Better Bounded Bisimulation Contractions

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Abstract

Bisimulations are standard in modal logic and, more generally, in the theory of state-transition systems. The quotient structure of a Kripke model with respect to the bisimulation relation is called a bisimulation contraction. The bisimulation contraction is a minimal model bisimilar to the original model, and hence, for (image-)finite models, a minimal model modally equivalent to the original. Similar definitions exist for bounded bisimulations (k-bisimulations) and bounded bisimulation contractions. Two finite models are k-bisimilar if and only if they are modally equivalent up to modal depth k. However, the quotient structure with respect to the k-bisimulation relation does not guarantee a minimal model preserving modal equivalence to depth k. In this paper, we remedy this asymmetry to standard bisimulations and provide a novel definition of bounded contractions called rooted k-contractions. We prove that rooted k-contractions preserve k-bisimilarity and are minimal with this property. Finally, we show that rooted k-contractions can be exponentially more succinct than standard k-contractions.

Keywords: Modal logic, Kripke models, Bounded bisimulations, Bisimulation contractions, Exponential succinctness.

1 Introduction

Bisimulation plays a central role in countless fields, such as modal logic, set theory, formal verification, concurrency theory, process calculus, and others. Two structures are bisimilar if they are indistinguishable with respect to some behavioral property. In the case of Kripke models, bisimilarity between two models mean that their accessibility relations are structurally equivalent, and it then follows that two (image-)finite models are bisimilar if and only if they are modally equivalent (modal equivalence means they satisfy the same formulas) [3]. When using Kripke models for computational purposes, it is often

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desirable to keep the models as small as possible while preserving their logical properties, *e.g.*, preserving modal equivalence. The quotient structure of a Kripke model with respect to the bisimulation relation is called a *bisimulation contraction*, giving us a minimal model modally equivalent to the original [4].

Bounded bisimulations only preserve structural equivalence up to some depth k. In the case of Kripke models, two models are k-bisimilar if their accessibility relations are structurally equivalent up to depth k. Two finite models are then k-bisimilar if and only if they are modally equivalent to modal depth k [3]. When the modalities are used to represent knowledge in an epistemic logic, k-bisimilarity means that higher-order reasoning is preserved to depth k. If we are only interested in reasoning to depth k, say in a setting with agents having bounded rationality, it seems intuitive to consider the quotient structure with respect to k-bisimilarity in the attempt to find a minimal model preserving modal equivalence to depth k. We call the quotient structure with respect to k-bisimilarity the *standard k-contraction* [17]. However, as we will show, the standard k-contraction does not guarantee a minimal model preserving modal equivalence to depth k. We provide an alternative notion of k-bisimulation contraction, called a *rooted k-contraction*, that indeed guarantees minimality.

The inspiration for this paper came from bounded rationality in epistemic planning [2]. Epistemic planning is concerned with computing plans for agents having incomplete information about the world and each other's knowledge, using epistemic logic as the underlying formalism [5]. Unfortunately, epistemic planning is in general undecidable, *i.e.*, it has an undecidable plan existence problem [5,7]. In the search for interesting decidable fragment of epistemic planning, we decided to limit the reasoning capabilities of agents to some fixed depth k (e.g. limiting to depth 2 would mean that agents can reason about what they know about the knowledge of others, but not what they know about what others know about them). In order to define such *depth-limited epistemic planning* formally, we needed a notion of contraction that would preserve modal equivalence to depth k. We originally started out using standard k-contractions, but soon discovered that they didn't guarantee minimality of the contracted models, in many cases actually quite far from it. We then set out on a quest to try to find a better notion of k-contraction that would guarantee minimality among k-bisimilarity preserving models, hence also reestablishing the symmetry to the corresponding existing results for standard bisimulations. In this paper we report on the results of that quest. We will report on the results of applying these notions to epistemic planning in a separate paper.

2 Preliminaries

In this section, we recall some basic notions in modal logic, *i.e.*, pointed Kripke models, bisimulation and bounded bisimulation [3]. Let $\mathcal{P}$ be a countable set of atomic propositions and $\mathcal{I}$ a finite set of modality indices. The language $\mathcal{L}$ of *multi-modal logic* is defined by the following BNF (where $p \in \mathcal{P}$ and $i \in \mathcal{I}$):

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i \varphi.$$

Symbols $\top$, $\bot$, $\vee$ and $\Diamond_i$ are defined as usual. *Modal depth* is defined inductively on the structure of formulas: $md(p) = 0$ (for all $p \in \mathcal{P}$), $md(\neg\varphi) = md(\varphi)$, $md(\varphi_1 \wedge \varphi_2) = \max\{md(\varphi_1), md(\varphi_2)\}$ and $md(\Box_i\varphi) = 1 + md(\varphi)$.

Definition 2.1 A *model* of $\mathcal{L}$ is a triple $M = (W, R, V)$ where:

- $W \neq \varnothing$ is a finite set of *(possible) worlds*;
- $R : \mathcal{I} \to 2^{W \times W}$ assigns to each $i \in \mathcal{I}$ an *accessibility relation* R_i;
- $V : \mathcal{P} \to 2^W$ is a *valuation function* assigning to each atom a set of worlds.

A *pointed model* $\mathcal{M}$ is a pair (M, w_d), where $w_d \in W$ is the *designated world.*

We also use wR_iv for $(w, v) \in R_i$. We call an *i-edge*, or simply an *edge*, such a pair of worlds. A *path* is a sequence of worlds connected by edges. Note that we have restricted our attention to finite models. All of our results generalize to infinite models, but for many of the results we then have to make additional assumptions such as the models being image-finite, the underlying set of propositional atoms being finite or the number of modalities being finite [3]. To avoid this additional layer of complexity, and since all of our intended applications are within finite models, we restrict to those throughout the paper.

Definition 2.2 Let $M = (W, R, V)$ be a model of $\mathcal{L}$ and let $w \in W$.

$$
\begin{array}{lll}
(M, w) \models p & \text{iff} & w \in V(p) \\
(M, w) \models \neg\varphi & \text{iff} & (M, w) \not\models \varphi \\
(M, w) \models \varphi \wedge \psi & \text{iff} & (M, w) \models \varphi \text{ and } (M, w) \models \psi \\
(M, w) \models \Box_i\varphi & \text{iff} & \text{for all } v \text{ if } wR_iv \text{ then } (M, v) \models \varphi
\end{array}
$$

We say that two pointed models $\mathcal{M}$ and $\mathcal{M}'$ *agree on* (the formulas of) a set $\Phi \subseteq \mathcal{L}$ if, for all $\phi \in \Phi$, $\mathcal{M} \models \phi$ iff $\mathcal{M}' \models \phi$. We recall below the notions of bisimulation and bounded bisimulation (k-bisimulations) [3,15].

Definition 2.3 Let (M, w_d) and (M', w'_d) be two pointed models, with $M = (W, R, V)$ and $M' = (W', R', V')$. A *bisimulation* between (M, w_d) and (M', w'_d) is a non-empty binary relation $Z \subseteq W \times W'$ with $(w_d, w'_d) \in Z$ and satisfying:

- [atom] If $(w, w') \in Z$, then for all $p \in \mathcal{P}$, $w \in V(p)$ iff $w' \in V'(p)$.
- [forth] If $(w, w') \in Z$ and wR_iv, then there exists $v' \in W'$ such that $w'R'_iv'$ and $(v, v') \in Z$.
- [back] If $(w, w') \in Z$ and $w'R'_iv'$, then there exists $v \in W$ such that wR_iv and $(v, v') \in Z$.

If a bisimulation between (M, w_d) and (M', w'_d) exists, we say that (M, w_d) and (M', w'_d) are *bisimilar*, denoted $(M, w_d) \leftrightarrows (M', w'_d)$. When $(M, w) \leftrightarrows (M, w')$ for some worlds w, w' of the same model M, we simply write $w \leftrightarrows w'$, and say that w and w' are *bisimilar.* Finally, we denote the *bisimulation (equivalence) class* of a world $w \in W$ as $[w]_{\leftrightarrows} = \{v \in W \mid w \leftrightarrows v\}$.

Proposition 2.4 ([3]) *Two pointed models are bisimilar iff they agree on* $\mathcal{L}$.

Definition 2.5 Let $k \geq 0$ and let (M, w_d) and (M', w'_d) be two pointed models, with $M = (W, R, V)$ and $M' = (W', R', V')$. A *k-bisimulation* between (M, w_d) and (M', w'_d) is a sequence of non-empty binary relations $Z_k \subseteq \cdots \subseteq Z_0 \subseteq W \times W'$ with $(w_d, w'_d) \in Z_k$ and satisfying, for all $h < k$:

- [atom] If $(w, w') \in Z_0$, then for all $p \in \mathcal{P}$, $w \in V(p)$ iff $w' \in V'(p)$.
- [forth$_h$] If $(w, w') \in Z_{h+1}$ and wR_iv, then there exists $v' \in W'$ such that $w'R'_iv'$ and $(v, v') \in Z_h$.
- [back$_h$] If $(w, w') \in Z_{h+1}$ and $w'R'_iv'$, then there exists $v \in W$ such that wR_iv and $(v, v') \in Z_h$.

If a k-bisimulation between (M, w_d) and (M', w'_d) exists, we say that (M, w_d) and (M', w'_d) are *k-bisimilar*, denoted $(M, w_d) \leftrightarrows_k (M', w'_d)$. When $(M, w) \leftrightarrows_k (M', w')$, we often simply write $w \leftrightarrows_k w'$, and say that w and w' are *k-bisimilar* (when M and M' are clear from the context). Finally, we denote the *k-bisimulation (equivalence) class* of a world $w \in W$ as $[w]_k = \{v \in W \mid w \leftrightarrows_k v\}$.

Note that a k-bisimulation between pointed models is also an h-bisimulation for all $h \leq k$, and hence that k-bisimilar worlds are also h-bisimilar for all $h \leq k$.

