Fast Searching in Packed Strings

Philip Bille[*]

Technical University of Denmark
phbi@imm.dtu.dk

Abstract. Given strings P and Q the (exact) string matching problem is to find all positions of substrings in Q matching P. The classical Knuth-Morris-Pratt algorithm [SIAM J. Comput., 1977] solves the string matching problem in linear time which is optimal if we can only read one character at the time. However, most strings are stored in a computer in a packed representation with several characters in a single word, giving us the opportunity to read multiple characters simultaneously. In this paper we study the worst-case complexity of string matching on strings given in packed representation. Let $m \leq n$ be the lengths P and Q, respectively, and let σ denote the size of the alphabet. On a standard unit-cost word-RAM with logarithmic word size we present an algorithm using time

$$O\left(\frac{n}{\log_\sigma n} + m + \mathrm{occ}\right).$$

Here occ is the number of occurrences of P in Q. For $m = o(n)$ this improves the $O(n)$ bound of the Knuth-Morris-Pratt algorithm. Furthermore, if $m = O(n/\log_\sigma n)$ our algorithm is optimal since any algorithm must spend at least $\Omega(\frac{(n+m)\log\sigma}{\log n} + \mathrm{occ}) = \Omega(\frac{n}{\log_\sigma n} + \mathrm{occ})$ time to read the input and report all occurrences. The result is obtained by a novel automaton construction based on the Knuth-Morris-Pratt algorithm combined with a new compact representation of subautomata allowing an optimal tabulation-based simulation.

1 Introduction

Given strings P and Q of length m and n, respectively, the *(exact) string matching problem* is to report all positions of substrings in Q matching P. The string matching problem is perhaps the most basic problem in combinatorial pattern matching and also one of the most well-studied, see e.g. [5, 7, 12, 14] for classical textbook algorithms and the surveys in [11, 17]. The first worst-case $O(n)$ algorithm (we assume w.l.o.g. that $m \leq n$) is the classical Knuth-Morris-Pratt algorithm [14]. If we assume that we can read only one character at the time this bound is optimal since we need $\Omega(n)$ time to read the input. However, most strings are stored in a computer in a *packed representation* with several characters in a single word. For instance, DNA-sequences have an alphabet of size 4 and are therefore typically stored using 2 bit per character with 32 characters in a

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64-bit word. On packed strings we can read multiple characters in constant time and hence potentially do better that the $\Omega(n)$ lower bound for string matching. In this paper we study the worst-case complexity of packed string matching and present an algorithm to beat the $\Omega(n)$ lower bound for almost all combinations of m and n.

1.1 Setup and Results

We assume a standard unit-cost word RAM with word length $w = \Theta(\log n)$ and a standard instruction set including arithmetic operations, bitwise boolean operations, and shifts. The space complexity is the number of words used by the algorithm, not counting the input which is assumed to be read-only. All strings in this paper are over an alphabet Σ of size σ. The *packed representation* of a string S is obtained by storing $\Theta(\log n/\log\sigma)$ characters per word thus representing S in $O(|S|\log\sigma/\log n) = O(|S|/\log_\sigma n)$ words. If S is given in the packed representation we simply say that S is a *packed string*. The *packed string matching problem* is defined as above except that P and Q are packed strings. In the worst-case any algorithm for packed string matching must examine all of the words in the packed representation of the input strings. The algorithm must also report all occurrences of P in Q and therefore must spend at least $\Omega\left(\frac{n}{\log_\sigma n} + \text{occ}\right)$ time, where occ denotes the number of occurrences of P in Q. In this paper we present an algorithm with the following complexity.

Theorem 1. *For packed strings P and Q of length m and n, respectively, with characters from an alphabet of size σ, we can solve the packed string matching problem in time $O\left(\frac{n}{\log_\sigma n} + m + \text{occ}\right)$ and space $O(n^\varepsilon + m)$ for any constant ε, $0 < \varepsilon < 1$.*

For $m = o(n)$ this improves the $O(n)$ bound of the Knuth-Morris-Pratt algorithm. Furthermore, if $m = O(n/\log_\sigma n)$ our algorithm matches the lower bound and is therefore optimal. In practical situations m is typically much smaller than n and therefore this condition is almost always satisfied.

