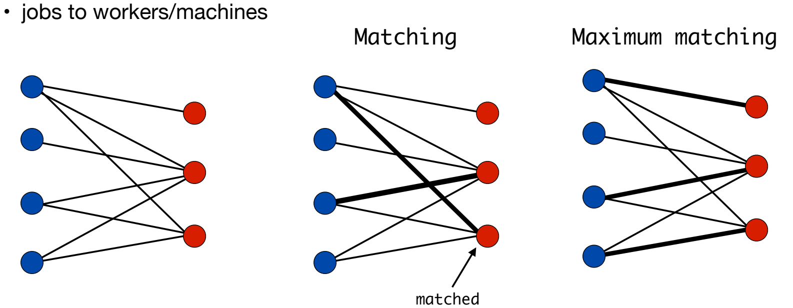
Maximum flow algorithms

- Edmonds-Karp: $O(m^2 n)$
- Scaling: $O(m^2 \log C)$
- Ford-Fulkerson $O(m\, v(f))$.
- Preflow-push $O(n^3)$
- Other algorithms: $O(mn \log n)$ or $O(\min(n^{2/3}, m^{1/2}) m \log n \log U)$.

Maximum Bipartite Matching

- Bipartite graph: Can color vertices red and blue such that all edges have a red and a blue endpoint.
- Matching: Subset of edges $M \subseteq E$ such that no edges in M share an endpoint.
- Maximum matching: matching of maximum cardinality.

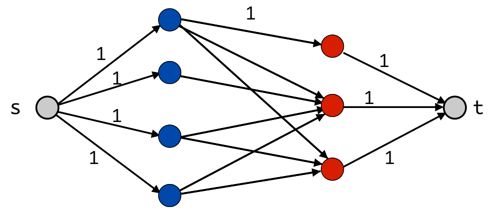
- Applications:
  - planes to routes
  - jobs to workers/machines



Maximum Bipartite Matching

- Bipartite graph: Can color vertices red and blue such that all edges have a red and a blue endpoint.
- Matching: Subset of edges M ⊆ E such that no edges in M share an endpoint.
- Maximum matching: matching of maximum cardinality.

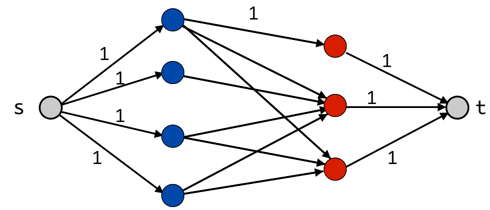
- Solve via flow:



Maximum Bipartite Matching

- Bipartite graph: Can color vertices red and blue such that all edges have a red and a blue endpoint.
- Matching: Subset of edges M ⊆ E such that no edges in M share an endpoint.
- Maximum matching: matching of maximum cardinality.

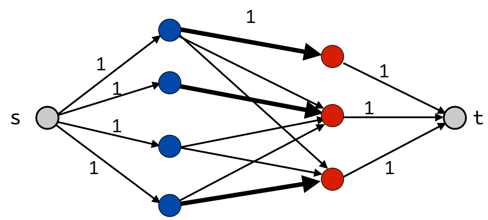
- Solve via flow:
  - Matching M => flow of value |M|



Maximum Bipartite Matching

- Bipartite graph: Can color vertices red and blue such that all edges have a red and a blue endpoint.
- Matching: Subset of edges M ⊆ E such that no edges in M share an endpoint.
- Maximum matching: matching of maximum cardinality.

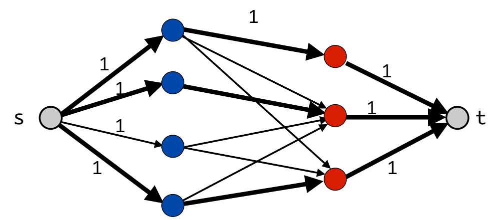
- Solve via flow:
  - Matching M => flow of value |M|



Maximum Bipartite Matching

- Bipartite graph: Can color vertices red and blue such that all edges have a red and a blue endpoint.
- Matching: Subset of edges M ⊆ E such that no edges in M share an endpoint.
- Maximum matching: matching of maximum cardinality.

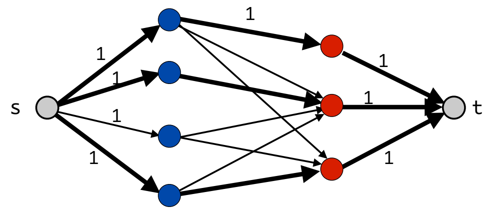
- Solve via flow:
  - Matching M => flow of value |M|



Maximum Bipartite Matching

- Bipartite graph: Can color vertices red and blue such that all edges have a red and a blue endpoint.
- Matching: Subset of edges M ⊆ E such that no edges in M share an endpoint.
- Maximum matching: matching of maximum cardinality.