Proposition 2.6 ([3]) *Two pointed models are k-bisimilar iff they agree on* $\{\phi \in \mathcal{L} \mid md(\phi) \leq k\}$*,* i.e.*, on all of formulas up to modal depth k.*

Definition 2.7 Let (M, w_d) be a pointed model. The *depth* $d(w)$ of a world w is the length of the shortest path from w_d to w (∞ if no such path exists). Given $k \geq 0$, the *restriction* $M \upharpoonright k$ of M to k is the sub-model containing all worlds with depth at most k (and preserving all edges between them).

Lemma 2.8 ([3]) *Let M and k be as above. Then, for every world w of $M \upharpoonright k$, we have* $(M \upharpoonright k, w) \leftrightarrows_{k-d(w)} (M, w)$*.*

3 Defining Rooted k-Contractions

The notion of bisimulation contraction is well-known in modal logic. The *(bisimulation) contraction* of a pointed model $\mathcal{M} = ((M, R, V), w_d)$, that we denote with $\lfloor \mathcal{M} \rfloor$, is defined as the *quotient structure* of $\mathcal{M}$ with respect to $\leftrightarrows$, *i.e.*, $\lfloor \mathcal{M} \rfloor = ((W', R', V'), [w_d]_{\leftrightarrows})$, where $W' = \{[w]_{\leftrightarrows} \mid w \in W\}$, $R'_i = \{([w]_{\leftrightarrows}, [v]_{\leftrightarrows}) \mid wR_iv\}$, and $V'(p) = \{[w]_{\leftrightarrows} \in W' \mid w \in V(p)\}$ [15]. It is relatively straightforward to prove that: (i) $\lfloor \mathcal{M} \rfloor$ is bisimilar to $\mathcal{M}$; and (ii) $\lfloor \mathcal{M} \rfloor$ is a minimal model bisimilar to $\mathcal{M}$. A similar definition exists for *k-(bisimulation) contractions.* Namely, the k-contraction of $\mathcal{M}$, that we denote with $\lfloor \mathcal{M} \rfloor_k$, has been defined as the quotient structure of $\mathcal{M}$ with respect to $\leftrightarrows_k$ [8,17]. We call this the *standard k-contraction* of $\mathcal{M}$. However, although such a contracted model is k-bisimilar to the original one [8,17], in general it is not minimal, as the following example shows.

Example 3.1 Consider the chain model $\mathcal{M}$ in Figure 1 (left). Since p is true in all worlds, and the length of the chain is k, a minimal model k-bisimilar to $\mathcal{M}$ is a singleton pointed model with a loop (Figure 1, right). This is because the loop model preserves all formulas up to depth k, cf. Proposition 2.6. However,

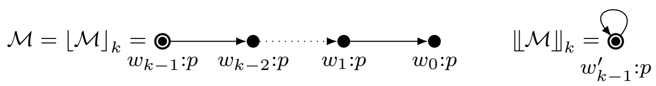

Fig. 1. Standard ($\lfloor \mathcal{M} \rfloor_k$) and rooted ($\llbracket \mathcal{M} \rrbracket_k$) k-contractions of chain $\mathcal{M}$ (symbol $\llbracket \mathcal{M} \rrbracket_k$ is borrowed from Definition 3.10). Each world w is denoted by a bullet labeled by its name, followed by the atomic propositions that hold in w. An arrow labeled with i from w to v means that wR_iv. We omit the labels on arrows whenever $|\mathcal{I}| = 1$. The designated world is represented by a circled bullet.

d	b (k=1)	b (k=2)	b (k=3)
0	1	2	3
1	0	1	2
2	−1	0	1

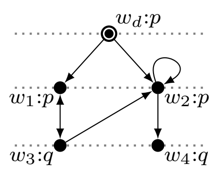

Fig. 2. Depth (d) and bound (b) of worlds for $k = 1, 2$ and 3.

the standard k-contraction of $\mathcal{M}$ is simply $\mathcal{M}$ itself: First note that for all $h \leq k-1$, the formula $\Diamond^h \Box \bot$ is true only in world w_h of $\mathcal{M}$ (it expresses the existence of a path of length h to a world from which no world is accessible). Hence, any two worlds of $\mathcal{M}$ can be distinguished by a formula $\Diamond^h \Box \bot$ of depth $h \leq k-1$. This implies that no two distinct worlds of $\mathcal{M}$ are modally equivalent to modal depth k, and hence cannot be part of the same k-bisimulation class. Thus, the standard k-contraction of $\mathcal{M}$ is $\mathcal{M}$ itself (or, more precisely, the k-contraction of $\mathcal{M}$ is isomorphic to $\mathcal{M}$).

We now move to introduce our *rooted k-contractions*. First, in this section, we show how to define a notion of a rooted k-contraction that guarantees the resulting model to have the smallest number of worlds among any model k-bisimilar to the original one (we call this property *world minimality*). Later we then define a stronger notion of rooted k-contraction that additionally guarantees the set of edges of the contracted model to be minimal (called *edge minimality*). In what follows, we fix a constant $k \geq 0$ and a pointed model $\mathcal{M} = (M, w_d)$ with $M = (W, V, R)$. Recall the notion of the depth $d(w)$ of a world w (Definition 2.7). We now introduce the notion of *bound* of a world.

Definition 3.2 The *bound* of a world w is $b(w) = k - d(w)$.

Lemma 3.3 *If xR_iy, then $b(y) \geq b(x) - 1$.*

Proof. $d(y)$ is the length of the shortest path from the designated world to y. Such a path either goes through the edge $(x, y) \in R_i$, or there is a shorter path to y. Hence, $d(y) \leq d(x) + 1 \Leftrightarrow k - b(y) \leq k - b(x) + 1 \Leftrightarrow b(y) \geq b(x) - 1$. □

Example 3.4 The notion of *bound* of a world will play a key role in the definition of rooted k-contractions. Figure 2 shows an example of bound of worlds for $k = 1, 2$ and 3. Now consider the pointed models $\mathcal{N}_1$ and $\mathcal{N}_2$ of Figure 3.

Fig. 3. Two 2-bisimilar pointed models: $\mathcal{N}_1$ (left) and $\mathcal{N}_2$ (right).

Taking the standard 2-contraction of $\mathcal{N}_1$ would result in the model $\mathcal{N}_1$ itself, by a similar argument as in Example 3.1 (no two worlds of $\mathcal{N}_1$ are 2-bisimilar, actually not even 1-bisimilar). However, $\mathcal{N}_1$ is not world minimal among models 2-bisimilar to $\mathcal{N}_1$. That is true for $\mathcal{N}_2$, however. Any model 2-bisimilar to $\mathcal{N}_1$ must have at least three worlds (one for each atomic proposition), which is exactly what $\mathcal{N}_2$ has. The 2-bisimulation between $\mathcal{N}_1$ and $\mathcal{N}_2$ is defined by: $Z_2 = \{(w_d, w'_d)\}$; $Z_1 = Z_2 \cup \{(w_1, w'_1), (w_2, w'_2)\}$; and $Z_0 = Z_1 \cup \{(w_3, w'_2)\}$. Notice that $\mathcal{N}_2$ has been obtained by $\mathcal{N}_1$ by redirecting all incoming edges of w_3 to w_2 and deleting the worlds that are no longer reachable from the designated world. In standard bisimulation contractions, we simply identify worlds that are bisimilar, meaning that we can merge them into one world. In the case of $\mathcal{N}_1$ and $\mathcal{N}_2$, we cannot just trivially merge w_2 and w_3 into one world, as they don't have the same successor worlds (we might be left with w_4 as a successor to w_2, which destroys 2-bisimilarity). The idea of redirecting edges rather than merging worlds forms a crucial part of the intuition behind our rooted k-contractions. The point is that w_2 can be used as a "representative" for w_3 when we perform the contraction, and hence we can get rid of w_3. The reason that w_2 works as a representative for w_3 is that w_3 is at depth 2, so has bound 0. Intuitively this means that w_3 can be represented by any world that it is 0-bisimilar to, as to maintain k-bisimilarity at the designated worlds, we only need to require $(k-d)$-bisimilarity of the worlds at depth d, *i.e.*, worlds of bound b only need to be b-bisimilar. These intuitions are made formally precise in the following.

Lemma 3.5 *Let $k \geq 0$, let (M, w_d) be a pointed model, with $M = (W, R, V)$ and let $x, y \in W \setminus \{w_d\}$ be two distinct worlds such that $b(x) \geq b(y) \geq 0$ and $x \leftrightarroweq_{b(y)} y$. Let (M', w_d), with $M' = (W', R', V')$, be the pointed model obtained by deleting y from (M, w_d) and redirecting its incoming edges to x. More precisely:*

- $W' = W \setminus \{y\}$;
- $R'_i = (R_i \cap (W' \times W')) \cup \{(w, x) \mid wR_i y\}$;
- $V'(p) = V(p) \cap W'$, *for all* $p \in \mathcal{P}$.

Then $(M, w_d) \leftrightarroweq_k (M', w_d)$.

Proof. To avoid confusion, we remark that, for all w, $b(w)$ refers to the bound that w has in $\mathcal{M}$ (and not in $\mathcal{M}'$). Similarly, $w \leftrightarroweq_h w'$ means $(M, w) \leftrightarroweq_h$

(M, w'). For all $0 \le h \le k$, let $Z_h \subseteq W \times W'$ be the following binary relation:

$$Z_h = \{(w, w') \in W \times W' \mid w \leftrightarroweq_h w', b(w) \ge h \text{ and } b(w') \ge h\}$$

We now show that $Z_k, \dots, Z_0$ is a k-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. Clearly, $Z_k \subseteq \cdots \subseteq Z_0$. Also, trivially, $(w_d, w_d) \in Z_k$ (recalling that $w \leftrightarroweq_h w'$ means $(M, w) \leftrightarroweq_h (M, w')$). We get [atom] since $(w, w') \in Z_0$ implies $w \leftrightarroweq_0 w'$ and thus $w \in V(p)$ iff $w' \in V'(p)$ (by [atom] of Definition 2.5).