1.2 Techniques

The KMP-algorithm [14] may be viewed as simulating an automaton K according to the characters from Q in a left-to-right order. At each character in Q we use K to maintain the longest prefix of P matching the current suffix of Q. Improvements of automaton-based algorithms can often be obtained by partitioning the automaton into many small subautomata, tabulate relevant information for the subautomata, and use the tables to speed-up the simulation in each subautomaton [15, 16, 21]. This idea is also known as the "Four Russian Technique" after Arlazarov et al. [4].

However, if we attempt to apply this idea to the KMP-algorithm two major problems appear. First, the structure of the transitions in K does not in general allow us to partition K into subautomata such that a simulation does not change subautomata too often. Indeed, for any partition we might be forced to repeatedly change subautomaton after every group of $O(1)$ characters of Q and hence end up using $\Omega(n)$ time. Secondly, even if we could design a suitable partition of K into subautomata we have to compactly encode the transitions of the subautomata in order for the tabulation to be efficient. An explicit list of such transitions will not suffice to achieve the bound of Theorem 1. The main contribution of this paper are two new ideas to overcome these problems.

First, we present the *segment automaton*, C, derived from K. In C, the states of K are grouped into overlapping intervals of $r = \Theta(\log n/\log \sigma)$ states from K such that (almost all of) the states in K are duplicated in C. We show how to selectively "copy" the transitions from K to C such that the total number of transitions between subautomata never exceeds $O(n/r)$ in the simulation on Q. Secondly, we show how to exploit structural properties of the transitions to represent subautomata optimally. This allows us to tabulate paths of transitions for all subautomata of size $< r$ using $O(\sigma^r + m) = O(n^\varepsilon + m)$ space and preprocessing time for a suitably chosen r. The simulation can then be performed in time $O(n/r + \text{occ}) = O(n/\log_\sigma n + \text{occ})$ leading to Theorem 1.

This main contribution of this paper is theoretical, however, we believe that both the segment automaton and the compact representation of automata may prove very useful in practice if combined with ideas from other algorithms for packed matching.

1.3 Related Work

Exploiting packed string representations to speed-up string matching is not a new idea and is even mentioned in the early papers by Knuth et al. and Boyer and Moore [7, 14]. More recently, several packed string matching algorithms have appeared [6, 8, 9, 10, 13, 19]. However, none of these improve the worst-case $O(n)$ bound of the classical KMP-algorithm.

It is possible to extend the "super-alphabet" technique by Fredriksson [9, 10] to obtain a simple trade-off for packed string matching. The idea is to build an automaton that, similar to the KMP-automaton, maintains the longest prefix of P matching the current suffix of Q but allows Q to be processed in groups of r characters. Each state has σ^r outgoing transitions corresponding to all combinations of r characters. This algorithm uses $O(n/r + m\sigma^r)$ time and $O(m\sigma^r)$ space. Choosing $r = \varepsilon \log_\sigma n$ this is $O(n/\log_\sigma n + mn^\epsilon)$ time and $O(mn^\epsilon)$ space. Compared to Theorem 1 this is a factor $\Theta(m)$ worse in space and only improves the $O(n)$ time bound of the KMP-algorithm when $m = o(n^{1-\epsilon})$.

Packed string matching is closely related to the area of *compressed pattern matching* introduced by Amir and Benson [1, 2]. Here the goal is to search for a uncompressed pattern in a compressed text without decompressing it first. Furthermore, the search should be faster than the naive approach of decompressing the text first and then using the fastest algorithm for the uncompressed problem. In *fully compressed pattern matching* the pattern is also given in compressed form. Several algorithms for (fully) compressed string matching are known, see e.g., the survey by Rytter [18]. For instance, if Q is compressed with the

Ziv-Lempel-Welch scheme [20] into a string Z of length z, Amir et al. [3] showed how to find all occurrences of P in time $O(m^2 + z)$. The packed representation of a string may be viewed as the most basic way to compress a string. Hence, in this perspective we are studying the fully compressed string matching problem for packed strings. Note that our result is optimal if the pattern is not packed.

