Improved Extended Regular Expression Matching

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Extended Regular Expressions

- Extended regular expressions.
  - Patterns formed by characters from an alphabet Σ combined with operators:
    - union S|T
    - concatenation S·T
    - Kleene star S*
    - Intersection S∩T
    - Complement ¬S

  extended operators

- Regular languages.
  - L(α) = {α}
  - L(S|T) = L(S) ∪ L(T)
  - L(S·T) = L(S) · L(T)
  - $L(S^*) = \{\varepsilon\} \cup L(S) \cup L(S)^2 \cup L(S)^3 \cdots$
  - L(S∩T) = L(S) ∩ L(T)
  - L(¬S) = Σ* - L(S)

- Example.
  - Σ = {a, b}
  - R = (a|b)* ab (a|b)* ∩ (a|b)* ba (a|b)*
  - L(R) = {abba, baab, aabba, abaab, ... }

Extended Regular Expression Matching

- Regular Expression Matching.
  - Given an extended regular expression R and a string Q, decide if $Q \in L(R)$.

Extended Regular Expression Matching

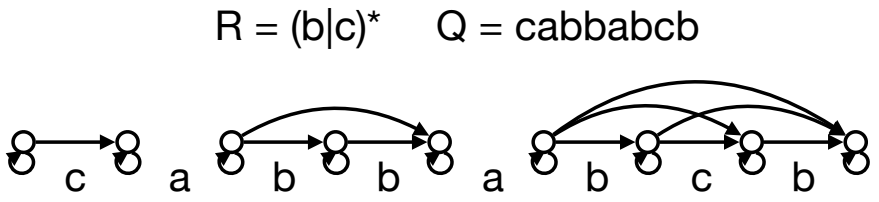
Time	Space	Reference
$O(n^3m)$	$O(n^2m)$	Hopcroft and Ullman, 1979
$O(n^\omega m)$	$O(n^2m)$	Hopcroft and Ullman, 1979 + matrix mul.
$O\left(\frac{n^3k + n^2m}{w} + n + m\right)$	$O\left(\frac{n^2k + nm}{w} + n + m\right)$	Yamamoto and Miyazaki, 2003
$O(n^3\lambda + n^2m)$	$O(n^2\lambda + nm)$	Yamamoto, 2001
$\Omega(n^{\omega-\epsilon}\mathrm{poly}(m))$		Nogami and Terauchi, 2026
$O\left(n^\omega k + n^2m\right)$	$O(n^2 + m)$	**This paper**
$\left(n^\omega k + \frac{n^2m}{\max(w/\log w, \log(n+m))} + m\right)$	$O(n^2 + m)$	**This paper**

m = |R|, n = |Q|, k = number of extended operators, λ = number of critical modules.
Word RAM with word length w ≥ log n.