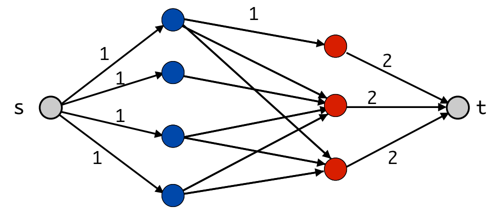
- Solve via flow:
  - Matching M => flow of value |M|
  - Flow of value v(f) => matching of size v(f)



Maximum Bipartite Matching

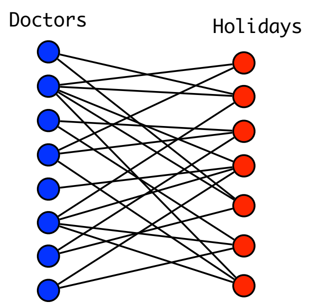
- Bipartite graph: Can color vertices red and blue such that all edges have a red and a blue endpoint.
- Matching: Subset of edges M ⊆ E such that no edges in M share an endpoint.
- Maximum matching: matching of maximum cardinality.

- Solve via flow:
- Can generalize to general matchings



Scheduling of doctors

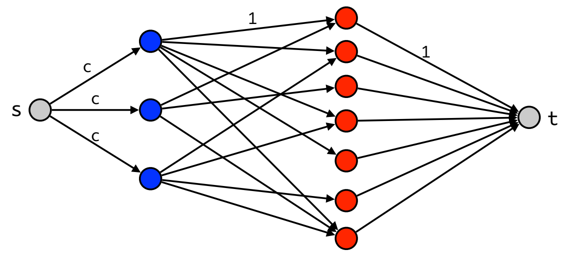
- X doctors, Y holidays, each doctor should work at at most 1 holiday, each doctor is available at some of the holidays.



- Same problem, but each doctor should work at most c holidays?

Scheduling of doctors

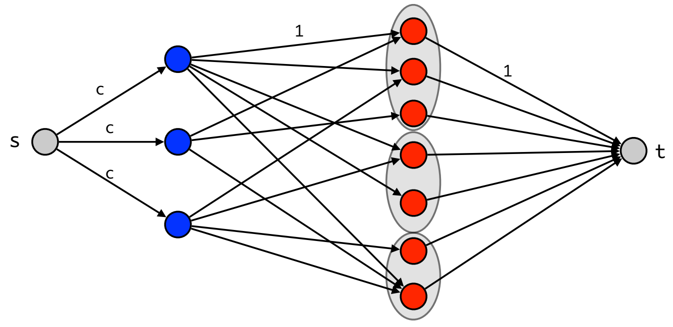
- X doctors, Y holidays, each doctor should work at at most c holidays, each doctor is available at some of the holidays.



- Same problem, but each doctor should work at most one day in each vacation period?

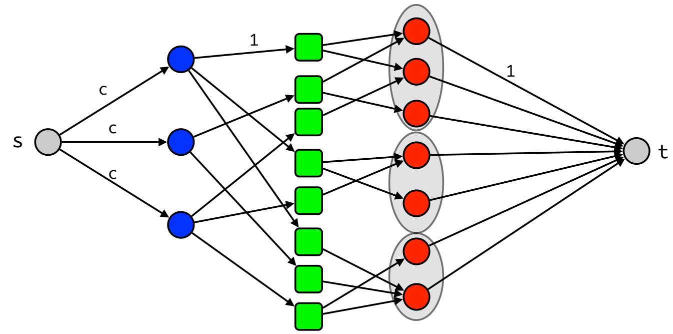
Scheduling of doctors

- X doctors, Y holidays, each doctor should work at at most c holidays, each doctor is available at some of the holidays.
- Same problem, but each doctor should work at most one day in each vacation period?



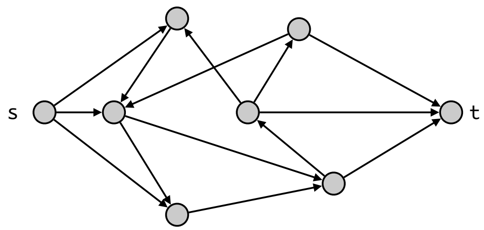
Scheduling of doctors

- X doctors, Y holidays, each doctor should work at at most c holidays, each doctor is available at some of the holidays.
- Same problem, but each doctor should work at most one day in each vacation period?