1.4 Outline

In Section 2 we first review the KMP-algorithm before presenting the segment automaton in Section 3. In Section 4 we show how to compactly represent and efficiently tabulate subautomata and finally, in Section 5 we present the complete algorithm.

2 The Knuth-Morris-Pratt Automaton and String Matching

In this section we briefly review KMP-algorithm [14], which will be the starting point of our new algorithm.

Let S be a string of length $|S|$ on an alphabet Σ. The character at position i in S is denoted $S[i]$ and the substring from position i to j is denoted by $S[i, j]$. The substrings $S[1, j]$ and $S[i, |S|]$ are the *prefixes* and *suffixes* of S, respectively.

The Knuth-Morris-Pratt automaton (KMP-automaton), denoted $K(P)$, for P consists of $m+1$ states identified by the integer $\{0, \dots, m\}$ each corresponding to a prefix of P. From state s to state $s+1$, $0 \leq s < m$ there is a *forward transition* labeled $P[s]$. We call the rightmost forward transition from $m-1$ to m the *accepting transition*. From state s, $0 < s \leq m$, there is a *failure transition* to a state denoted $\text{fail}(s)$ such that $P[1, \text{fail}(s)]$ is the longest prefix of P matching a proper suffix of $P[1, s]$. Fig. 1(a) depicts the KMP-automaton for the pattern $P = \text{ababca}$.

The failure transitions form a tree with root in state 0 and with the property that $\text{fail}(s) < s$ for any state s. Since the longest prefixes of $P[1, s]$ and $P[1, s+1]$ matching a suffix of P can increase by at most one character we have the following property of failure transitions.

Lemma 1. *Let P be a string of length m and $K(P)$ be the KMP-automaton for P. For any state* $1 < s < m$, $\text{fail}(s+1) \leq \text{fail}(s) + 1$.

We will exploit this property in Section 4.1 to compactly encode subautomata of the KMP-automaton. The KMP-automaton can be constructed in time $O(m)$ [14].

To find the occurrences of P in Q we read the characters of Q from left-to-right while traversing $K(P)$ to maintain the longest prefix of P matching a suffix of the current prefix of Q as follows. Initially, we set the state of $K(P)$ to 0. Suppose that we are in state s after reading the $k-1$ characters of Q, i.e., the longest prefix of P matching a suffix of $Q[1, k-1]$ is $P[1, s]$. We process the next character $\alpha = Q[k]$ as follows. If α matches the label of the forward transition

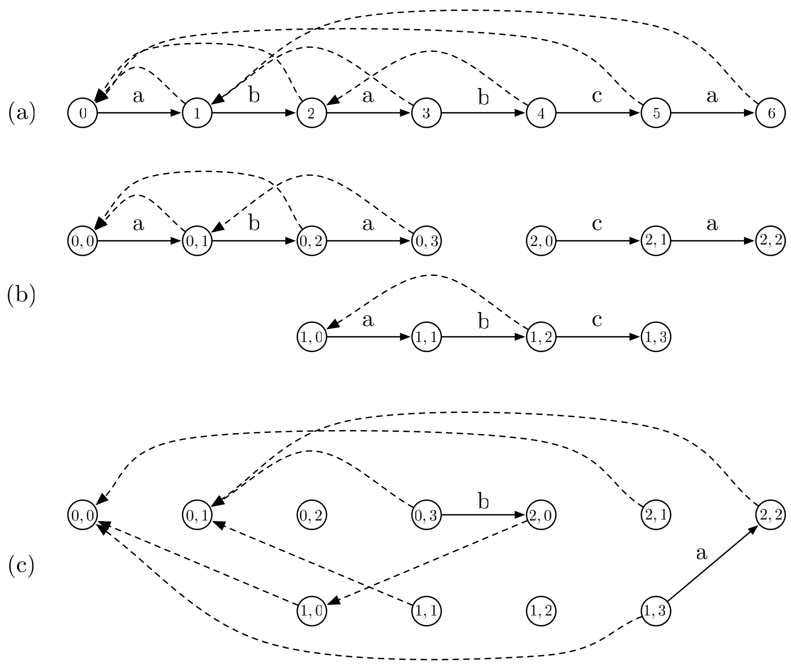

Fig. 1. (a) The Knuth-Morris-Pratt automaton $K(P)$ for the pattern $P =$ ababca. Solid lines are forward transitions and dashed lines are failure transitions. (b)-(c) The corresponding segment automaton $C(P, 4)$ for P consisting of 3 segments with 4, 4, and 3 states. The light transitions are shown in (b) and the heavy transition transitions in (c).