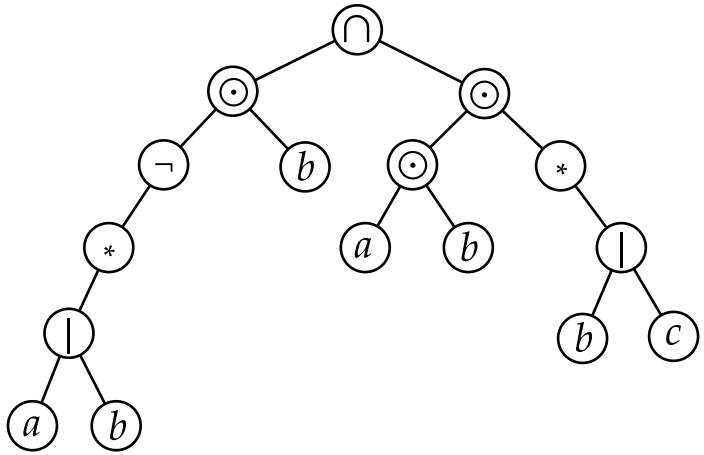
Dynamic Programming Algorithm

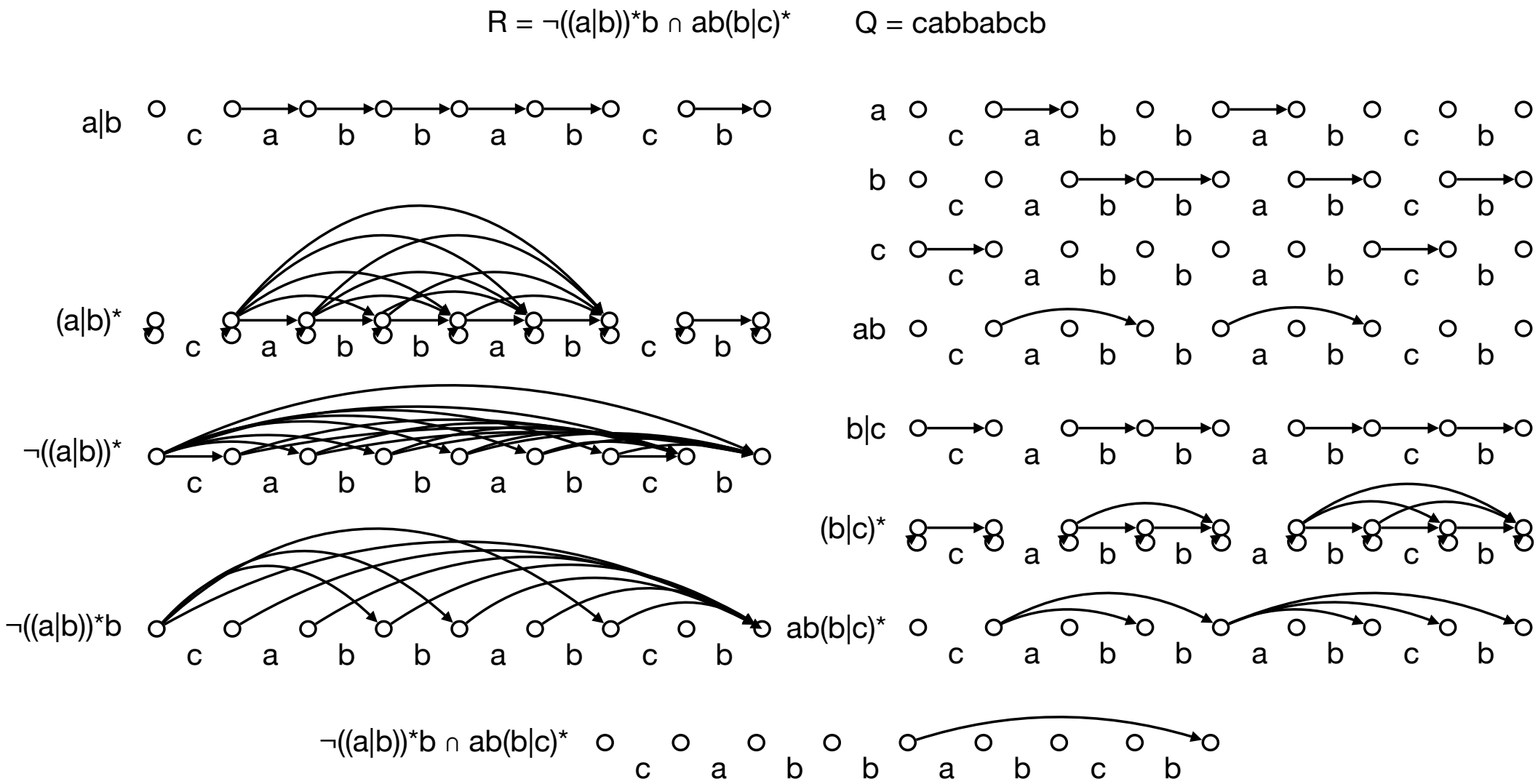
Dynamic Programming Algorithm

- Match graph of R and Q.
  - Encodes all substrings of Q that match R.
  - Directed graph on n+1 vertices 1, 2, ..., n+1.
  - Edge (i,j) ⇔ Q[i, j-1] matches R.
  - Represent as a (n+1) x (n+1) boolean matrix.