We now show [forth$_h$]. Let $h < k$, $(w, w') \in Z_{h+1}$ and wR_iv. We need to find a $v' \in W'$ such that $w'R'_iv'$ and $(v, v') \in Z_h$. Since $(w, w') \in Z_{h+1}$, we have $w \leftrightarroweq_{h+1} w'$, $b(w) \ge h+1$ and $b(w') \ge h+1$. From $w \leftrightarroweq_{h+1} w'$, there exists $u \in W$ such that $w'R_iu$ and $v \leftrightarroweq_h u$. Since wR_iv, $w'R_iu$, $b(w) \ge h+1$ and $b(w') \ge h+1$, by Lemma 3.3, we get $b(v) \ge h$ and $b(u) \ge h$. We have two cases. (i) If $u \ne y$, then by construction of R'_i, we get $w'R'_iu$. Since $v \leftrightarroweq_h u$, $b(v) \ge h$ and $b(u) \ge h$, letting $v' = u$ we get $(v, v') \in Z_h$. (ii) If $u = y$, then by construction of R'_i, we get $w'R'_ix$. From $b(y) = b(u) \ge h$ and $x \leftrightarroweq_{b(y)} y$, we get $x \leftrightarroweq_h y$. Since $v \leftrightarroweq_h y \leftrightarroweq_h x$, $b(v) \ge h$ and $b(y) \ge h$, letting $v' = x$ we get $(v, v') \in Z_h$. This concludes [forth$_h$].

Now for [back$_h$]. Let $h < k$, $(w, w') \in Z_{h+1}$ and $w'R'_iv'$. We need to find v such that wR_iv and $(v, v') \in Z_h$. Since $(w, w') \in Z_{h+1}$, we have $w \leftrightarroweq_{h+1} w'$, $b(w) \ge h+1$ and $b(w') \ge h+1$. We have two cases. (i) If $v' \ne x$, then by construction of R'_i, we get $w'R_iv'$. Since $w \leftrightarroweq_{h+1} w'$, there exists $v \in W$ such that wR_iv and $v \leftrightarroweq_h v'$. As in [forth$_h$], Lemma 3.3 gives $b(v) \ge h$ and $b(v') \ge h$. Thus, $(v, v') \in Z_h$. (ii) If $v' = x$, then by construction of R'_i, we get $w'R_ix$ or $w'R_iy$. If $w'R_ix$, we can reason as in (i). If $w'R_iy$, pick v with wR_iv and $v \leftrightarroweq_h y$. As before, $b(v) \ge h$ and $b(y) \ge h$. Since $b(x) \ge b(y) \ge h$ and $x \leftrightarroweq_{b(y)} y$, we get $x \leftrightarroweq_h y$, and thus $v \leftrightarroweq_h x$. Hence $(v, x) \in Z_h$, as required. □

The lemma tells us that if we're only interested in preserving k-bisimilarity, a world y can be deleted from a model if there exists a distinct world x such that $b(x) \ge b(y)$ and $x \leftrightarroweq_{b(y)} y$. This leads us to the following definition.

Definition 3.6 Let x, y be two worlds with non-negative bound. We say that x *represents* y, denoted by $x \succeq y$, iff $b(x) \ge b(y)$ and $x \leftrightarroweq_{b(y)} y$. If furthermore $b(x) > b(y)$, we say that x *strictly represents* y, denoted by $x \succ y$. The set of *maximal representatives* of W is the set of worlds $W^{\max} = \{x \in W \mid b(x) \ge 0 \text{ and } \neg\exists y \in W (y \succ x)\}$. We say that a world x is a *maximal representative* of y if $x \in W^{\max}$ and $x \succeq y$.

Note that every world $w \in W$ with $b(w) \ge 0$ has at least one maximal representative: Any chain $w \prec w' \prec w'' \prec \cdots$ is finite (since W is finite) and must hence end in a maximal representative of w. We are going to build our rooted k-contractions on the maximal representatives, the intuition being that all other worlds can be represented by one of these and hence be deleted, cf. Lemma 3.5.

Proposition 3.7 *For any pointed model $((W, R, V), w_d)$, we have:*

1) $w_d \in W^{\max}$;

2) if $w \succ v$, then $v \notin W^{\max}$;

3) if $b(w) < 0$, then $w \notin W^{\max}$;

4) if $w, v \in W^{\max}$ and $w \leftrightarrows_{b(w)} v$ then $b(w) = b(v)$.

Proof. Item 1 follows since w_d has bound $b(w_d) = k$ and there can not be any world with a greater bound. Items 2 and 3 immediately follow by definition of $W^{\max}$. Item 4 is by contradiction: Suppose that $w \leftrightarrows_{b(w)} v$ and $b(w) > b(v)$ (the case $b(v) > b(w)$ being symmetric). Since $w \leftrightarrows_{b(w)} v$, it follows that $w \leftrightarrows_{b(v)} v$, which implies that $w \succ v$, contradicting the fact that $v \in W^{\max}$. □

Example 3.8 Let $\mathcal{M}$ the pointed model in Figure 2, let W be its set of worlds and let $k = 2$. Using Proposition 2.6, we can show that for all worlds $w, v \in W$, if $b(w) > b(v)$ then $w \not\leftrightarrows_{b(v)} v$: w_3 and w_4 of bound 0 are not propositionally equivalent to any world of greater bound (they are the only ones satisfying q); w_1 and w_2 of bound 1 both satisfy $\Diamond q$ of modal depth 1, which is not satisfied by the only world of greater bound, w_d. Hence, all worlds are maximal representatives, *i.e.*, $W^{\max} = W$. We immediately get $W^{\max} = W$ for $k = 3$ as well, since if worlds w and v are not n-bisimilar, they are also not $(n+1)$-bisimilar. For $k = 1$, Proposition 3.7(3) gives $w_3, w_4 \notin W^{\max}$, and since $w_d \succ w_1, w_2$ (they satisfy the same atomic propositions), we get $W^{\max} = \{w_d\}$.

Definition 3.9 The *representative class* of a world w is the class $[w]_{b(w)}$, which we denote with the compact notation $[\![w]\!]$.

Definition 3.10 Let $\mathcal{M} = ((W, R, V), w_d)$ and let $k \geq 0$. The *rooted k-contraction* of $\mathcal{M}$ is the pointed model $[\![\mathcal{M}]\!]_k = ((W', R', V'), [\![w_d]\!])$, where:

- $W' = \{[\![x]\!] \mid x \in W^{\max}\}$;
- $R'_i = \{([\![x]\!], [\![y]\!]) \mid x, y \in W^{\max}, \exists z(xR_iz \text{ and } y \leftrightarrows_{b(x)-1} z) \text{ and } b(x) > 0\}$;[3]
- $V'(p) = \{[\![x]\!] \mid x \in W^{\max} \text{ and } x \in V(p)\}$.[4]

The definition of R'_i requires some explanation. At first, one might think that defining the set of i-edges as $\{([\![x]\!], [\![y]\!]) \mid x, y \in W^{\max} \text{ and } xR_iy\}$ would be sufficient. However, this is not the case. To show this, consider the pointed model $\mathcal{N}_1$ in Figure 3 and let $k = 2$. One can easily show that $W^{\max} = \{w_d, w_1, w_2\}$, hence the rooted 2-contraction has worlds $W' = \{[\![w_d]\!], [\![w_1]\!], [\![w_2]\!]\}$. Since $(w_1, w_2) \notin R$, defining the accessibility relation

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[3] The definition of R'_i doesn't depend on the choice of maximal representatives: If $[\![x]\!] = [\![x']\!]$ and $[\![y]\!] = [\![y']\!]$ and $x, x', y, y' \in W^{\max}$, then $b(x) > 0$ iff $b(x') > 0$ and $\exists z(xR_iz$ and $y \leftrightarrows_{b(x)-1} z)$ iff $\exists z'(x'R_iz'$ and $y' \leftrightarrows_{b(x')-1} z')$. To prove this, first note that since $[\![x]\!] = [\![x']\!]$, we get $x \in [\![x']\!]$ and $x' \in [\![x]\!]$. Similarly for y and y'. Since $x \in [\![x']\!] = [x']_{b(x')}$, we then get $x \leftrightarrows_{b(x')} x'$, and hence $b(x) = b(x')$, by Proposition 3.7(4). Hence, $b(x) > 0$ iff $b(x') > 0$. Now suppose $\exists z(xR_iz$ and $y \leftrightarrows_{b(x)-1} z)$. We need to show that $\exists z'(x'R_iz'$ and $y' \leftrightarrows_{b(x')-1} z')$. Since $x \leftrightarrows_{b(x')} x'$ and xR_iy, there exists a z' such that $x'R_iz'$ and $y \leftrightarrows_{b(x')-1} z'$. Since xR_iy then $b(y) \geq b(x) - 1$, by Lemma 3.3. As $b(x) = b(x')$, we get $b(y) \geq b(x') - 1$. Since $y' \in [\![y]\!] = [y]_{b(y)}$, we get $y' \leftrightarrows_{b(y)} y$ and hence $y' \leftrightarrows_{b(x')-1} y$. Combining $y \leftrightarrows_{b(x')-1} z'$ and $y' \leftrightarrows_{b(x')-1} y$, we get $y' \leftrightarrows_{b(x')-1} z'$, as required.

[4] The definition of $V'(p)$ is well-defined since from $x' \in [\![x]\!]$ we get that $x' \leftrightarrows_{b(x)} x$ and, thus, $x \in V(p)$ iff $x' \in V'(p)$, by [atom] of Definition 2.5.