from s the next state is $s + 1$. Furthermore, if this transition is the accepting transition then $k + 1$ is the endpoint of a substring of Q matching P and we therefore report an occurrence. Otherwise, (α does not match the label of the forward transition from s to $s + 1$) we recursively follow failure transitions from s until we find a state s' whose forward transition is labeled α in which case the next state is $s' + 1$, or if no such state exist we set the next state to be 0. We define the *simulation* of $K(P)$ on Q to be sequence of transitions traversed by the algorithm.

Each time the simulation on Q follows a forward transition we continue to the next character and hence the total number of forward transitions is at most n. Each failure transition strictly decrease the current state number while forward transitions increase the state number by 1. Since we start in state 0 the number of failure transition is therefore at most the number of forward transitions. Hence, the total number of transitions is at most $2n$ and therefore the searching takes $O(n)$ time. In total the KMP-algorithm uses time $O(n + m) = O(n)$.

3 The Segment Automaton

In this section we introduce a simple automaton called the *segment automaton*. The segment automaton for P is equivalent to $K(P)$ in the sense that the simulation on Q at each step provides the longest prefix P matching a suffix of the current prefix of Q. The segment automaton allows to easily decompose $K(P)$ into subautomata of a given size r such that the simulation on Q passes through at most $O(n/r)$ subautomata.

Let $K = K(P)$ be the KMP-automaton for P. For a even integer parameter r, $1 < r \leq m + 1$ we define the segment automaton with parameter r, denoted $C(P, r)$, as follows. Define a *segment* S to be an interval $S = [l, r]$, $0 \leq l \leq r \leq m$, of states in $K(P)$ and let $|S| = r - l + 1$ denote the size of S. Divide the $m + 1$ states of K into a set of $z = \lceil (m + 1)/r \rceil$ overlapping segments, denoted $SS = \{S_0, \ldots, S_{z-1}\}$, where $S_i = [l_i, r_i]$ is defined by

$$l_i = i \cdot \frac{r}{2} \qquad r_i = \min(l_i + r - 1, m).$$

Thus, each segment in SS consists of r consecutive states from K, except the last segment, S_{z-1}, which may be smaller. Any state s in K appears in at most 2 segments and adjacent segments share $r/2$ states.

The segment automaton $C = C(P, r)$ is obtained by adding $|S|$ states for each segment $S \in SS$ and then selectively "copying" transitions from K to C. Specifically, the states of C is the set of pairs given by

$$\{(i, j) \mid 0 \leq i < z, 0 \leq j < |S_i|\}.$$

We view each state (i, j) C as the jth state of the ith segment, i.e., state (i, j) corresponds to the state $l_i + j$ in K. Hence, each state in K is represented by 1 or 2 states in C and each state in C uniquely corresponds to a state in K.

We copy transitions from K to C in the following way. Let $t = (s, s')$ be a transition in K. For each segment S_i such that $s \in [l_i, r_i]$ we have the following transitions in C:

– If $s' \in [l_i, r_i]$ there is a *light transition* from $(i, s - l_i)$ to $(i, s' - l_i)$.
– If $s' \notin [l_i, r_i]$ there is a *heavy transition* from $(i, s - l_i)$ to $(i', s' - l_{i'})$, where either $S_{i'}$ is the unique segment containing s' or if two segments contain s, then $S_{i'}$ is the segment such that $s' \in [l_{i'}, l_{i'} + r/2]$, i.e., the segment containing s' in the leftmost half.

If t is a forward transition with label $\alpha \in \Sigma$ it is also a forward transition in C with label α, if t is a failure transition it is also a failure transition in C, and if t the accepting transition it is also an accepting transition in C. The segment automaton with $r = 4$ corresponding to the KMP-automaton of Fig. 1(a) is shown in Fig. 1(b) and (c) showing the light and heavy transitions, respectively. From the correspondence between C and K we have that each accepting transition in a simulation of C on Q corresponds to an occurrence of P in Q. Hence, we can solve string matching by simulating C instead of K.