- Dynamic programming algorithm.
  - Compute match graphs for all subexpressions of R in a bottom-up traversal of the parse tree.
  - Combine match graphs using fast matrix operations.




R = ¬((a|b))*b ∩ ab(b|c)*
Q = cabbabcb
a|b
(a|b)*
¬((a|b))*
¬((a|b))*b
a
b
c
ab
b|c
(b|c)*
ab(b|c)*
¬((a|b))*b ∩ ab(b|c)*
c a b b a b c b

Dynamic Programming Algorithm

- Match graph operations.

Expression	Match graph matrix operation	Time
S|T	or	$O(n^2)$
S·T	multiplication	$O(n^\omega)$
¬S	complement	$O(n^2)$
S*	transitive closure	$O(n^\omega)$
S ∩ T	and	$O(n^2)$

- ⇒ $O(n^\omega m)$ time and $O(n^2 m)$ space algorithm.

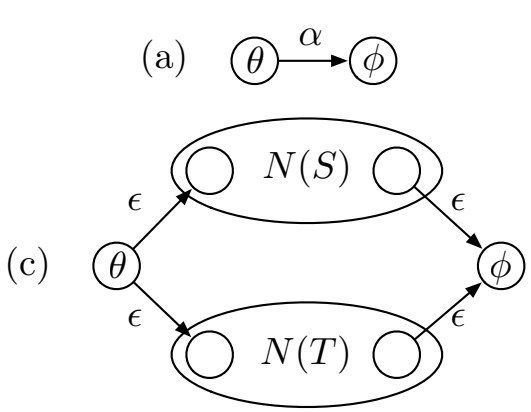
Improved Dynamic Programming Algorithm

Improved Dynamic Programming Algorithm

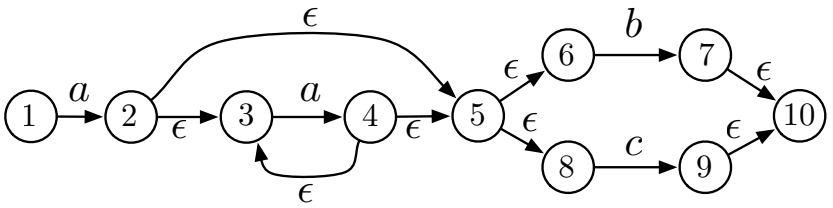
- Goal.
  - $O(n^{\omega}k + n^2m)$ time and $O(n^2m)$ space.

- Algorithmic idea.
  - Combine automata-based techniques for non-extended parts of the regular expression with a dynamic programming approach for the extended parts.

Non-extended Regular Expression Matching



R = a·(a*)·(b|c)

