Semi-Convergence and Relaxation Parameters for a Class of SIRT Algorithms

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Joint work with:

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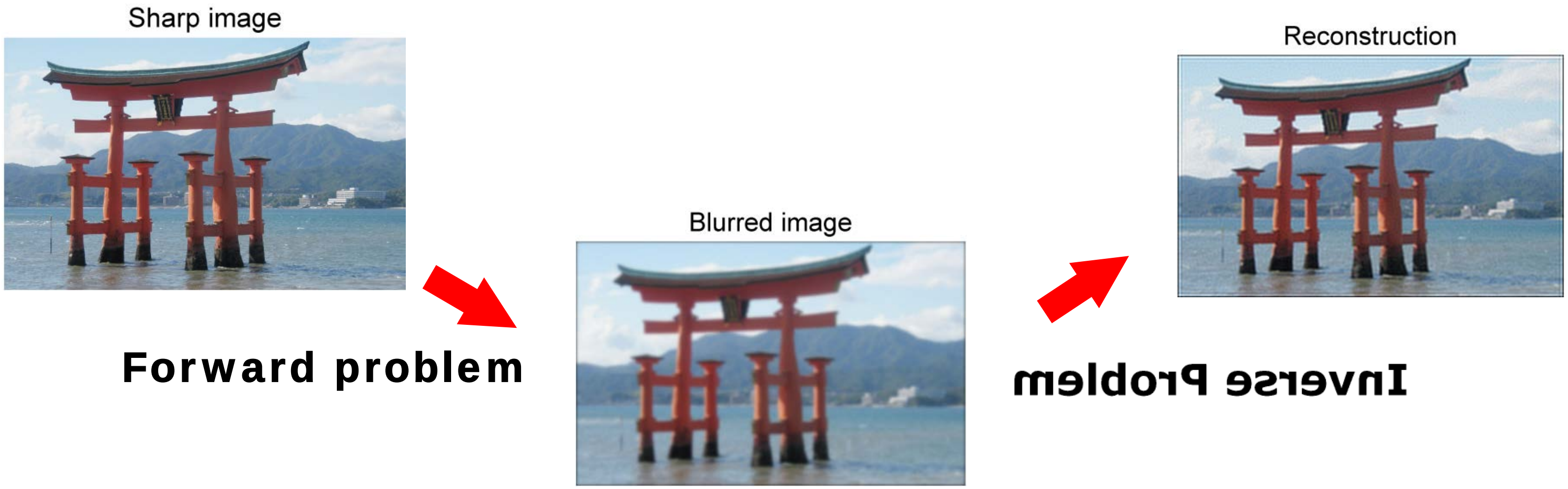
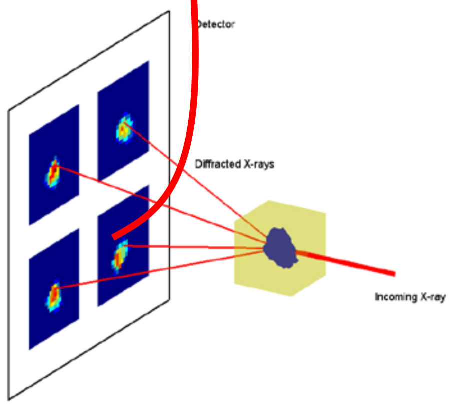
Overview of Talk

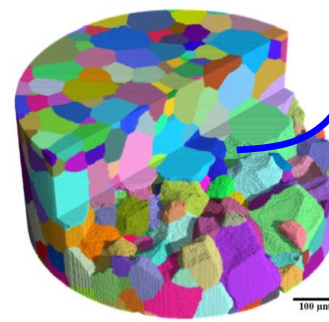


- Inverse problems and reconstruction algorithms
- Iterative SIRT methods and semi-convergence
- Strategies for the relaxation parameter (step size)
- A few results
- If time permits: AIR Tools – a new MATLAB® package

Inverse Problems

Goal: use measured **data** to ***compute*** "hidden" **information**.





