Weekplan: Priority Queues and Heaps

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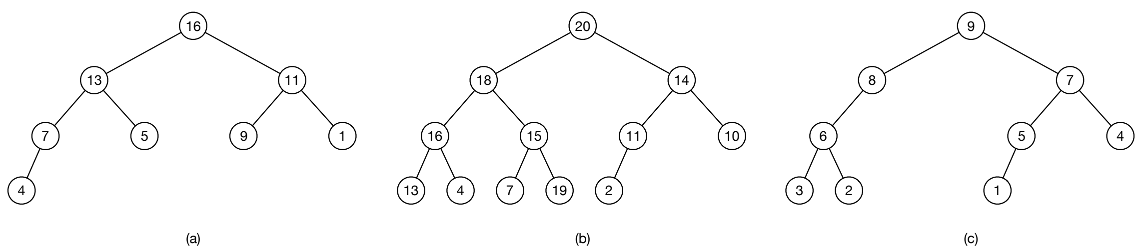
Reading

Introduction to Algorithms, 4th edition, Cormen, Rivest, Leisersons, and Stein (CLRS): Chapter 6 + Appendix B.5

Exercises

1 Heap Properties and Simulation Solve the following exercises.

1.1 [w] Which of the following trees are heaps?



1.2 [w] Which of the following arrays are heaps? Index 0 is not used and is therefore marked with −

$$A = [-, 9, 7, 8, 3, 4] \qquad B = [-, 12, 4, 7, 1, 2, 10] \qquad C = [-, 5, 7, 8, 3]$$

1.3 [w] Let $S = 4, 8, 11, 5, 21, \star, 2, \star$ be a sequence of operations where a number corresponds to an insertion of that number and $\star$ corresponds to an ExtractMax operation. Starting with an empty heap H, show how H looks after each operation in S.

1.4 Is a sorted array a heap?

1.5 Where can the minimum element be found in a (max-)heap?

1.6 Show that Insert, ExtractMax and IncreaseKey maintains the heap property.

1.7 [$*$] Suppose we have k sorted arrays containing in total n elements. Show how to merge the array into a single sorted array in time $O(n \log k)$.

2 Priority Queue Operations We now want to extend the set of operations on priority queues. We are interested in the following operations.

- RemoveLargest(m): remove the m largest elements in the priority queue.
- Delete(x): remove the element x from the priority queue.
- Fusion(x, y): remove x and y from the priority queue and add the element z with key $x.key + y.key$.
- Threshold(k): return the elements in the priority queue with key $\geq k$.
- ExtractMin: Remove and return the element with the lowest key.

We want to support these operations efficiently while maintaining the complexities of the standard operations. Let n be the number of elements in the priority queue. Solve the following exercises.

2.1 Extend the priority queue to support REMOVELARGEST(m) in $O(m \log n)$ time.

2.2 Extend the priority queue to support DELETE and FUSION in $O(\log n)$ time.

2.3 [$*$] Extend the priority queue to support THRESHOLD in $O(m)$ time, where m is the number of elements with key $\geq k$.

2.4 [$*$] Extend the priority queue to support EXTRACTMIN in $O(\log n)$ time.

3 Satellite Data Let $A[0..n]$ be a heap represented as an array. Each element x in the heap is represented by an index i and the key stored in $A[i]$. It is often useful to store some extra information (called *satellite data*) associated with an element (for instance, if we want to store persons in a heap, the satellite data could be age, gender, height, weight, etc). Show how to support access to satellite data in $O(1)$ time only given the index i while maintaining the running times for the standard heap operations.

4 Heap Properties Let T be a complete binary tree of height h. Solve the following exercises.

4.1 Show the number of nodes in T is $n = 2^{h+1} - 1$. *Hint:* Argue that the number of nodes in T is $n = 1 + 2 + 4 + \cdots 2^h$ and consider the binary representation of this number.

4.2 Show that the sum, $S = n/4 \cdot 1 + n/8 \cdot 2 + n/16 \cdot 3 + n/32 \cdot 4 + \cdots = \Theta(n)$. *Hint:* Calculate $S - S/2$.

5 [†] **Task Delegation** Josefine is in charge of the local student organization at The University of Algorithms. The organization gets tasks they must complete. Each task has a unique id and a unique difficulty. Over time, new tasks are given to the organization, and Josefine is responsible for delegating these tasks to the organization's members. When a member is ready to do a new task, he/she asks Josefine for a new task. Josefine likes to challenge her members, so she always picks the most difficult currently available task when a member requests a new task. More precisely, we want to support the following operations:

- NEWTASK(i, d): Add the task with id i and difficulty d to the set of tasks.
- REQUESTTASK(): Remove the task with the highest difficulty from the set of tasks and return its id.

Give a data structure for this problem and implement it.

Input Line 1 contains the integer N that is the total number of tasks. Line $2 \ldots N+1$ is can be either a new task indicated by the letter `N` followed by two integers id and diff (corresponding to the id of the task and the difficulty) or a member that is ready to receive a new task indicated by the letter `R`.

Output Line i is the id of the task given to the ith recieve request.

6 Sums Let $A[0..n-1]$ be an array of integers. We are interested in the following operations on A.

- SUM(i, j): compute $A[i] + A[i+1] + \cdots + A[j]$.
- CHANGE(i, x): set $A[i] = x$.

Solve the following exercises.

6.1 [w] Give a simple data structure that supports SUM in $O(1)$ time and uses $O(n^2)$ space.

6.2 [$*$] Give a data structure that supports SUM in $O(1)$ time and uses $O(n)$ space.

6.3 [$**$] Give a data structure that supports both SUM and CHANGE in $O(\log n)$ time and uses $O(n)$ space.