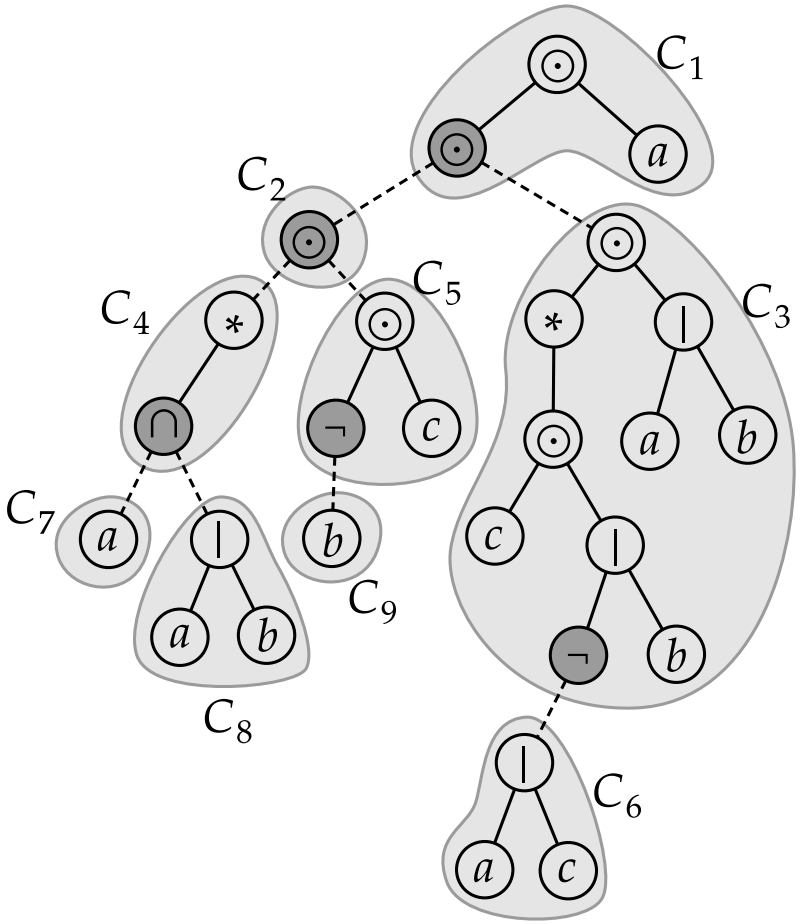


- Thompson's NFA [Thompson, 1968].
  - Recursively construct non-deterministic finite automaton (NFA) A from R.
  - A accepts the strings in L(R). Each path from the start state to the accept state corresponds to a string in L(R).
  - O(m) states and transitions.

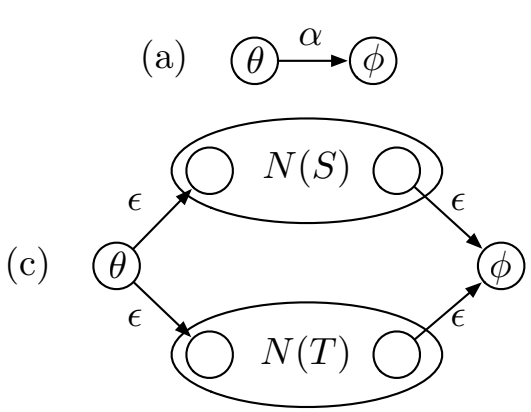
- Regular expression matching.
  - Simulate A on Q one character at a time.
  - O(m) per character ⇒ O(nm) time algorithm.

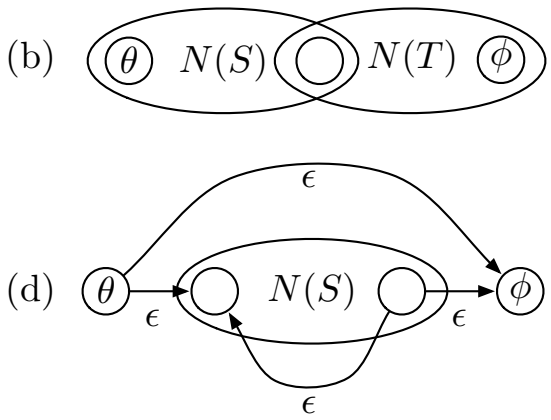
Clustering

- Mark nodes in the parse tree.
  - Nodes with an extended operator.
  - Lowest common ancestor of such nodes.
  - Cut edges to children of nodes.
- Clusters properties.
  - O(k) node-disjoint connected subtrees.
  - Leaf cluster: non-extended regular expression.
  - Internal cluster: non-extended regular expression + 1 marked node.
- Improved dynamic programming algorithm.
  - Dynamic programming on the cluster tree.