Blurring process

Sharp image

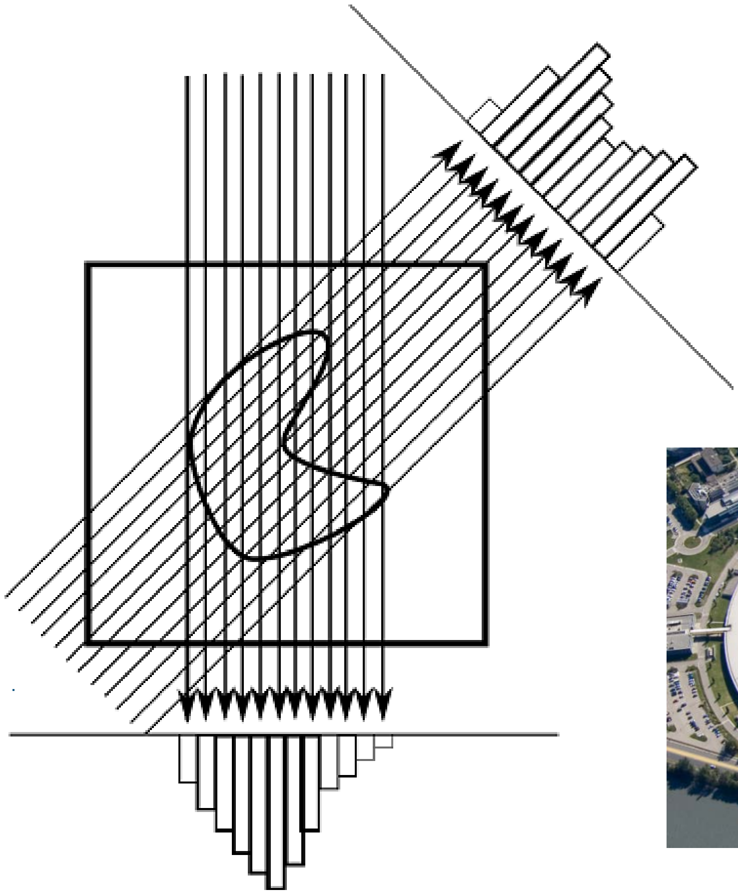
Our model: $A\,x = b$

Blurred image

Tomography = Our Main Application Area

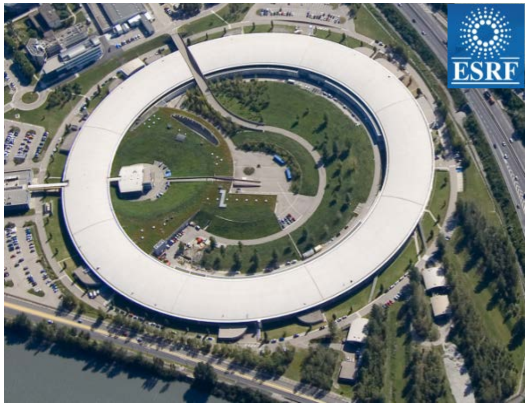
Image reconstruction

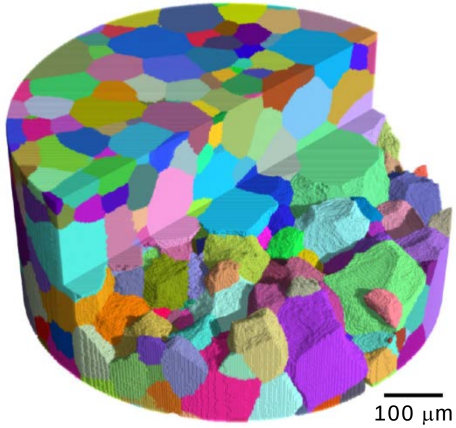
from projections



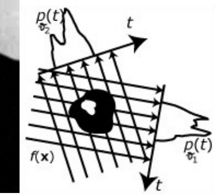
Medical scanning

Mapping of metal grains





The Origin of Tomography



Johan Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte Längs gewisser Mannings-faltigkeiten*, Berichte Sächsische Akadamie der Wissenschaften, Leipzig, Math.-Phys. Kl., 69, pp. 262-277, 1917.