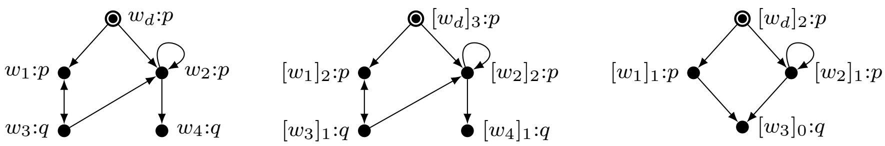

Fig. 4. Pointed model $\mathcal{M}$ (left) of Figure 2, $\llbracket\mathcal{M}\rrbracket_3$ (center) and $\llbracket\mathcal{M}\rrbracket_2$ (right).

of the contracted model by $R' = \{(\llbracket x \rrbracket, \llbracket y \rrbracket) \mid x, y \in W^{\max}$ and $xRy\}$ would imply $(\llbracket w_1 \rrbracket, \llbracket w_2 \rrbracket) \notin R'$. But then the formula $\Diamond r$ would be true in w_1 and not in $\llbracket w_1 \rrbracket$, *i.e.*, $w_1 \not\leftrightarrows_1 \llbracket w_1 \rrbracket$, implying that $\mathcal{N}_1$ is not even 2-bisimilar to its own rooted 2-contraction! With our current definition of R', we actually get that the rooted 2-contraction of $\mathcal{N}_1$ is exactly the model $\mathcal{N}_2$ of Figure 3 that we in Example 3.4 showed to be a world-minimal model 2-bisimilar to $\mathcal{N}_1$.

Lemma 3.11 *Let $\mathcal{M} = ((W, R, V), w_d)$ and $\llbracket\mathcal{M}\rrbracket_k = ((W', R', V'), w'_d)$. For all $i \in \mathcal{I}$ we have $R'_i \supseteq \{(\llbracket x \rrbracket, \llbracket y \rrbracket) \mid x, y \in W^{\max}, xR_iy$ and $b(x) > 0\}$.*

Proof. Let $x, y \in W^{\max}$ be such that xR_iy and $b(x) > 0$. We need to show that $(\llbracket x \rrbracket, \llbracket y \rrbracket) \in R'_i$. By definition of R'_i, this is true if there exists a $z \in W$ such that xR_iz and $y \leftrightarrows_{b(x)-1} z$. Since xR_iy, we can take $z = y$. □

Example 3.12 Let $\mathcal{M} = (M, w_d)$ with $M = (W, R, V)$ be the pointed model of Figure 2, shown again in Figure 4 (left). Let $\llbracket\mathcal{M}\rrbracket_3 = (M', w'_d)$, with $M' = (W', R', V')$, shown in Figure 4 (center). In Example 3.8, we showed that, when $k = 3$, we have $W^{\max} = W$. From Definition 3.10, we then have $W' = \{[w_d]_3, [w_1]_2, [w_2]_2, [w_3]_1, [w_4]_1\}$ and $w'_d = [w_d]_3$, where $[w_d]_3 = \{w_d\}$, $[w_1]_2 = \{w_1\}$, $[w_2]_2 = \{w_2\}$, $[w_3]_1 = \{w_3\}$ and $[w_4]_1 = \{w_4\}$. Since $W^{\max} = W$ and $\llbracket w \rrbracket = w$ for all $w \in W$, it is easy to check that we also get $R' = \{\llbracket x \rrbracket, \llbracket y \rrbracket \mid xRy\}$. Let now $\llbracket\mathcal{M}\rrbracket_2 = (M', w'_d)$, with $M' = (W', R', V')$, shown in Figure 4 (right). Again, from Example 3.8, we have $W^{\max} = W$. From Definition 3.10, we then have $W' = \{[w_d]_2, [w_1]_1, [w_2]_1, [w_3]_0\}$ and $w'_d = [w_d]_2$, where $[w_d]_2 = \{w_d\}$, $[w_1]_1 = \{w_1\}$, $[w_2]_1 = \{w_2\}$ and $[w_3]_0 = \{w_3, w_4\}$. In this case we get $R' = \{(\llbracket x \rrbracket, \llbracket y \rrbracket) \mid xRy$ and $b(x) > 0\}$.

4 Properties of Rooted k-Contractions

The first crucial property to show is that rooted k-contractions are k-bisimilar to their original models.

Theorem 4.1 *Let $\mathcal{M}$ be a pointed model and let $k \geq 0$. Then, $\mathcal{M} \leftrightarrows_k \llbracket\mathcal{M}\rrbracket_k$.*

Proof. Let $\mathcal{M} = ((W, R, V), w_d)$ and $\llbracket\mathcal{M}\rrbracket_k = ((W', R', V'), w'_d)$. For all $0 \leq h \leq k$, let $Z_h \subseteq W \times W'$ be the following binary relation:

$$Z_h = \{(x, \llbracket x' \rrbracket) \mid x' \in W^{\max}, x' \leftrightarrows_h x \text{ and } b(x) \geq h\}.$$

We now show that the sequence $Z_k, \ldots, Z_0$ is a k-bisimulation between $\mathcal{M}$ and $\llbracket\mathcal{M}\rrbracket_k$. Clearly, $Z_k \subseteq \cdots \subseteq Z_0$. Moreover, since $w_d \in W^{\max}$ (by Proposition 3.7(1)) and $b(w_d) = k$, it follows that $(w_d, \llbracket w_d \rrbracket) \in Z_k$.

We first show [atom]. Let $(x, x'') \in Z_0$. Then, by definition of Z_0, we have that $x'' = [\![x']\!]$ for some $x' \in W^{\max}$ such that $x' \leftrightarroweq_0 x$ and $b(x) \geq 0$. From $x'' = [\![x']\!]$, we get $x'' \leftrightarroweq_{b(x')} x'$ and hence $x'' \leftrightarroweq_0 x'$. Thus, $x'' \leftrightarroweq_0 x' \leftrightarroweq_0 x$ and, by [atom] of Definition 2.5, we then get $x'' \in V'(p)$ iff $x \in V(p)$, as required.

Before moving to [forth$_h$] and [back$_h$], we show the following claim.

Claim 1. Let $h \leq k$, $x \in W$ and $x' \in W^{\max}$ be such that $x \leftrightarroweq_h x'$ and $b(x) \geq h$. Then $b(x') \geq h$.

Proof of claim. Since $x' \in W^{\max}$, we have $x \not\succ x'$, hence either $b(x) \not> b(x')$ or $x \not\leftrightarroweq_{b(x')} x'$. If $b(x) \not> b(x')$, we get $b(x') \geq b(x) \geq h$, as required. If $x \not\leftrightarroweq_{b(x')} x'$, then since $x \leftrightarroweq_h x'$, we must have $b(x') > h$. This proves the claim.

We now show [forth$_h$]. Let $h < k$, $(x, x'') \in Z_{h+1}$, and xR_iy. Then $x'' = [\![x']\!]$, where $x' \in W^{\max}$, $x' \leftrightarroweq_{h+1} x$ and $b(x) \geq h + 1$. From Claim 1, we get $b(x') \geq h + 1$. We need to find a world $y'' \in W'$ such that $x''R'_iy''$ and $(y, y'') \in Z_h$. Since xR_iy and $x \leftrightarroweq_{h+1} x'$, there exists $z \in W$ such that $x'R_iz$ and $y \leftrightarroweq_h z$. Let y' be a maximal representative of z. Then, $y' \in W^{\max}$, $b(y') \geq b(z)$ and $y' \leftrightarroweq_{b(z)} z$. Since $x'R_iz$, by Lemma 3.3 we have $b(z) \geq b(x') - 1$. Since $y' \leftrightarroweq_{b(z)} z$, then we also get $y' \leftrightarroweq_{b(x')-1} z$. We now have $x', y' \in W^{\max}$, $x'R_iz$, $y' \leftrightarroweq_{b(x')-1} z$ and $b(x') \geq h+1 > 0$, which by Definition 3.10 means $[\![x']\!]R'_i[\![y']\!]$. Letting $y'' = [\![y']\!]$, we have hence found a y'' such that $x''R'_iy''$. The only thing left to prove now is that $(y, y'') \in Z_h$. By definition of Z_h, it suffices to prove $y' \in W^{\max}$, $y' \leftrightarroweq_h y$ and $b(y) \geq h$. We already have $y' \in W^{\max}$. Since $y' \leftrightarroweq_{b(x')-1} z$ and $b(x') \geq h+1$, we get $y' \leftrightarroweq_h z$. As we also have $y \leftrightarroweq_h z$, we get $y' \leftrightarroweq_h y$, as required. Finally, since xR_iy and $b(x) \geq h + 1$, Lemma 3.3 gives $b(y) \geq h$. This concludes [forth$_h$].