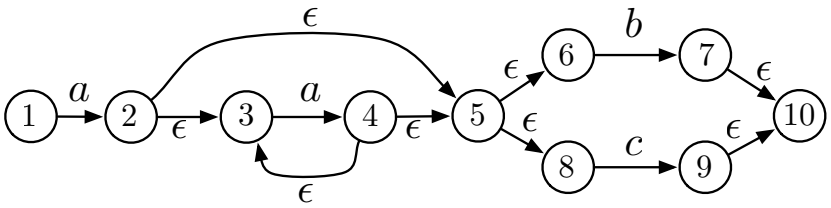


Leaf Cluster





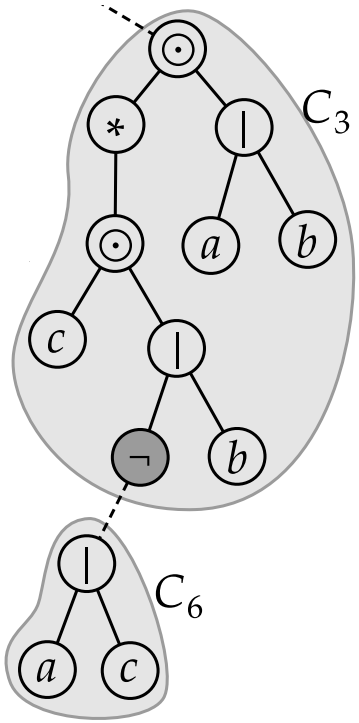
$R_C = a\cdot(a_*)\cdot(b|c)$

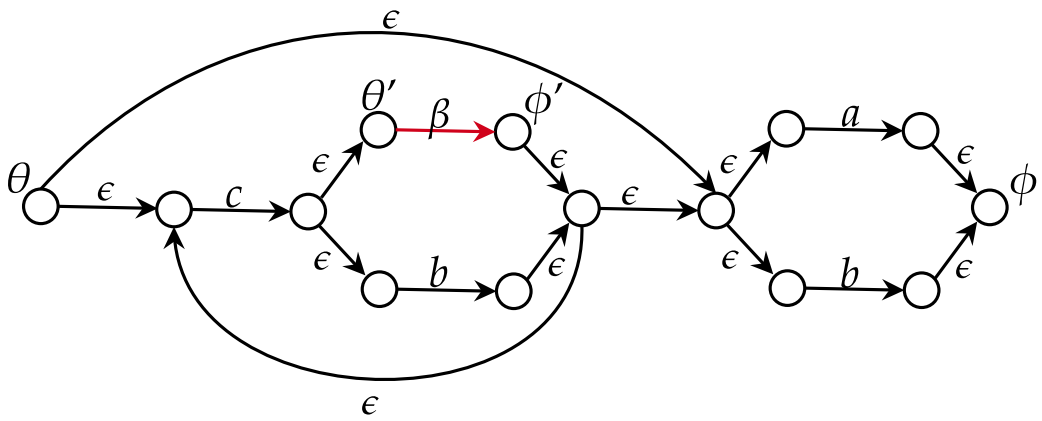


- Leaf cluster representation.
  - Construct Thompson's NFA A_C from the subexpression R_C.

- Match graph computation for cluster.
  - For each suffix Q[i,n]:
    - Traverse A on Q one character at a time and record whenever we reach the accept state.
  - $O(m_C)$ per char ⇒ $O(nm_C)$ time per suffix ⇒ $O(n^2m_C)$ time to compute match graph for C.