We will use the following key property of the C.

Lemma 2. *For a string P of length m and even integer parameter $1 < r \leq m+1$, the simulation of the segment automaton, $C(P,r)$, on a string Q of length n contains at most $O(n/r + \mathrm{occ})$ heavy and accepting transitions.*

Proof. Consider the sequence T of transitions in the simulation of $C = C(P,r)$ on Q. Let N_{accept} denote the number of accepting transitions, and let N_{hforward} and N_{hfail} denote the number of heavy forward and heavy failure transitions, respectively. Each accepting transition in T corresponds to an occurrence and therefore $N_{\mathrm{accept}} = \mathrm{occ}$. For a state (i,j) in C we will refer to i as the *segment number*. Since a forward transition in K increases the state number by 1 in K a heavy forward transition increases the segment number by 1 or 2 in C. A heavy failure transition strictly decrease the segment number. Hence, since we start the simulation in segment 0, we can have at most 2 heavy failure transitions for each heavy forward transition in T and therefore

$$N_{\mathrm{hfail}} \leq 2N_{\mathrm{hforward}}. \tag{1}$$

If $N_{\mathrm{hforward}} = 0$ the results trivially follows. Hence, suppose that $N_{\mathrm{hforward}} > 0$. Before the first heavy forward transition in T there must be at least $r-1$ light transitions in order to reach state $(0, r-1)$. Consider the subsequence of transitions t in T between an arbitrary heavy transition h and a forward heavy transition f. The heavy transition h cannot end in segment $z-1$ since there is no heavy forward transition from here. All other heavy transitions have an endpoint in the leftmost half of a segment and therefore at least $r/2$ light transitions are needed before a heavy forward transition can occur. Recall that the total number of transition in T is at most $2n$ and therefore the number of heavy forward transitions in T is bounded by

$$N_{\mathrm{hforward}} \leq 2n/(r/2) = 4n/r. \tag{2}$$

Combining the bound on N_{accept} with (1) and (2) we have that the total number of heavy and accepting transitions is

$$N_{\mathrm{hforward}} + N_{\mathrm{hfail}} + N_{\mathrm{accept}} \leq 3N_{\mathrm{hforward}} + \mathrm{occ} = O(n/r + \mathrm{occ}). \qquad \square$$

4 Representing Segments

4.1 A Compact Encoding

Let S be a segment with r states over an alphabet of size σ. We show how to compactly represent all light transitions in S using $O(r \log \sigma)$ bits. To represent forward transitions we simply store the labels of the $r-1$ light forward transitions in S using $(r-1)\log\sigma = O(r\log\sigma)$ bits. Next consider the failure transitions. A straightforward approach is to explicitly store for each state $s \in S$ a bit indicating if its failure pointer is light or heavy and, if it is light, a pointer to $\mathrm{fail}(s)$. Each pointer requires $\lceil \log r \rceil$ bits and hence the total cost for this representation is $O(r \log r)$ bits. We show how to improve this to $O(r)$ bits in the following.

First, we locally enumerate the states in S to $[0, r-1]$. Let $I = \{i_1, \ldots, i_\ell\}$, $0 \leq i_1 < \cdots < i_\ell < r$, be the set of states in S with a light failure transition and let $F = \{f_{i_1}, \ldots, f_{i_\ell}\}$ be the set of failure pointers for the states in I. We encode I as a bit string B_I of length r such that $B_I[j] = 1$ iff $j \in I$. This uses r bits. To represent F compactly we encode f_1 and the sequence of differences between consecutive elements $D = d_{i_2}, \ldots, d_{i_\ell}$, where $d_{i_j} = f_{i_j} - f_{i_{j-1}}$. We represent f_1 explicitly using $\lceil \log r \rceil$ bits. Our representation of D consists of 2 bit strings. The first string, denoted B_D, is the concatenation of the binary encoding of the numbers in D, i.e., $B_D = \text{bin}(d_{i_2}) \cdots \text{bin}(d_{i_\ell})$, where $\text{bin}(\cdot)$ denotes standard two's complement binary encoding (the differences may be negative) and $\cdot$ denotes concatenation. Each number d_j uses at most $1 + \log |d_j|$ bits and therefore the size of the B_D is at most

