Roots of the chromatic polynomial, spanning trees and minors

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The Chromatic Polynomial

To each graph $G$, we can associate a polynomial $P(G, t)$ which counts, for each non-negative integer $t$, the number of proper $t$-colourings of $G$.

\[ P(K_3, t) = t(t - 1)(t - 2) \]
**Chromatic Roots**

**Definition**
A real number $t$ is a *chromatic root of $G$* if $P(G, t) = 0$.

Clearly $0, 1, \ldots, \chi(G) - 1$ are always chromatic roots of $G$.

Can chromatic polynomials have non-integer roots?
Chromatic Polynomial of $K_{2,3}$
No chromatic roots

Theorem (Tutte/Woodall 1974)

*No graph has a chromatic root in the intervals* $(-\infty, 0)$ *and* $(0, 1)$.
No chromatic roots

Theorem (Tutte/Woodall 1974)

_No graph has a chromatic root in the intervals $(-\infty, 0)$ and $(0, 1)$. _

Every bipartite graph with an odd number of vertices has a chromatic root in $(1, 2)$. 

No chromatic roots

Theorem (Jackson 1993)

No graph has a chromatic root in the interval \((1, \frac{32}{27}]\).
No chromatic roots

Theorem (Jackson 1993)

No graph has a chromatic root in the interval $(1, \frac{32}{27}]$.

$32/27$ is the infimum in the interval $(1, 2]$ of the chromatic roots of all graphs.
Chromatic roots in \((1, 2)\)

For a class of graphs \(\mathcal{G}\), let \(\omega(\mathcal{G})\) be the infimum in \((1, 2]\) of the chromatic roots of \(G \in \mathcal{G}\).
Chromatic roots in $(1, 2)$.

**Theorem (Jackson 1993)**

If $\mathcal{G}$ is the class of all graphs, then $\omega(\mathcal{G}) = \frac{32}{27}$. 
Planar triangulations

**Theorem (Birkhoff and Lewis, 1946)**

*If $\mathcal{P}$ is the class of planar triangulations, then $\omega(\mathcal{P}) = 2$.***
Theorem (Thomassen 2000)

If \( \mathcal{H} \) is the class of graphs with a Hamiltonian path, then \( \omega(\mathcal{H}) = t_0 \) where \( t_0 \approx 1.296 \) is the unique real root of the polynomial \( t^3 - 2t^2 + 4t - 4 \).

\[ (-\infty, 0) \quad (0, 1) \quad (1, t_0) \]

\[ 0 \quad 1 \quad 2 \]
3-leaf spanning trees

Theorem (P. 2015+)

If $\mathcal{T}$ is the class of graphs containing a spanning tree with at most 3 leaves, then $\omega(\mathcal{T}) = t_1$, where $t_1 \approx 1.290$ is the smallest real root of the polynomial $t^6 - 8t^5 + 27t^4 - 56t^3 + 82t^2 - 76t + 31$. 

\[
\begin{align*}
(\infty, 0) & \quad (0, 1) & \quad (1, t_1)
\end{align*}
\]
Minors

Theorem (Dong 2010)
If $G$ is the class of graphs not containing $K_{2,3}$ as a minor, then $\omega(G) = 2$.

Theorem (Dong 2010)
If $G$ is the class of graphs not containing $K_{2,4}$ as a minor, then $\omega(G) = \gamma$ where $\gamma \approx 1.430$ is the chromatic root of $K_{2,3}$.
Theorem (P. 2015+)

If $G$ is the class of graphs which are \{H_0, H_1, H_2\} -minor-free, then $\omega(G) = t_0$, where $t_0 \approx 1.296$ is the unique real root of the polynomial $t^3 - 2t^2 + 4t - 4$. 