<u>Main result:</u>

An object can be perfectly reconstructed from a full set of projections.

NOBELFÖRSAMLINGEN KAROLINSKA INSTITUTET

THE NOBEL ASSEMBLY AT THE KAROLINSKA INSTITUTE

11 October 1979

The Nobel Assembly of Karolinska Institutet has decided today to award the Nobel Prize in Physiology or Medicine for 1979 jointly to

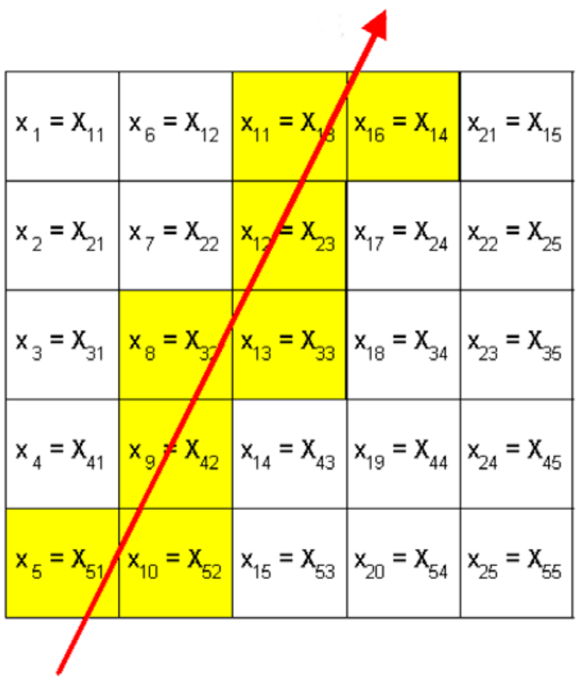
Allan M Cormack and Godfrey Newbold Hounsfield

for the "development of computer assisted tomography".

Setting Up the Algebraic Model

Damping of *i*-th X-ray through domain:

$$b_i = \int_{\mathrm{ray}_i} \chi(\mathbf{s})\, d\ell, \qquad \chi(\mathbf{s}) = \text{attenuation coef.}$$



Discretization leads to a large, sparse, ill-conditioned system:

$$A\,x = b \qquad b = b^* + e$$

Geometry

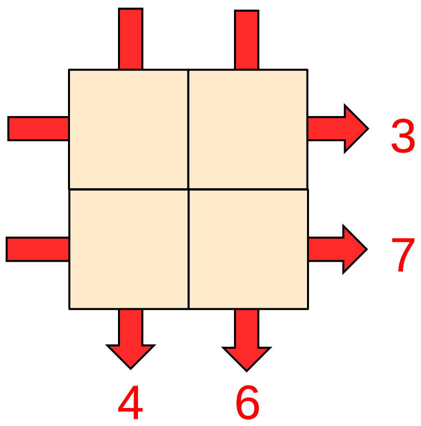
Image

Projections

Noise

$$b^* = A\,x^*$$

Analogy: the "Sudoku" Problem – 数独



$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 4 \\ 6 \end{pmatrix}$$

0	3
4	3

1	2
3	4

2	1
2	5

3	0
1	6

Infinitely many solutions ($c \in \mathbb{R}$):

$$\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + c \times \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Prior: solution is integer and non-negative

Some Large-Scale Inversion Algorithms

Transform-Based Methods

The forward problem is formulated as a certain transform $\rightarrow$
formulate a stable way to compute the inverse transform.

Example: the inverse Radon transform for tomography.

Krylov Subspace Methods

Use the forward model to produce a Krylov subspace $\rightarrow$
inversion amounts to projecting on this “signal subspace” & using prior information. Examples: CGLS, RRGMRES.