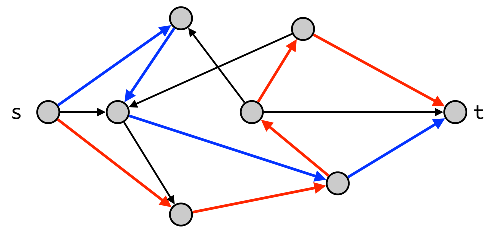
Edge Disjoint paths

- Problem: Find maximum number of edge-disjoint paths from s to t.
- Two paths are edge-disjoint if they have no edge in common.



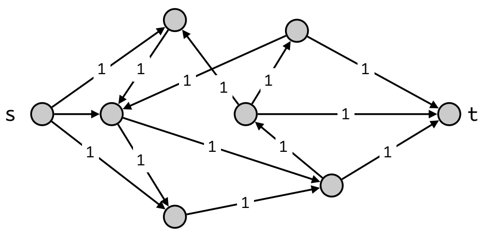
Edge Disjoint paths

- Edge-disjoint path problem. Find the maximum number of edge-disjoint paths from s to t.
- Two paths are edge-disjoint if they have no edge in common.



Edge Disjoint Paths

- Reduction to max flow: assign capacity 1 to each edge.

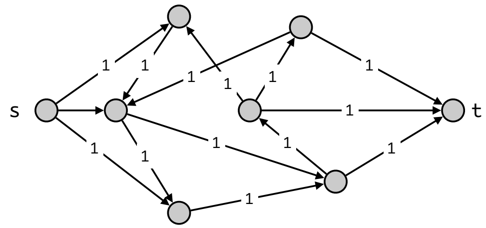


- Thm. Max number of edge-disjoint s-t paths is equal to the value of a maximum flow.
  - Suppose there are k edge-disjoint paths: then there is a flow of k (let all edges on the paths have flow 1).
  - Other way (graph theory course).

- Ford-Fulkerson: v(f) ≤ n (no multiple edges and therefore at most n edges out of s) => running time O(nm).

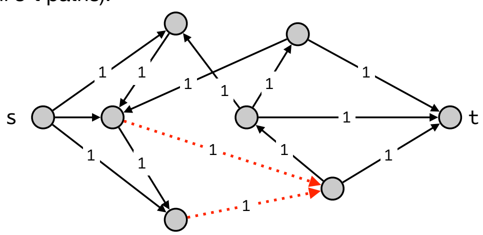
Network Connectivity

- Network connectivity. Find minimum number of edges whose removal disconnects t from s (destroys all s-t paths).



Network Connectivity

- Network connectivity. Find minimum number of edges whose removal disconnects t from s (destroys all s-t paths).



- Set all capacities to 1 and find minimum cut.
- Thm. (Menger) The maximum number of edge-disjoint s-t paths is equal to the minimum number of edges whose removal disconnects t from s.

Baseball elimination

Team	Wins	Games left	Against			
			NY	Bal	Tor	Bos
New York	92	2	-	1	1	0
Baltimore	91	3	1	-	1	1
Toronto	91	3	1	1	-	1
Boston	90	2	0	1	1	-

- Question: Can Boston finish in first place (or in tie of first place)?

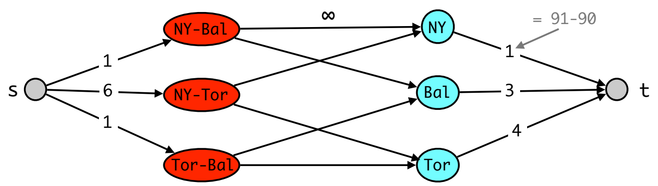
- No: Boston must win both its remaining 2 and NY must loose. But then Baltimore and Toronto both beat NY so winner of Baltimore-Toronto will get 93 points.

- Other argument: Boston can finish with at most 92. Cumulatively the other three teams have 274 wins currently and their 3 games against each other will give another 3 points => 277. 277/3 = 92,33333 => one of them must win > 92.

Baseball elimination

Team	Wins	Games left	Against			
			NY	Bal	Tor	Bos
New York	90	11	-	1	6	4
Baltimore	88	6	1	-	1	4
Toronto	87	11	6	1	-	4
Boston	79	12	4	4	4	-

- Question: Can Boston finish in first place (or in tie of first place)?



Boston can get at most 79 + 12 = 91 points

- Boston is eliminated ⇔ max s-t flow < 8.

Node capacities

- Capacities on nodes.