We now show [back$_h$]. Let $h < k$, $(x, x'') \in Z_{h+1}$, and $x''R'_iy''$. We need to find a world $y \in W$ such that xR_iy and $(y, y'') \in Z_h$. Since $x''R'_iy''$, by Definition 3.10 there exist $x', y' \in W^{\max}$ and $z \in W$ such that $x'' = [\![x']\!]$, $y'' = [\![y']\!]$, $x'R_iz$, $y' \leftrightarroweq_{b(x')-1} z$ and $b(x') > 0$. Since $(x, x'') \in Z_{h+1}$, then $x'' = [\![\hat{x}]\!]$, where $\hat{x} \in W^{\max}$, $\hat{x} \leftrightarroweq_{h+1} x$ and $b(x) \geq h + 1$. By Claim 1, we get $b(\hat{x}) \geq h + 1$. Since $x', \hat{x} \in W^{\max}$ and $[\![x']\!] = [\![\hat{x}]\!]$, by Proposition 3.7(4) we get $b(x') = b(\hat{x})$. From $x' \leftrightarroweq_{b(x')} \hat{x}$ and $b(x') \geq h + 1$ we get $x' \leftrightarroweq_{h+1} \hat{x}$ and, hence, $x' \leftrightarroweq_{h+1} x$. Since $x' \leftrightarroweq_{h+1} x$ and $x'R_iz$, there exists $y \in W$ such that xR_iy and $y \leftrightarroweq_h z$. Only left to show is that $(y, y'') \in Z_h$. From $y' \leftrightarroweq_{b(x')-1} z$ and $b(x') \geq h + 1$, we get $y' \leftrightarroweq_h z$ and, thus, $y' \leftrightarroweq_h y$. Since xR_iy and $b(x) \geq h + 1$, Lemma 3.3 gives $b(y) \geq h$. We now have $y' \in W^{\max}$, $y' \leftrightarroweq_h y$ and $b(y) \geq h$. Thus, $(y, y'') \in Z_h$, as required. This concludes [back$_h$]. □

We now prove world minimality. To show this property, it is useful to group the worlds of a rooted k-contraction wrt. to their bound and analyze them separately. Specifically, we prove that each such group of worlds is minimal. To this end, we first show an intermediate result, namely that a maximal representative x of $\mathcal{M}$ and its representative class $[\![x]\!]$ have the same bound (wrt. $\mathcal{M}$ and $[\![\mathcal{M}]\!]_k$, respectively). This result highlights the link between the notions of maximal representatives and representative classes, since a representative class $[\![x]\!]$ maintains the same bound as the maximal representative x.

Lemma 4.2 *Let $\mathcal{M} = ((W, R, V), w_d)$ be a pointed model with rooted k-contraction $\llbracket\mathcal{M}\rrbracket_k = ((W', R', V'), w'_d)$ and let $x \in W^{\max}$. Then $b(x) = b(\llbracket x\rrbracket)$.*

Proof. The proof is by induction on $h = b(x)$. For the base case, we consider $h = k$ (we do induction from $h = k$ down to $h = 0$). By definition, only the designated world of a model has bound k, so we immediately get $b(w_d) = b(\llbracket w_d\rrbracket) = k$, concluding the base case. Assume now by induction hypothesis (I.H.) that for all $x \in W^{\max}$ with $b(x) = h > 0$ we have $b(x) = b(\llbracket x\rrbracket)$. Let $y \in W^{\max}$ with $b(y) = h-1$. We need to show $b(\llbracket y\rrbracket) = h-1$. Since $b(y) = h-1$, there must exist an x with xR_iy and $b(x) = h$. We now prove $x \in W^{\max}$ by contradiction: Assuming $x \notin W^{\max}$, there exists $x' \in W^{\max}$ such that $x' \succ x$, *i.e.*, $b(x') > b(x)$ and $x' \leftrightarrows_{b(x)} x$. Since xR_iy, there exists y' such that $x'R_iy'$ and $y \leftrightarrows_{b(x)-1} y'$. As $b(x) = h$ and $b(y) = h-1$ we get $y \leftrightarrows_{b(y)} y'$. Since $x'R_iy'$ and $b(x') > b(x) = h$, Lemma 3.3 gives $b(y') > h-1$ and thus $b(y') > b(y)$. We now have $b(y') > b(y)$ and $y' \leftrightarrows_{b(y)} y$, which means $y' \succ y$, contradicting $y \in W^{\max}$. Thus, $x \in W^{\max}$. Since $x, y \in W^{\max}$, xR_iy and $b(x) = h > 0$, Lemma 3.11 gives $\llbracket x\rrbracket R'_i \llbracket y\rrbracket$. Since $b(\llbracket x\rrbracket) = h$ (by I.H.), Lemma 3.3 then gives $b(\llbracket y\rrbracket) \geq h-1$. We also have $b(\llbracket y\rrbracket) \leq h-1$, since if $b(\llbracket y\rrbracket) \geq h$, then I.H. would give $b(y) \geq h$, contradicting $b(y) = h-1$. Thus $b(\llbracket y\rrbracket) = h-1$, as required. □

Corollary 4.3 *If $x \neq y$ are worlds of $\llbracket\mathcal{M}\rrbracket_k$ and $b(x) = b(y) = h$ then $x \not\leftrightarrows_h y$.*

Proof. Let $\llbracket\mathcal{M}\rrbracket_k = (M, w_d)$ and $\mathcal{M} = (M', w'_d)$. By Definition 3.10, we have $x = \llbracket x'\rrbracket$ and $y = \llbracket y'\rrbracket$ for some worlds $x', y' \in W^{\max}$. By Lemma 4.2 we get $b(x) = b(x')$ and $b(y) = b(y')$, and hence $b(x') = b(y') = h$. Since $x \neq y$ and $b(x') = b(y') = h$, we get $x' \not\leftrightarrows_h y'$. From the proof of Theorem 4.1 we get $(M', x') \leftrightarrows_h (M, x)$ and $(M', y') \leftrightarrows_h (M, y)$, and hence $x \not\leftrightarrows_h y$. □

For a model $M = (W, R, V)$ and $h \geq 0$, let W_h denote the subset of worlds with bound h, *i.e.*, $W_h = \{w \in W \mid b(w) = h\}$ (given a $k \geq 0$).

Lemma 4.4 *Let $k \geq 0$, let $\mathcal{M} = ((W, V, R), w_d)$ be a rooted k-contraction, and $\mathcal{M}' = ((W', V', R'), w'_d)$ be a world-minimal pointed model k-bisimilar to $\mathcal{M}$. Then, for any $0 \leq h \leq k$, the relation $\leftrightarrows_h$ is a bijection between W_h and W'_h.*

Proof. Since $\mathcal{M} = ((W, R, V), w_d)$ is a rooted k-contraction, we have $\mathcal{M} = \llbracket\mathcal{M}''\rrbracket_k$ for some $\mathcal{M}'' = ((W'', R'', V''), w''_d)$. We first show that for each $x \in W_h$, there is a unique $x' \in W'_h$ such that $x \leftrightarrows_h x'$. Given $x \in W_h$, we have $b(x) = h$, which implies the existence of a path of length $k-h$ from w_d to x. Since $w_d \leftrightarrows_k w'_d$, by repeated application of [forth], we get a path of length $k-h$ from w'_d to a world x' with $x \leftrightarrows_h x'$. Since x' is reachable by a path of length $k-h$ from w'_d, we have $b(x') \geq k-(k-h) = h$. We now show that $b(x') \leq h$ by contradiction, which together with $b(x') \geq h$ gives $b(x') = h$, thus obtaining $x' \in W'_h$. Assume $b(x') > h$, *i.e.*, $d(x') < k-h$. Then, there is a path of length $< k-h$ from w'_d to x' and, since $w_d \leftrightarrows_k w'_d$, by repeated applications of [back], we get a path of length $< k-h$ from w_d to a world y with $y \leftrightarrows_h x'$. As above, since y is reachable by such a path, we have $b(y) > h$. From Definition 3.10, there exists $x'', y'' \in W^{\max}$ such that $x = \llbracket x''\rrbracket$ and $y = \llbracket y''\rrbracket$. Lemma 4.2 now gives $b(x'') = b(x) = h$ and $b(y'') = b(y) > h$. Since $x'', y'' \in W^{\max}$ and since both have bound $\geq h$, we have that $(x'', [\![x'']\!]), (y'', [\![y'']\!]) \in Z_h$, where Z_h is the binary relation defined in the proof of Theorem 4.1. The proof of Theorem 4.1 shows that $Z_0, \ldots, Z_h$ is an h-bisimulation, so $x'' \leftrightarrows_h [\![x'']\!]$ and $y'' \leftrightarrows_h [\![y'']\!]$. We then get $y'' \leftrightarrows_h [\![y'']\!] = y \leftrightarrows_h x' \leftrightarrows_h x = [\![x'']\!] \leftrightarrows_h x''$, showing that $y'' \leftrightarrows_h x''$. We now have $b(y'') > h = b(x'')$ and $y'' \leftrightarrows_h x''$, which implies $y'' \succ x''$, contradicting $x'' \in W^{\max}$. This gives the required proof by contradiction that $x' \in W'_h$. So far, we showed that for any $x \in W_h$ there exists a world $x' \in W'_h$ such that $x' \leftrightarrows_h x'$. Since $\mathcal{M}'$ is world minimal, there is no $y' \in W'_h$ such that $x' \leftrightarrows_h y'$. Therefore, such an x' is unique in W'_h, as required.

We now show that for each $x' \in W'_h$, there exists a unique $x \in W_h$ with $x \leftrightarrows_h x'$. Letting $x' \in W'_h$, we can first reason symmetrically as above to show that there exists a world $x \in W_h$ such that $x \leftrightarrows_h x'$ and $b(x) \geq h$ (using [back] instead of [forth]). Symmetrically to before, we now show that $b(x) \leq h$ by contradiction, from which we can conclude $x \in W_h$. Assume $b(x) > h$, *i.e.*, $d(x) < k - h$. Then there is a path of length $< k - h$ from w_d to x and, since $w_d \leftrightarrows_k w'_d$, by repeated applications of [forth], we get a path of length $< k - h$ from w'_d to a world y' with $x \leftrightarrows_h y'$. As above, since y' is reachable by such a path, we have $b(y') > h$. We now have $b(y') > h = b(x')$ and $y' \leftrightarrows_h x \leftrightarrows_h x'$. By Lemma 3.5, this implies that there exists a pointed model $\mathcal{N}$ with world set $W' \setminus \{x'\}$ such that $\mathcal{N} \leftrightarrows_k \mathcal{M}'$, contradicting the fact that $\mathcal{M}'$ is a world-minimal pointed model k-bisimilar to $\mathcal{M}$. This complete the proof by contradiction that $x \in W_h$. The only thing left to prove is uniqueness of x. Suppose x' was h-bisimilar to another world $y \in W_h$. Then we would have $x \neq y$, $b(x) = b(y) = h$, and $x \leftrightarrows_h y$, contradicting Corollary 4.3. □

Theorem 4.5 *Let $\mathcal{M}$ be a pointed model and let $k \geq 0$. Then $\lfloor\!\lfloor \mathcal{M} \rfloor\!\rfloor_k$ is a world-minimal model k-bisimilar to $\mathcal{M}$,* i.e.*, it has the least number of worlds among all models k-bisimilar to $\mathcal{M}$.*

Proof. Let $\mathcal{M}' = \lfloor\!\lfloor \mathcal{M} \rfloor\!\rfloor_k$, let $\mathcal{M}''$ be a world-minimal pointed model with $\mathcal{M}' \leftrightarrows_k \mathcal{M}''$ and let W' and W'' be the world sets of $\mathcal{M}'$ and $\mathcal{M}''$, respectively. We need to show that $|W'| = |W''|$. From Lemma 4.4, we immediately get that $|W'_h| = |W''_h|$ for all $0 \leq h \leq k$, and hence $|W'| = |W''|$, as required. □

5 Minimal Contractions

We have defined rooted k-contractions and shown them to be world minimal. However, Definition 3.10 does not guarantee that the resulting k-contraction is also edge minimal, as we now exemplify.