Algebraic Iterative Methods

This work

Formulate the forward problem as a discretized problem $\rightarrow$
inversion amounts to solving $A\,x = b$ using algebraic properties of A & using prior information.

Some Algebraic Iterative Methods

ART – Algebraic Reconstruction Techniques

- Kaczmarz's method + variants.
- *Sequential* row-action methods that update the solution using one row of *A* at a time.

SIRT – Simultaneous Iterative Reconstruction Techniques

- Landweber, Cimmino, CAV, DROP, SART, …
- These methods use all the rows of *A simultaneously* in one iteration (i.e., they are based on matrix multiplications).

Making the methods useful

- Relaxation parameter (step length) choice.
- Stopping rules.
- Nonnegativity constraints.

This work

SIRT Methods

Diagonally Relaxed Orthogonal Projection

Simultaneous Algebraic Reconstruction Technique

The general form:

$$x^{k+1} = x^k + \lambda_k T A^T M (b - A x^k), \qquad k = 0, 1, 2, \ldots$$

Some methods use the row norms $\|a^i\|_2$.

Landweber: $T = I$ and $M = I$.

Cimmino: $T = I$ and $M = D = \frac{1}{m} \mathrm{diag}\left(\frac{1}{\|a^i\|_2^2}\right)$.

CAV (component averaging method): $T = I$ and $M = D_S = \mathrm{diag}\left(\frac{1}{\|a^i\|_S^2}\right)$ with $S = \mathrm{diag}(\mathbf{nnz}(\text{column } j))$.

DROP: $T = S^{-1}$ and $M = mD$.

SART: $T = \mathrm{diag}(\text{row sums})^{-1}$ and $M = \mathrm{diag}(\text{column sums})^{-1}$.

Semi-Convergence of the SIRT Methods

During the first iterations, the iterates x^k capture the "important" information in the noisy right-hand side b.

- In this phase, the iterates x^k approach the exact solution.

At later stages, the iterates starts to capture undesired noise components.

- Now the iterates x^k diverge from the exact solution and they approach the undesired solution $A^{-1}b$ or $A^\dagger b$.

The iteration number k plays the role of the regularization parameter.