$H_0, H_1, H_2$
Theorem (P. 2015+)

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Theorem (Thomassen 2000)

If $\mathcal{H}$ is the class of graphs with a Hamiltonian path, then $\omega(\mathcal{H}) = t_0$ where $t_0 \approx 1.296$ is the unique real root of the polynomial $t^3 - 2t^2 + 4t - 4$. 
Generalised triangles

A very important class of graphs in the study of chromatic roots.

\[ \mathcal{K} = \{ \text{generalised triangles} \} \]
Double subdivision
Generalised triangles

Any graph you can obtain from a triangle by a sequence of double subdivisions.
Generalised triangles

The generalised triangles form a partially ordered set under the double subdivision operation: For generalised triangles $G$ and $H$, we say $H \leq G$ if $G$ can be obtained from $H$ by a sequence of double subdivisions.
Generalised triangles
Generalised triangles
Key observation

Lemma

If $G$ and $H$ are generalised triangles, then $H$ is a minor of $G$ if and only if $H \leq G$. 
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A downwards closed subset $\mathcal{K}'$ of $\mathcal{K}$ can be characterised within $\mathcal{K}$ by forbidding a collection of minors.
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If $G$ and $H$ are generalised triangles, then $H$ is a minor of $G$ if and only if $H \leq G$.

A downwards closed subset $\mathcal{K}'$ of $\mathcal{K}$ can be characterised within $\mathcal{K}$ by forbidding a collection of minors.
There is a minor-closed class of graphs $\mathcal{G}$ such that $\mathcal{G} \cap \mathcal{K} = \mathcal{K}'$.

**Lemma (Dong 2010)**

*If $\mathcal{G}$ is a minor-closed class of graphs, then $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$. Using Dong’s result: $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K}) = \omega(\mathcal{K}')$.***
The families $\mathcal{K}_1$ and $\mathcal{K}_2$

**Definition**

Let $G$ be a generalised triangle.

- $G \in \mathcal{K}_1$ if for every 2-cut $\{x, y\}$, and every component $C$ of $G - x - y$, at least one of $x$ and $y$ has precisely one neighbour in $C$.

- $G \in \mathcal{K}_2$ if for every 2-cut $\{x, y\}$, at least one component of $G - x - y$ is a single vertex.
The family $\mathcal{K}_1$

<table>
<thead>
<tr>
<th>Forbidden minors</th>
<th>$H_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega(\mathcal{K}_1)$</td>
<td>$5/4$</td>
</tr>
</tbody>
</table>

**Theorem**

*If $\mathcal{G}$ is the class of $H_0$-minor-free graphs, then $\omega(\mathcal{G}) = 5/4$.***

**Conjecture (Dong and Jackson 2011)**

*If $\mathcal{G}$ is the family of graphs such that some vertex is contained in every 2-cut, then $\omega(\mathcal{G}) = 5/4$.***
### The family $\mathcal{K}_2$

<table>
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<tr>
<th>Forbidden minors</th>
<th>$\omega(\mathcal{K}_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>$H_2$</td>
</tr>
</tbody>
</table>

$q \approx 1.225$ root of $t^4 - 4t^3 + 4t^2 - 4t + 4$.

**Theorem**

If $\mathcal{G}$ is the class of $\{H_1, H_2\}$-minor-free graphs, then $\omega(\mathcal{G}) = q$.

**Conjecture (Dong and Jackson 2011)**

If $\mathcal{G}$ is the family of 2-connected plane graphs such that every two cut is contained in the outer cycle, then $\omega(\mathcal{G}) = q$. 
The family $\mathcal{K}_1 \cap \mathcal{K}_2$

<table>
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<tr>
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</table>

Lemma (Thomassen 2000)

If $\mathcal{H}$ is the class of graphs with a Hamiltonian path, then $\omega(\mathcal{H}) = \omega(\mathcal{H} \cap \mathcal{K}) = t_0$.

We show $\{ P(H, t) : H \in \mathcal{H} \cap \mathcal{K} \} = \{ P(G, t) : G \in \mathcal{K}_1 \cap \mathcal{K}_2 \}$.

Theorem

If $\mathcal{G}$ is the class of $\{ H_0, H_1, H_2 \}$-minor-free graphs, then $\omega(\mathcal{G}) = t_0$. 
Further work

Conjecture

Let $H$ be a generalised triangle. If $\mathcal{G}$ is the class of $H$-minor-free graphs, then $\omega(\mathcal{G}) > 32/27$. 
Thanks.