Example 5.1 Let $\mathcal{M} = ((W, R, V), w_d)$ be the pointed model in Figure 4 left and let $\lfloor\!\lfloor \mathcal{M} \rfloor\!\rfloor_3 = ((W', R', V'), w'_d)$ be its rooted 3-contraction (Figure 4 center). Recall from Example 3.12 that $W^{\max} = W$ and $b(w_3) = 1$. Since $w_3 R w_1$ and $w_3 R w_2$, Lemma 3.11 hence gives us $[\![w_3]\!] R' [\![w_1]\!]$ and $[\![w_3]\!] R' [\![w_2]\!]$. However, including only one of those edges in R' is sufficient to guarantee 3-bisimilarity to $\mathcal{M}$: $b(w_3) = 1$ and thus $[\![w_3]\!]$ only needs to preserve 1-bisimilarity to w_3.

When not all edges are required, we need to decide which to preserve. To this end, we introduce the notion of *least h-representative* of a world.

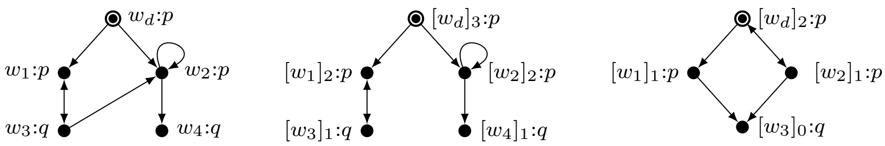

Fig. 5. Pointed model $\mathcal{M}$ (left) of Example 5.5, $\llbracket \mathcal{M} \rrbracket_3^<$ (center) and $\llbracket \mathcal{M} \rrbracket_2^<$ (right).

Definition 5.2 Let $<$ be a total order on W and let $0 \leq h \leq k$. The *least h-representative* of $w \in W$ is the world $\min_h(w) = \min_<\{v \in W^{\max} \mid v \leftrightarroweq_h w\}$.

The least h-representative of a world w is the minimal (wrt. $<$) maximal representative v such that $v \leftrightarroweq_h w$. We now present the revised definition of rooted k-contraction guaranteeing minimality among k-bisimilar models both in number of worlds and edges (and hence minimality in terms of overall size).

Definition 5.3 Let $\mathcal{M} = ((W, R, V), w_d)$, let $k \geq 0$ and let $<$ be a total order on W. The *rooted k-contraction* of $\mathcal{M}$ wrt. $<$ is the pointed model $\llbracket \mathcal{M} \rrbracket_k^< = ((W', R', V'), \llbracket w_d \rrbracket)$, where:

- $W' = \{\llbracket x \rrbracket \mid x \in W^{\max}\}$;
- $R'_i = \{(\llbracket x \rrbracket, \llbracket \min_{b(x)-1}(y) \rrbracket) \mid x \in W^{\max}, xR_iy \text{ and } b(x) > 0\}$;
- $V'(p) = \{\llbracket x \rrbracket \mid x \in W^{\max} \text{ and } x \in V(p)\}$.

Well-definedness of the definition (independence of choice of representatives) is guaranteed in the same way as for Definition 3.10.

Lemma 5.4 *Let $\mathcal{M} = ((W, R, V), w_d)$, $\llbracket \mathcal{M} \rrbracket_k^< = ((W', R', V'), w'_d)$ and $\llbracket \mathcal{M} \rrbracket_k = ((W'', R'', V''), w''_d)$. Then, $R'_i \subseteq R''_i$.*

Proof. Let $x'R'_iy'$. Then by Definition 5.3, there exists $x, y \in W$ such that $x' = \llbracket x \rrbracket$, $y' = \llbracket \min_{b(x)-1}(y) \rrbracket$, $x \in W^{\max}$, xR_iy and $b(x) > 0$. From Definition 5.2, we immediately get $\min_{b(x)-1}(y) \leftrightarroweq_{b(x)-1} y$ and $\min_{b(x)-1}(y) \in W^{\max}$. Since $x, \min_{b(x)-1}(y) \in W^{\max}$, xR_iy, $\min_{b(x)-1}(y) \leftrightarroweq_{b(x)-1} y$, and $b(x) > 0$, choosing $z = y$ in Definition 3.10, we get $(\llbracket x \rrbracket, \llbracket \min_{b(x)-1}(y) \rrbracket) \in R''_i$, *i.e.*, $x'R''_iy'$. □

Example 5.5 Let $\mathcal{M} = ((W, R, V), w_d)$ be the pointed model of Figure 2 (also shown in Figure 5 left). Let $<$ be a total order on W such that $w_d < w_1 < w_2 < w_3 < w_4$. The pointed models $\llbracket \mathcal{M} \rrbracket_3^<$ and $\llbracket \mathcal{M} \rrbracket_2^<$, also shown in Figure 5, will now be analyzed. First let $\llbracket \mathcal{M} \rrbracket_3^< = ((W', R', V'), w'_d)$. From Example 3.12, we have $W^{\max} = W$, $W' = \{[w_d]_3, [w_1]_2, [w_2]_2, [w_3]_1, [w_4]_1\}$ and $w'_d = [w_d]_3$. Now for the edges of $\llbracket \mathcal{M} \rrbracket_3^<$. Notice that for all xRy of $\mathcal{M}$ such that $b(x) \geq 2$, we have that $\min_{b(x)-1}(y) = y$. Thus, we have $[w_d]_3R'[w_1]_2$, $[w_d]_3R'[w_2]_2$, $[w_1]_2R'[w_3]_1$, $[w_2]_2R'[w_2]_2$ and $[w_2]_2R'[w_4]_1$. Finally, consider the edges w_3Rw_1 and w_3Rw_2 as discussed in Example 5.1. Since $\min_{b(w_3)-1}(w_1) = \min_{b(w_3)-1}(w_2) = w_1$, we have $[w_3]_1R'[w_1]_2$ (and not $[w_3]_1R'[w_2]_2$).

Now let $\llbracket \mathcal{M} \rrbracket_2^< = ((W', V', R'), w'_d)$. From Example 3.12, we have $W^{\max} = W$, $W' = \{[w_d]_2, [w_1]_1, [w_2]_1, [w_3]_0\}$ and $w'_d = [w_d]_2$. From $\min_{b(w_d)-1}(w_1) = w_1$ and $\min_{b(w_d)-1}(w_2) = w_2$ we get $[w_d]_2R'[w_1]_1$ and $[w_d]_2R'[w_2]_1$. From

$\min_{b(w_1)-1}(w_3) = \min_{b(w_2)-1}(w_4) = w_3$ we get $[w_1]_1 R' [w_3]_0$ and $[w_2]_1 R' [w_3]_0$. Finally, since $\min_{b(w_2)-1}(w_2) = w_d$, we have $[w_2]_1 R' [w_d]_2$.

Theorem 5.6 *Let $\mathcal{M}$ be a pointed model and let $k \geq 0$. Then $\mathcal{M} \leftrightarrows_k \llbracket \mathcal{M} \rrbracket_k^<$.*

Proof. Let $\mathcal{M} = ((W, R, V), w_d)$ and $\llbracket \mathcal{M} \rrbracket_k^< = ((W', R', V'), w'_d)$. For all $0 \leq h \leq k$, let $Z_h \subseteq W \times W'$ be as in the proof of Theorem 4.1:

$$Z_h = \{(x, \llbracket x' \rrbracket) \mid x' \in W^{\max}, x' \leftrightarrows_h x \text{ and } b(x) \geq h\}.$$

We now show that $Z_k, \ldots, Z_0$ is a k-bisimulation between $\mathcal{M}$ and $\llbracket \mathcal{M} \rrbracket_k^<$. From the proof of Theorem 4.1, we immediately get $Z_k \subseteq \cdots \subseteq Z_0$, $(w_d, \llbracket w_d \rrbracket) \in Z_k$ and [atom]. Moreover, from the same proof, we also get [back$_h$], since by Lemma 5.4 we have $R'_i \subseteq R''_i$, where $((W'', R'', V''), w''_d)$ is the k-contraction of $\mathcal{M}$ of Definition 3.10.