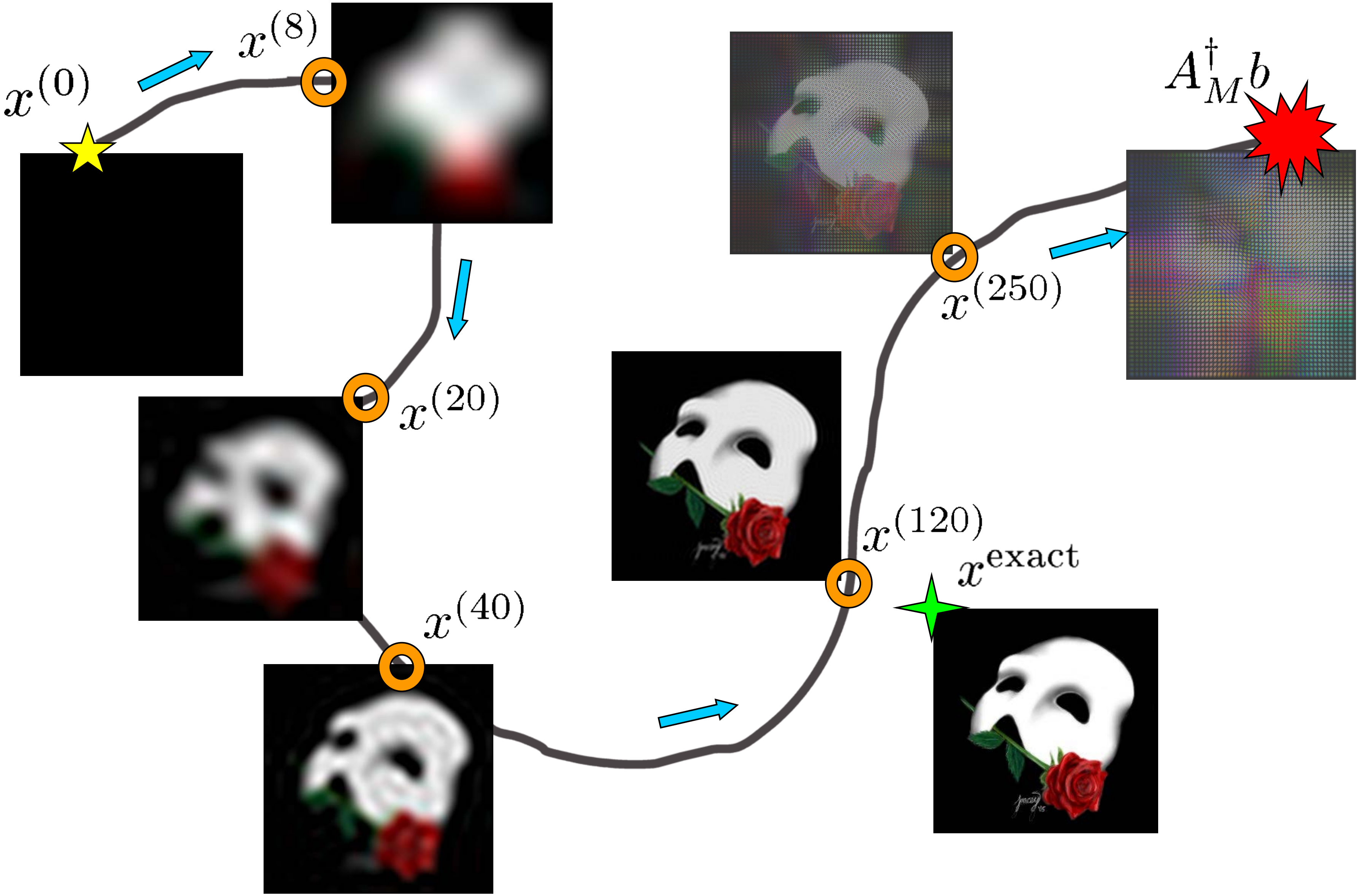
This behavior is called *semi-convergence.*

Studies of Semi-Convergence

Semi-convergence has been analyzed by several authors:

- F. Natterer, *The Mathematics of Computerized Tomography* (1986)
- A. van der Sluis & H. van der Vorst, *SIRT- and CG-type methods for the iterative solution of sparse linear least-squares problems* (1990)
- M. Bertero & P. Boccacci, *Inverse Problems in Imaging* (1998)
- M. Kilmer & G. W. Stewart, *Iterative Regularization And Minres* (1999)
- H. W. Engl, M. Hanke & A. Neubauer, *Regularization of Inverse Problems* (2000)