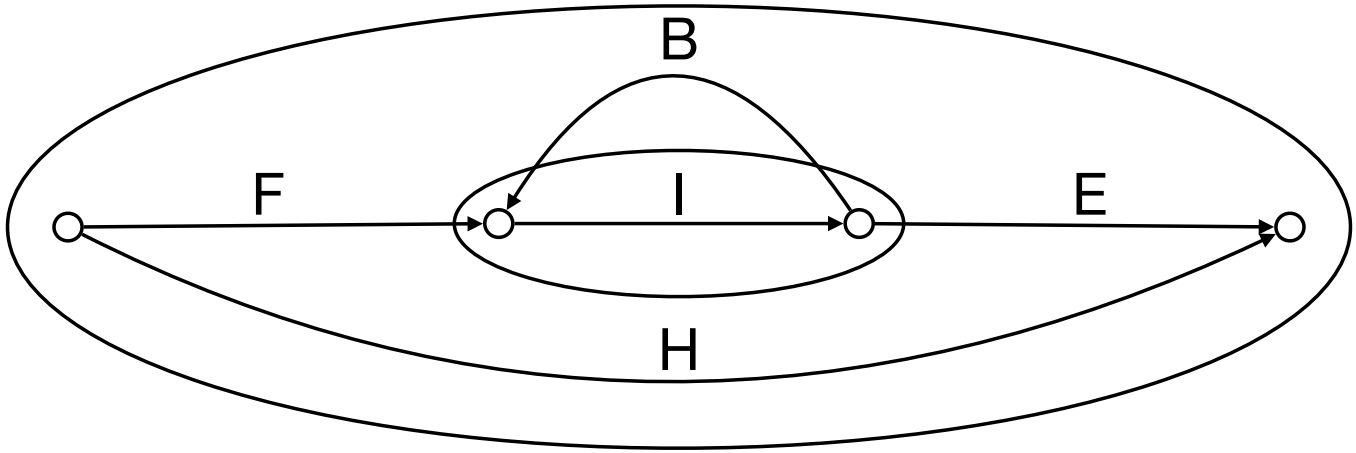
Internal Cluster





- Internal cluster representation.
  - Thompson's NFA with a special transition labeled $\beta \notin \Sigma$ that represents the subtree rooted at the marked node.
  - Match graph I that represents the substrings of Q that can replace β.

Match Graph Computation for Cluster

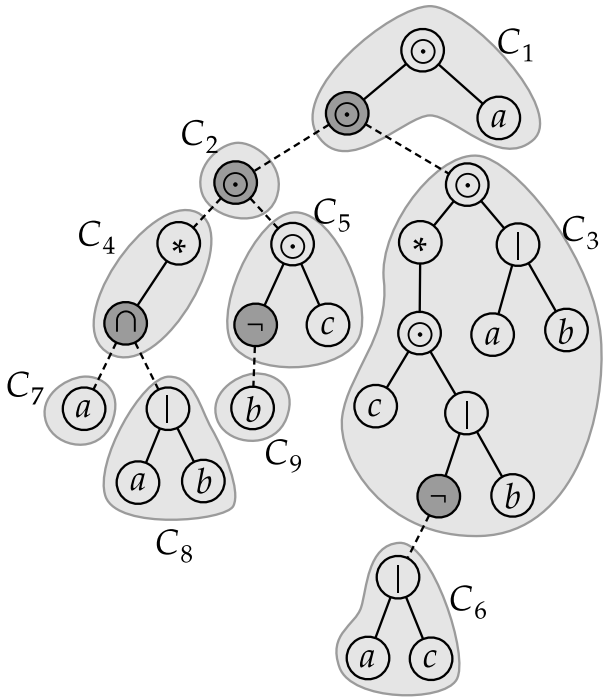


$$G = H \mid F \cdot I \cdot (B \cdot I)^* \cdot E$$

Improved Dynamic Programming Algorithm

- Match graph computation.
  - $O(n^2 m_C)$ time for leaf cluster.
  - $O(n^\omega + n^2 m_C)$ time for internal cluster.

- $\Rightarrow O(n^\omega k + n^2 m)$ time and $O(n^2 k + m)$ space algorithm.



Improved Dynamic Programming Algorithm

- Space reduction.
  - Improve space via heavy-path decomposition and traversal of cluster tree.
  - $\Rightarrow O(n^{\omega}k + n^2m)$ time and $O(n^2 + m)$ space algorithm.

- Black-box NFA simulation reduction.
  - Plug in any state-of-the-art algorithm for simulating TNFAs.
  - $\Rightarrow O\left(n^{\omega}k + \frac{n^2m}{\max(w/\log w, \log(n+m)}} + m\right)$ time and $O(n^2 + m)$ space algorithm.

- Intervals.
  - The interval operator $R^{[i,j]}$ represents between i and j copies of R (aka counting operator).
  - $L(R^{[i,j]}) = L(R)^i \cup L(R)^{i+1} \cup L(R)^{i+2} \cup \ldots\ L(R)^j$.
  - Include $R^{[i,j]}$ in the algorithm with logarithmic overhead per interval operator.

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