$$|B_D| \leq \sum_{j \in I'} (|\log(d_j)| + 1) < r + \sum_{j \in I'} |\log(d_j)|, \tag{3}$$

where $I' = I \backslash \{i_1\}$. The second bit string, denoted $B_{D'}$, represents the boundaries of the numbers in B_D, i.e., $B_{D'}[k] = 1$ iff k is the start of a new number in B_D. Thus, $|B_{D'}| = |B_D|$. Note that with f_1, B_D, and $B_{D'}$ we can uniquely decode F. The total size of the representation is $\lceil \log r \rceil + 2|B_D|$ bits.

To bound the size of the representation we show that $|B_D| = O(r)$ implying that the representation uses $\lceil \log r \rceil + 2 \cdot O(r) = O(r)$ bits as desired. We first bound the sum $\sum_{j \in I'} |d_j|$. Recall from Lemma 1 that the failure function increases by at most 1 between consecutive states in K. Hence, over the subsequence F of $< r$ of failure pointers in the range $[0, r-1]$ the total increase of the failure function can be at most r. Hence, $\sum_{j \in I'} d_j \leq r$. Furthermore, if $f_1 = x$, for some $x \in [0, r-1]$, the total decrease of F over a segment of r states is at most x plus the total increase and therefore $\sum_{j \in I'} d_j \geq -(x + r) \geq -2r$. Hence,

$$\sum_{j \in I'} |d_j| \leq 2r \tag{4}$$

Combining (3) and (4) we have that

$$\begin{aligned} |B_D| &< r + \sum_{j \in I'} \log |d_j| = r + \log \left(\prod_{j \in I'} |d_j| \right) \leq r + \log \left((2r/|I'|)^{|I'|} \right) \\ &< r + \log\left((2r/r)^r\right) = O(r). \end{aligned}$$

Thus, we have shown the following result.

Lemma 3. *All light forward and failure transitions of a segment of size r can be encoded using $O(r \log \sigma)$ bits.*

4.2 Simulating Light Transitions

Let $C = C(P, r)$ be the segment automaton, and consider the path p of states in the simulation on C from a state (i, j) on some string q. Then, the *longest light* *path* from (i, j) on q is defined as the longest prefix of p consisting entirely of light non-accepting transitions in segment i. For example, consider state $(1, 1)$ in segment 1 in Fig. 1. The longest light path on the string $q =$ bac is the path $p = (1, 1), (1, 2), (1, 0), (1, 1)$. The transition on c from state $(1, 1)$ is heavy and therefore not included in p.

We show how to quickly compute the length and endpoint of longest light paths. Let S^{enc} be the compact encoding of a segment S as described above including the label of the forward heavy transition from the rightmost state in S (if any) and a bit indicating whether or not the rightmost light transition is accepting or not. Furthermore, let j be a state in S, let q be a string, and define

$\textsc{Next}(S^{\text{enc}}, j, q)$: Return the pair (l, j'), where l and j' is the length and final state, respectively, of the longest light path in S from j matching a prefix of q.

We can efficiently tabulate $\textsc{Next}$ for arbitrary strings q of length r as follows. Let b be the total number of bits needed to represent the input to $\textsc{Next}$. The string q uses $r\lceil\log\sigma\rceil$ bits and by Lemma 3 S^{enc} uses $O(r\log\sigma + \log\sigma + 1) = O(r\log\sigma)$ bits. Furthermore, the state number j uses $\lceil\log r\rceil$ bits and hence $b = O(r\log\sigma + \log r) = O(r\log\sigma)$. Using a table T with 2^b entries we can store all results of $\textsc{Next}$. Each entry is computed using a standard simulation in $O(r)$ time and therefore we can construct T in $2^b \cdot O(r) = 2^{O(b)}$ time and space. Hence, if we have $t < 2^w$ space available for T we may set $r = \frac{1}{c} \cdot \frac{\log t}{\log\sigma}$, where $c > 0$ is an upper bound on the constant appearing in the $2^{O(b)}$ expression above. Hence, the total space and preprocessing time now becomes $2^{O(b)} = 2^{\frac{1}{c}\frac{c\log t}{\log\sigma}\log\sigma} = O(t)$.