To show [forth$_h$], let $h < k$, $(x, x'') \in Z_{h+1}$, and $x R_i y$. Then $x'' = \llbracket x' \rrbracket$, where $x' \in W^{\max}$, $x' \leftrightarrows_{h+1} x$ and $b(x) \geq h+1$. We need to find $y'' \in W'$ such that $x'' R'_i y''$ and $(y, y'') \in Z_h$. Since $x R_i y$ and $x \leftrightarrows_{h+1} x'$, there exists $z \in W$ such that $x' R_i z$ and $y \leftrightarrows_h z$. Let $y' = \min_{b(x')-1}(z)$. Then $y' \in W^{\max}$ and $y' \leftrightarrows_{b(x')-1} z$. By Claim 1 of Theorem 4.1, we get $b(x') \geq h+1$, and thus $y' \leftrightarrows_h z \leftrightarrows_h y$. Since $x R_i y$ and $b(x) \geq h+1$, Lemma 3.3 gives $b(y) \geq h$. Let $y'' = \llbracket y' \rrbracket$. We now have $y' \in W^{\max}$, $y' \leftrightarrows_h y$ and $b(y) \geq h$, which by definition of Z_h means that $(y, y'') \in Z_h$. By Definition 5.3, from $x' \in W^{\max}$, $x' R_i z$, $y' = \min_{b(x')-1}(z)$ and $b(x') \geq h+1 > 0$, we get $x'' R'_i y''$, as required. □

Lemma 5.7 *Let $\mathcal{M} = ((W, R, V), w_d)$ be a pointed model with rooted k-contraction $\llbracket \mathcal{M} \rrbracket_k^< = ((W', R', V'), w'_d)$ and let $x \in W^{\max}$. Then $b(x) = b(\llbracket x \rrbracket)$.*

Proof. The proof mimics the proof of Lemma 4.2, except we have fewer edges in $\llbracket \mathcal{M} \rrbracket_k^<$ than in $\llbracket \mathcal{M} \rrbracket_k$, so we cannot rely on Lemma 3.11. The proof is again by induction on $h = b(x)$ (from $h = k$ down to $h = 0$), and the base case is identical to the proof of Lemma 4.2. Assume now by induction hypothesis (I.H.) that for all $x \in W^{\max}$ with $b(x) = h > 0$ we have $b(x) = b(\llbracket x \rrbracket)$. Let $y \in W^{\max}$ with $b(y) = h - 1$. We need to show $b(\llbracket y \rrbracket) = h - 1$. Since $b(y) = h - 1$, there must exist an x with $x R_i y$ and $b(x) = h$. From this it follows that $x \in W^{\max}$ exactly as in the proof of Lemma 4.2 (we are here reasoning about $\mathcal{M}$ only). Since $x \in W^{\max}$, $x R_i y$ and $b(x) = h > 0$, Definition 5.3 gives $\llbracket x \rrbracket R'_i \llbracket \min_{b(x)-1}(y) \rrbracket$, *i.e.*, $\llbracket x \rrbracket R'_i \llbracket \min_{b(y)}(y) \rrbracket$. Since $b(\llbracket x \rrbracket) = h$, Lemma 3.3 then gives $b(\llbracket \min_{b(y)}(y) \rrbracket) \geq h - 1$. Since $\min_{b(y)}(y) \leftrightarrows_{b(y)} y$, we get that $\min_{b(y)}(y) \succeq y$. Since $y \in W^{\max}$, we must then have $b(\min_{b(y)}(y)) \leq b(y)$, since otherwise we would have $\min_{b(y)}(y) \succ y$, contradicting the maximality of y. Since $\min_{b(y)}(y) \in W^{\max}$, we must also have $b(\min_{b(y)}(y)) \geq b(y)$, since otherwise we would have $y \succ \min_{b(y)}(y)$, contradicting the maximality of $\min_{b(y)}(y)$. We can thus conclude $b(\min_{b(y)}(y)) = b(y)$. We also have $b(\llbracket \min_{b(y)}(y) \rrbracket) \leq h - 1$, since if $b(\llbracket \min_{b(y)}(y) \rrbracket) \geq h$, then I.H. would give $b(\min_{b(y)}(y)) \geq h > h - 1 = b(y)$, contradicting what we just concluded. We can thus conclude $b(\llbracket \min_{b(y)}(y) \rrbracket) = h - 1$. Now note that since $\min_{b(y)}(y) \leftrightarrows_{b(y)} y$ and $b(\min_{b(y)}(y)) = b(y)$, we get $\llbracket \min_{b(y)}(y) \rrbracket = \llbracket y \rrbracket$, and hence $b(\llbracket y \rrbracket) = b(\llbracket \min_{b(y)}(y) \rrbracket) = h - 1$, as required. □

Corollary 5.8 *If $x \neq y$ are worlds of $\llbracket\mathcal{M}\rrbracket_k^<$ and $b(x) = b(y) = h$ then $x \not\leftrightarrows_h y$.*

Proof. The proof is identical to that of Corollary 4.3 by using Lemma 5.7 instead of Lemma 4.2. □

Lemma 5.9 *Let $k \geq 0$, let $\mathcal{M}$ be a rooted k-contraction wrt. $<$, $\mathcal{M}'$ be a world-minimal pointed model k-bisimilar to $\mathcal{M}$ and let W and W' be their world sets. Then, for any $0 \leq h \leq k$, the relation $\leftrightarrows_h$ is a bijection between W_h and W'_h.*

Proof. The proof is identical to that of Lemma 4.4 by using Definition 5.3, Lemma 5.7 and Theorem 5.6 instead of Definition 3.10, Lemma 4.2 and Theorem 4.1, respectively. □

Theorem 5.10 *Let $\mathcal{M}$ be a pointed model and $k \geq 0$. Then $\llbracket\mathcal{M}\rrbracket_k^<$ is a minimal pointed model k-bisimilar to $\mathcal{M}$ (i.e., it is both world and edge minimal).*

Proof. For any model $M = (W, R, V)$, let $(R_i)_h$ denote the set of i-edges outgoing from worlds in W_h, *i.e.*, $(R_i)_h = R_i \cap (W_h \times W)$. Let $\mathcal{M} = ((W, R, V), w_d)$, $\mathcal{M}' = \llbracket\mathcal{M}\rrbracket_k^< = ((W', R', V'), w'_d)$, and let $\mathcal{M}'' = ((W'', R'', V''), w''_d)$ be a minimal pointed model with $\mathcal{M}' \leftrightarrows_k \mathcal{M}''$. We can prove that $|W''| = |W'|$ as in Theorem 4.5 by using Lemma 5.9 and Corollary 5.8 instead of Lemma 4.4 and Corollary 4.3, respectively. We then only need to show that $|R'_i| \leq |R''_i|$ for all $i \in \mathcal{I}$. To achieve a contradiction, assume $|(R'_i)_h| > |(R''_i)_h|$ for some i and h. Then $(x', y') \in (R'_i)_h$ for some x', y'. This implies $x' \in W'_h$ and hence $b(x') = h$. By Lemma 5.9, there then exists $x'' \in W''_h$ such that $x' \leftrightarrows_h x''$. By [back] and [forth], this implies that each i-successor of x'' is $(h-1)$-bisimilar to an i-successor of x' and vice versa. Since $|(R'_i)_h| > |(R''_i)_h|$, this is only possible if there exist two distinct i-successors y'_1 and y'_2 of x' that are $(h-1)$-bisimilar. We can now reason as follows for $n = 1, 2$. Since $b(x') = h$ and $(x', y'_n) \in (R'_i)_h$, Lemma 3.3 gives us $b(y'_n) \geq h - 1$; and Definition 5.3 further gives us the existence of $x \in W^{\max}$ and $y_n \in W$ such that $x R_i y_n$, $x' = \llbracket x \rrbracket$ and $y'_n = \llbracket \min_{h-1}(y_n) \rrbracket$. Since, by definition, $\min_{h-1}(y_n) \in W^{\max}$, and since $b(y'_n) \geq h - 1$, Lemma 5.7 gives $b(\min_{h-1}(y_n)) \geq h - 1$. From $\min_{h-1}(y_n) \in W^{\max}$ and $b(\min_{h-1}(y_n)) \geq h - 1$, we then get $(\min_{h-1}(y_n), \llbracket \min_{h-1}(y_n) \rrbracket) \in Z_{h-1}$, where Z_{h-1} is the binary relation defined in the proof of Theorem 5.6. The proof of that theorem shows that $Z_0, \dots, Z_{h-1}$ is an $(h-1)$-bisimulation, so $\min_{h-1}(y_n) \leftrightarrows_{h-1} \llbracket \min_{h-1}(y_n) \rrbracket$. We now get $y_1 \leftrightarrows_{h-1} \min_{h-1}(y_1) \leftrightarrows_{h-1} \llbracket \min_{h-1}(y_1) \rrbracket = y'_1 \leftrightarrows_{h-1} y'_2 = \llbracket \min_{h-1}(y_2) \rrbracket \leftrightarrows_{h-1} \min_{h-1}(y_2) \leftrightarrows_{h-1} y_2$. This shows that $y_1 \leftrightarrows_{h-1} y_2$, and hence $\min_{h-1}(y_1) = \min_{h-2}(y_2)$ and thus $y'_1 = \llbracket \min_{h-1}(y_1) \rrbracket = \llbracket \min_{h-1}(y_2) \rrbracket = y'_2$, contradicting $y'_1 \neq y'_2$. □

6 Exponential Succinctness

In this section, we show that, for any $k \geq 0$, rooted k-contractions can be exponentially more succinct than standard k-contractions. This means that we can create models of arbitrary size for which the rooted k-contraction is exponentially smaller than the corresponding standard k-contraction.

A binary tree of height k has $2^{k+1} - 1$ nodes. We will build a model $\mathcal{M}_k = ((W_k, R_k, V_k), \varepsilon)$ on a binary tree of height k such that the standard

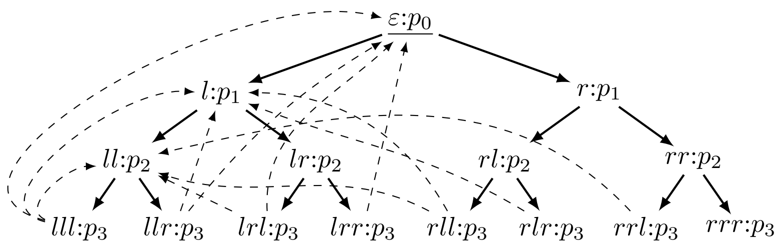

Fig. 6. The pointed model $\mathcal{M}_k = ((W_k, R_k, V_k), \varepsilon)$ with designated world ε, for $k = 3$. The solid and dashed edges are the accessibility edges of $\Box_s$ and $\Box_d$, respectively.

k-contraction of $\mathcal{M}_k$ still has all $2^{k+1} - 1$ nodes as worlds, but the rooted k-contraction is just a path in the tree, so a model with $k+1$ worlds. Hence the rooted k-contraction is exponentially smaller.

