The non-central Nakagami distribution

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Abstract

The Nakagami distribution describe the square root of a random variable drawn from a Gamma distribution. Equivalently, the Gamma distribution can be seen as a one-dimensional Wishart distribution. In this note, we consider the distribution of the square root of a random variable drawn from a non-central one-dimensional Wishart distribution. We present the probability density function of this distribution along with closed-form expressions for its moments.

1 The Nakagami distribution

Let $z_d \sim \mathcal{N}(0, \sigma^2), d = 1, \ldots, D$ be iid samples from a zero-mean normal distribution. Then

$$x = \sum_{d=1}^{D} z_d^2 \in \mathbb{R}^+$$ (1.1)

follows a Gamma distribution. Equivalently, and more suitable for our purposes, $x$ also follows a one-dimensional Wishart distribution [1],

$$x \sim \mathcal{W}_1(D, \sigma^2)$$ (1.2)

$$p(x) = \mathcal{W}_1(x \mid D, \sigma^2) = \frac{x^{D-2}}{2^{D/2} \sigma^{2D}} \Gamma \left( \frac{D}{2} \right)^{-1} \exp \left( -\frac{x}{2\sigma^2} \right).$$ (1.3)

This distribution can be seen as describing the squared norm of a zero-mean normally distributed vector. The norm of this vector is then given by $y = \sqrt{x}$, which follows a Nakagami distribution [2],

$$y \sim \text{Nakagami} \left( \frac{D}{2}, D\sigma^2 \right)$$ (1.4)

$$p(y) = \text{Nakagami} \left( y \mid \frac{D}{2}, D\sigma^2 \right) = \frac{2}{\Gamma \left( \frac{D}{2} \right) \left( \sqrt{2} \sigma \right)^D} y^{D-1} \exp \left( -\frac{y^2}{2\sigma^2} \right).$$ (1.5)

This expression is easily derived by the change of value theorem; see next section.

The expectation and variance of a Nakagami distributed variable is

$$\mathbb{E} [y] = \frac{\Gamma \left( \frac{D+1}{2} \right)}{\Gamma \left( \frac{D}{2} \right)} \sqrt{2\sigma}$$ (1.6)

$$\text{var} [y] = \sigma^2 \left( D - 2 \left( \frac{\Gamma \left( \frac{(D+1)}{2} \right)}{\Gamma \left( \frac{D}{2} \right)} \right)^2 \right).$$ (1.7)

We will derive a more general version of these results in the next section.
2 The non-central Nakagami distribution

Now, let $z_d \sim N(\mu_d, \sigma^2), d = 1, \ldots, D$ be independent samples from normal distributions with variance $\sigma^2$. Then

$$x = \sum_{d=1}^{D} z_d^2 \in \mathbb{R}^+$$ \hspace{1cm} (2.1)

follows a one-dimensional non-central Wishart distribution [1],

$$x \sim \mathcal{W}_1(D, \sigma^2, \Omega)$$ \hspace{1cm} (2.2)

$$\Omega = \sum_{d=1}^{D} \mu_d^2 \sigma^2$$ \hspace{1cm} (2.3)

$$p(x) = \mathcal{W}_1(x \mid D, \sigma^2, \Omega)$$ \hspace{1cm} (2.4)

Here $\mathcal{W}_1$ is a generalized hypergeometric function; see Sec. 7.3 of Muirhead’s book [1]. This distribution can be seen as describing the squared norm of a normal distributed vector with isotropic variance. The norm of this vector is then given by $y = \sqrt{x}$, which we say follows a non-central Nakagami distribution. This distribution can be derived by change-of-variables to be

$$p(y) = \mathcal{W}_1(y^2 \mid D, \sigma^2, \Omega) \cdot 2y$$ \hspace{1cm} (2.5)

$$= \frac{y^{D-1}}{2^D \sigma^D \Gamma(D/2)} \Gamma_{q}(-k, D/2, -1/2\Omega) \exp \left( -\frac{y^2}{\sigma^2} \right) \exp \left( -\frac{\Omega}{2} \right).$$ \hspace{1cm} (2.6)

To compute the moments of $\sqrt{x}$ we recall the following result from Muirhead [1, Theorem 10.3.7]. Let $X \in \mathbb{R}^{D \times q}$ and $X \sim \mathcal{W}_q(D, \Sigma, \Omega)$ then

$$\mathbb{E} \left[ (\det(X))^k \right] = (\det(\Sigma))^{k2q} \frac{\Gamma(D/2 + k)}{\Gamma(D/2)} \Gamma_{q}(-k, D/2, -1/2\Omega),$$ \hspace{1cm} (2.7)

where $\Gamma_{q}$ is a generalized hypergeometric function. Since $y = \sqrt{x}$ is a positive scalar, its determinant is merely $y$, and we get

$$\mathbb{E} \left[ \sqrt{x}^{2k} \right] = \sigma^{2k/2} \frac{\Gamma(D/2 + k)}{\Gamma(D/2)} \Gamma_{q}(-k, D/2, -1/2\Omega).$$ \hspace{1cm} (2.8)

From this we see that the mean and the variance of $y = \sqrt{x}$ is

$$\mathbb{E} \left[ \sqrt{x} \right] = \sigma^{1/2} \frac{\Gamma((D+1)/2)}{\Gamma(D/2)} \Gamma_{q}(-1/2, D/2, -1/2\Omega)$$ \hspace{1cm} (2.9)

$$\mathbb{E} \left[ x \right] = \sigma^2 \frac{\Gamma((D+2)/2)}{\Gamma(D/2)} \Gamma_{q}(-1, D/2, -1/2\Omega)$$ \hspace{1cm} (2.10)

$$\text{var} \left[ \sqrt{x} \right] = \mathbb{E} \left[ x \right] - \mathbb{E} \left[ \sqrt{x} \right]^2$$ \hspace{1cm} (2.11)

$$= \frac{\sigma^2}{\Gamma(D/2)} \left( \Gamma \left( \frac{D + 2}{2} \right) \Gamma_{q}(-1, D/2, -1/2\Omega) \right)$$ \hspace{1cm} (2.12)

$$- \frac{\Gamma((D+1)/2)^2}{\Gamma(D/2)^2} \Gamma_{q}(-1/2, D/2, -1/2\Omega)^2.$$ \hspace{1cm} (2.13)

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References