Weekplan: Shortest Paths

Philip Bille Inge Li Gørtz

Reading

Introduction to Algorithms, 4th edition, Cormen, Rivest, Leisersons, and Stein (CLRS): Chapter 22.1-22.4 + *Competitive Programmer's Handbook*, Laaksonen (CSES): Chapter 13.

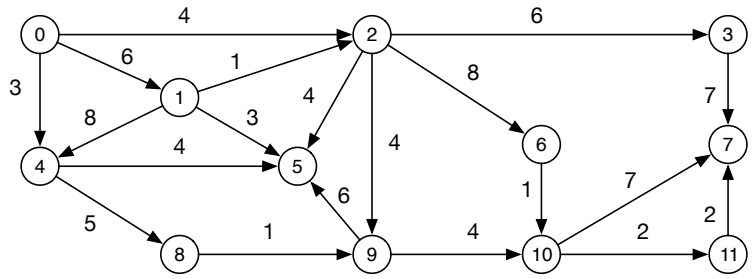


Figure 1: A directed weighted graph.

Exercises

1 Algorithms and Properties

1.1 [w] Consider the graph in Figure 1. Show a shortest path tree for the graph start at node 0. Write the length of the shortest path from 0 to each node.

1.2 Give an example graph with negative edges, but no negative cycles, results in an incorrect output from Dijkstra's algorithm.

1.3 Consider a graph G and a tree T in G rooted at a vertex s. Give an algorithm that determines if T is a shortest path tree from s in G.

1.4 Let T be a shortest path tree from a node s in a graph G. Assume we add a constant c to all edge weights in G. Is T still a shortest path tree?

2 Cable Routing (Exam 2012) The cable TV company AlgoNet broadcasts cable TV to all the houses in AlgoCity. They transmit the TV signals from their base station through a network of cables, where the length of each cable is known in meters. The cables are routed between a series of boxes. There is a box in each of the houses, one at the base station, and no boxes elsewhere. Each box may be connected to many cables. There are X houses and K cables in the network. Solve the following exercises.

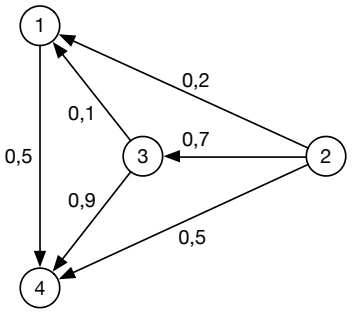
2.1 [†] AlgoNet wants all customers to get the best signal possible. The quality of the TV signal decreases proportional to the length of the cable. Give an algorithm to find a best way to route the signals to maximize the signal quality.

2.2 [†] Upon closer examination, AlgoNet discovered that when the signal goes through a box, its quality decreases as if it travelled through 5 meters of cable. Give an algorithm to find a best way to route the signals to maximize the signal quality in this scenario.

2.3 After cuts in government funding, AlgoNet is looking for ways to save money. Currently, they are spending 42 thousand kr. to maintain one meter of cable every year. Give an algorithm that finds a cheapest way to get a TV signal to all the houses in AlgoCity.

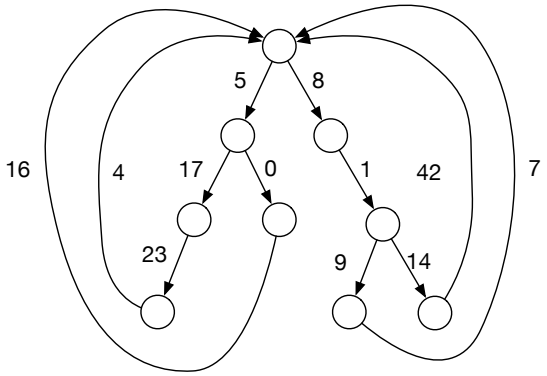
3 Longest Paths in DAGs Give an algorithm to find the *longest path* in a DAG.

4 [∗] **Zombie Travel** In the post-apocalyptic zombie world you need to know the safest travel between two cities such that you hopefully avoid being eaten by the zombies. You are given a graph G where each node represents a city and each edge a road between two citites. Each edge e has a *probability* $s(e)$, $0 \leq s(e) \leq 1$ for surviving traveling on that edge without being eaten. The probabilities on each edge are independent and the probability of surving the entire travel along a path P is the product of the probabilities of surviving on each edge of P.



As an example look at the above graph. If you travel directly from node 2 to 4 you have 50% chance of surviving. If you instead travel via node 3 you have $0.7 \cdot 0.9 = 63\%$ chance of surviving. If you travel via 3 and 1 you only have $0.7 \cdot 0.1 \cdot 0.5 = 3.5\%$ chance of surviving. Give an algorithm that computes the safest way from a node s to another node t.

5 Loopy Trees A *loopy tree* is a weighted directed graph constructed from a binary tree by adding an edge from each leaf to the root. All edges have non-negative weights.



Consider a loopy tree T with n nodes. Given a node s, we are interested in computing the shortest paths from s to all other nodes in T.

5.1 Suppose we use Dijkstra's algorithm to solve the problem. What is the running time in terms of parameter n?

5.2 [∗] Give a faster algorithm.