The nodes of a binary tree of height k can be represented as strings of length at most k over the alphabet $\{l, r\}$, where l and r represent the left and right child of a node, respectively. This is illustrated in Figure 6 for $k = 3$. The root is named ε (the empty string), and the left and right children of a node σ are σl and σr, respectively. More precisely, we let the set of worlds W_k of $\mathcal{M}_k$ be the set of strings of length at most k over the alphabet $\{l, r\}$, *i.e.*, $W_k = \{\sigma \in \{l, r\}^* \mid |\sigma| \leq k\}$. The tree edges (solid edges in Figure 6) are the edges from σ to σl (left child) and to σr (right child). We let $\mathcal{I} = \{s, d\}$, where the accessibility relation $(R_k)_s$ of the modality $\Box_s$ represents the solid edges. Hence we let $(R_k)_s = \{(\sigma, \sigma\alpha) \in W_k \times W_k \mid \alpha \in \{l, r\}\}$. The accessibility relation $(R_k)_d$ of the modality $\Box_d$ represents the dashed edges, described later.

As shown in Figure 6, the valuation function V_k of $\mathcal{M}_k$ is such that each world at depth n of the tree makes p_n true and all other propositions false. More precisely, the set of atomic propositions of $\mathcal{M}_k$ is $\mathcal{P} = \{p_0, \ldots, p_k\}$, and we let $V_k(p_n) = \{\sigma \in W_k \mid |\sigma| = n\}$. Suppose we decided to let $(R_k)_d = \emptyset$, *i.e.*, ignore the dashed edges. Then, the model is simply a binary tree where each world is labelled by an atomic proposition denoting the depth of the world. Clearly, any two worlds at the same depth are then bisimilar. Hence, the bisimulation contraction of $\mathcal{M}_k$ will simply be a chain model with $k+1$ worlds where the first world is labelled p_0, the second p_1, etc. We add the dashed edges to ensure that $\mathcal{M}_k$ cannot be contracted further when considering standard k-contractions, but where the rooted k-contraction is still the simple chain model.

We now describe the dashed edges of the model. The dashed edges are from the leaf nodes of the binary tree to nodes of the leftmost branch of the tree. The crucial property of these edges is that each leaf node has edges to a different subset of the nodes of the leftmost branch. Hence no two leaf nodes will satisfy the same formulas of modal depth 1. Our specific choice is to put an edge from a leaf node σ to the leftmost node at depth n iff the $(n+1)$st letter of σ is l, see Figure 6. More precisely, we let $(R_k)_d = \{(\alpha_1 \cdots \alpha_k, l^n) \in W_k \times W_k \mid \alpha_{n+1} = l\}$ (where l^n as usual denotes the string with n occurrences of l).

Lemma 6.1 *Let σ, τ be distinct worlds of $\mathcal{M}_k$ at depth n. Then $\sigma \leftrightarroweq_{k-n} \tau$ but not $\sigma \leftrightarroweq_{k-n+1} \tau$.*

Proof. Since σ and τ are worlds at depth n of the tree, they satisfy the same formulas up to modal depth $k - n$, since none of the dashed edges can be reached by formulas of modal depth $\leq k - n$ from worlds at depth n. From Proposition 2.6 we can then conclude that $\sigma \leftrightarroweq_{k-n} \tau$, as required.

Since $\sigma \neq \tau$, they differ in at least one of their positions, say position $m+1$ (*i.e.*, they differ in the $(m+1)$st letter). Suppose without loss of generality that σ has l in position $m+1$ and τ has r. Then by construction of $\mathcal{M}_k$ we have that the formula $\Diamond_s^{n-k} \Diamond_d\, p_m$ is true in σ but not in τ: There exists a solid path of length $n - k$ from σ to a leaf node that can see a p_m world via a dashed edge, but there is no such path from τ. Hence σ and τ are not $(n-k+1)$-bisimilar. $\Box$

We can now reason as follows. Since the standard k-contraction only identifies worlds that are k-bisimilar, it will never be able to identify worlds at the same depth of the tree since, according to the lemma, any such two distinct worlds σ and τ are not $(k - n + 1)$-bisimilar and hence not k-bisimilar. It will not be able to identify worlds at different levels either, as they have distinct valuations. Hence the standard k-contraction of $\mathcal{M}_k$ will contain all worlds of the original tree, so $2^{k+1} - 1$ worlds. Compare this to the rooted k-contraction. The worlds of the rooted k-contraction are of the form $[\![\sigma]\!]$ where $\sigma \in W_k^{\max}$. Note that if σ is a world of $\mathcal{M}_k$ at depth n, then $b(\sigma) = k - n$, and hence $[\![\sigma]\!] = [\sigma]_{k-n}$. By the lemma, we then get that any two worlds σ, τ at the same depth of $\mathcal{M}_k$ belong to the same representative class $[\![\sigma]\!]$ (since they are $(k-n)$-bisimilar). Hence the rooted k-contraction can have at most $k + 1$ worlds, one per level of the tree. We have hence proved the following succinctness result.

Theorem 6.2 (Exponential succinctness) *There exist models $\mathcal{M}_k$, $k \geq 0$, for which the rooted k-contraction has $\Theta(k)$ worlds whereas the standard k-contraction has $\Theta(2^k)$ worlds.*

7 Related and Future Work

Implementation and computational complexity The most commonly used technique to calculate bisimulation contractions is *partition refinement* [1,16]. In these algorithms, the worlds are initially partitioned into equivalence classes called *blocks*, for instance the blocks of the initial partition might consist of all worlds having the same valuation. The algorithms then refine the partition iteratively until some stopping condition is met, and the worlds of the contracted model are then (usually) the blocks of that final partition. Many partition refinement algorithms exist, the most famous probably being the one by Paige and Tarjan [16]. The Paige and Tarjan algorithm doesn't immediately lend itself to be adapted to do k-bisimulation contractions, as it is not partitioning blocks in a stratified manner. Other simpler partition refinement algorithms lend themselves more directly to be adapted to do k-bisimulation contractions. One such partition refinement algorithm for k-bisimulation contraction has been presented by Bolander and Lequen [9]. In that algorithm, the first h refinement steps lead to a partition consisting exactly of the h-bisimulation equivalence classes. Hence, by running the algorithm for k refinement steps, we compute all classes $[w]_h$ for all $w \in W$ and $h \leq k$. As shown in that paper, this can be done in polynomial time in the size of the model.

Having computed all classes $[w]_h$ for all $w \in W$ and $h \leq k$, we can easily compute the k-contraction (Definition 5.3). First we compute W', which can also be done in polynomial time. [5] We then compute the accessibility relations R'_i, also in polynomial time. [6] Computing V' is trivial, and we hence have a PTIME algorithm for computing rooted k-contractions. A version of this algorithm has been implemented and is further discussed in Bolander, Burigana and Montali [6] (under submission at the time of writing), as well as being applied to doing depth-bounded epistemic planning.

Rooted k-Contractions and Modal Structures A series of works by Fagin, Geanakoplos, Halpern and Vardi [10,11,12,13,14] explored an alternative semantics to modal logic, with a particular emphasis on epistemic logic. In these papers, they introduce and analyze objects called *modal structures*, which are essentially infinite sequences of functions $\mathbf{f} = \langle f_0, f_1, \dots \rangle$ where a function f_k describes the knowledge of a set of agents up to modal depth k. Among the many results they provide, the authors analyze the relation between pointed Kripke models and modal structures. Interestingly, it turns out that different k-bisimilar pointed models can be associated to the same subsequence $\mathbf{f}_{\leq k} = \langle f_0, \dots, f_k \rangle$ of functions, called the *k-prefix* of the modal structure $\mathbf{f}$. Hence, $\mathbf{f}_{\leq k}$ can be seen as representing a class of k-bisimilar pointed models.

In their works, the authors consider α-sequences of functions, where α is a generic ordinal. Moreover, they also consider an infinitary language of modal logic, where conjunctions of infinite formulas are allowed. Conversely, in this paper we worked under the assumption that k is a finite ordinal, and we only consider finite conjunctions. As a future avenue of research, we would like to investigate the possibility of closing the gap between rooted k-contractions and modal structures. We would like to generalize our notion of rooted k-contraction to account for generic ordinals and infinite models. This, in turn, would allow us to investigate the interplay between rooted k-contractions and modal structures.

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[5] A naive algorithm to do this runs as follows. First we compute $W^{\max}$. For each world x with $b(x) \geq 0$, we can iterate through all worlds y with $b(y) > b(x)$ and check whether $[y]_{b(x)} = [x]_{b(x)}$. If this is not true for any y, we put x in $W^{\max}$, since then no y strictly represents x, cf. Definition 3.6. From $W^{\max}$, we can then easily compute W', since it is simply the set of $[x]_{b(x)}$ for $x \in W^{\max}$. This algorithm runs in polynomial time since we already computed all the relevant equivalence classes $[w]_h$.

[6] A naive algorithm to compute R'_i is as follows. For each $x \in W^{\max}$ with $b(x) > 0$ we can iterate through all worlds $y \in W$ such that xR_iy. For each such y, we compute the least $(b(x)-1)$-representative z of y by taking the $<$-minimal world in the set $W^{\max} \cap [y]_{b(x)-1}$ (see Definition 5.2), and we add the pair $(\llbracket x \rrbracket, \llbracket z \rrbracket)$ to R'_i. Assuming we can compute whether $x < y$ in constant time, this also runs in polynomial time.

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