With T precomputed and stored in memory we can now answer arbitrary $\textsc{Next}$ queries for arbitrary encoded segments and strings of length at most q in constant time.

5 The Algorithm

We now put the pieces from the previous sections together to obtain our main result of Theorem 1. Assume that we have $t < 2^w$ space available and choose $r = \Theta(\log t/\log\sigma)$ as above for the tabulation. We first preprocess P by computing the following information:

- The segment automaton $C(P, r)$ with parameter r and $z = \lceil m + 1/r\rceil$ segments $SS = \{S_0, \ldots, S_{z-1}\}$.
- The compact encoding S^{enc} for each segment $S \in SS$.
- The tabulated $\textsc{Next}$ function for segments with r states and input string of length r.

We compute the segment automaton and the compact encodings in $O(m)$ time and space. The tabulation for $\textsc{Next}$ uses $O(t)$ time and space and hence the preprocessing uses $O(t + m)$ time and space.

We find the occurrences of P in Q using the algorithm described below. The main idea is to simulate the segment automaton using the tabulated $\textsc{Next}$

function with the segment automaton. At each iteration of the algorithm we traverse light transitions until we either have processed r characters from Q or encounter a heavy or accepting transition. We then follow the next transition reporting an occurrence if the transition is accepting and repeat until we have read all of Q.

Algorithm S (*Packed String Search*). Let P be a string of preprocessed for parameter r as above. Given a string Q of length n this algorithm finds all occurrences of P in Q.

S1. [Initialize] Set $(i, j) \leftarrow (0, 0)$ and $k \leftarrow 1$.

S2. [Do light transitions] Compute $(l, j') \leftarrow \textsc{Next}(S_i^{\mathrm{enc}}, j, Q[k, \min(k + r, n)])$. At this point (i, j') is the state in the traversal of C on Q after reading the prefix $Q[1, k + l]$. All transitions on the string $Q[k, k + l]$ are light and non-accepting by the definition of $\textsc{Next}$.

S3. [Done?] If $k = n$ the algorithm terminates.

S4. [Do next transition] Compute (i^*, j^*) by following the transition from (i, j') on character $Q[k + l + 1]$. If the transition is a failure transition we set $k^* \leftarrow k+l$ and otherwise set $k^* \leftarrow k+l+1$. If this is the accepting transition report an occurrence ending at position k^*.

S5. [Repeat] Update $(i, j) \leftarrow (i^*, j^*)$ and $k \leftarrow k^*$ and repeat from step S2.

It is straightforward to verify that Algorithm S simulates $C(P, r)$ on Q and reports occurrences whenever we encounter an accepting transition. In each iteration we either read r character from Q and/or perform a heavy or accepting transition. We can process r characters from Q on light transitions at most $\lceil n/r \rceil$ and by Lemma 2 the total number of heavy and accepting transitions is $O(n/r + \mathrm{occ})$. Hence, the total number of iterations is $O(n/r + \mathrm{occ})$. Since each iteration takes constant time this also bounds the running time. Adding the preprocessing time and plugging in $r = \Theta(\log t / \log \sigma)$ the time becomes

$$O\left(\frac{n}{r} + t + m + \mathrm{occ}\right) = O\left(\frac{n}{\log_\sigma t} + t + m + \mathrm{occ}\right)$$

with space $O(t + m)$. Hence we have the following result.

Theorem 2. *Let P and Q be packed strings of length m and n, respectively. For a parameter $t < 2^w$ we can solve the packed string matching problem in time $O\left(\frac{n}{\log_\sigma t} + t + m + \mathrm{occ}\right)$ and space $O(t + m)$.*

Note that the tabulation is independent of P and we therefore only need to compute it once for multiple searches. If we plugin $t = n^\varepsilon$, for $0 < \varepsilon < 1$, we obtain an algorithm using time $O\left(\frac{n}{\log_\sigma (n^\varepsilon)} + n^\varepsilon + m + \mathrm{occ}\right) = O\left(\frac{n}{\log_\sigma n} + m + \mathrm{occ}\right)$ and space $O(n^\varepsilon + m)$ thereby showing Theorem 1.

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