Illustration of Semi-Convergence

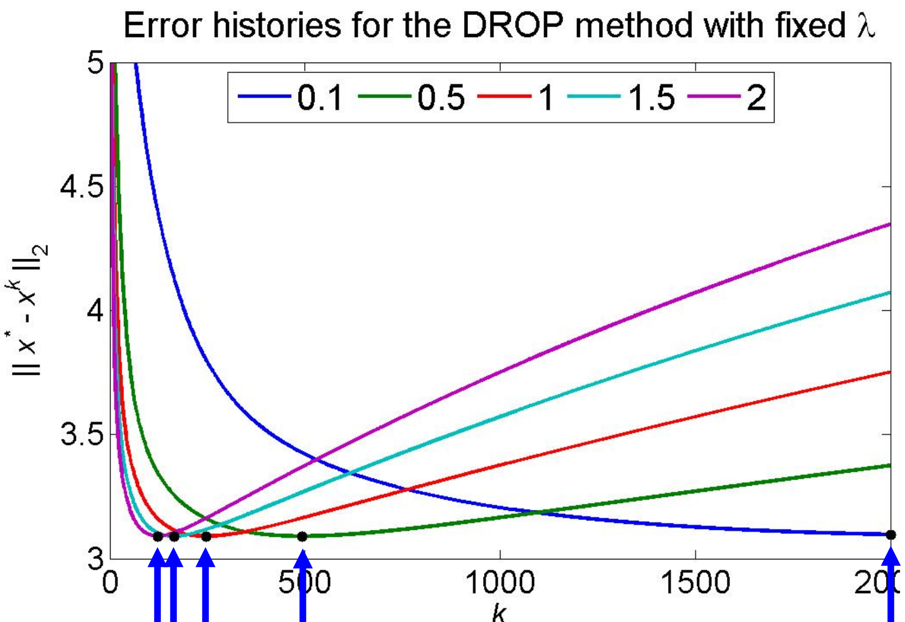


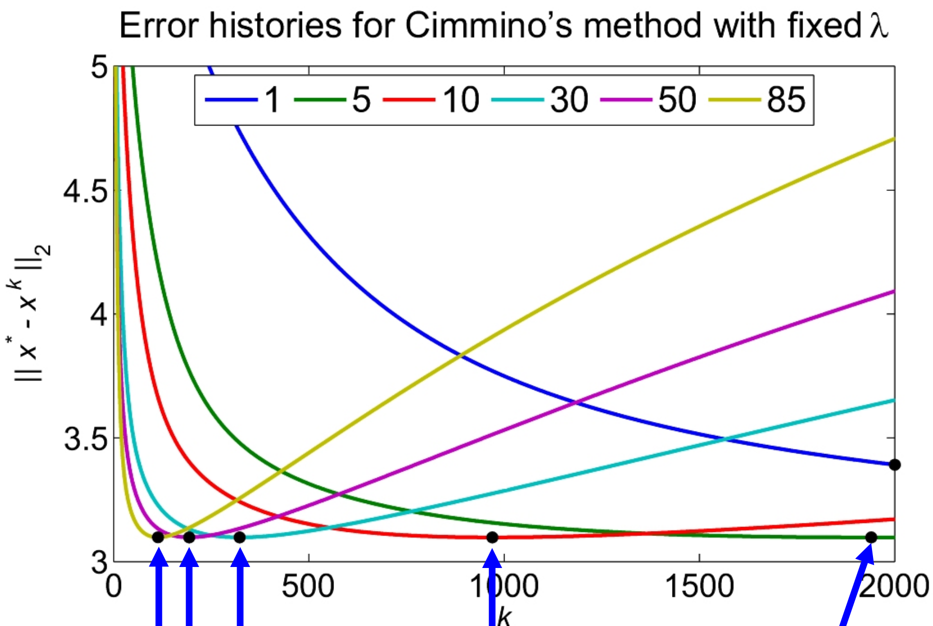
Another Look at Semi-Convergence

Notation: $b = A\,x^* + e$, $x^* =$ exact solution, $e =$ noise.

Initial iterations: the error $\|x^* - x^k\|_2$ decreases.

Later: the error increases as $x^k \to \mathrm{argmin}_x \|A\,x - b\|_M$.





The <u>minimum error</u> is *independent* of both λ and the method.

Analysis of Semi-Convergence

Consider the SIRT methods with $T = I$ and the SVD:

$$M^{1/2} A = U \, \Sigma \, V^T = \textstyle\sum_{i=1}^n u_i \, \sigma_i \, v_i^T \, .$$

Then x^k is a filtered SVD solution:

$$x^k = \textstyle\sum_{i=1}^n \varphi_i^{[k]} \, \frac{u_i^T (M^{\frac{1}{2}} b)}{\sigma_i} \, v_i \, , \qquad \varphi_i^{[k]} = 1 - \left(1 - \lambda \, \sigma_i^2\right)^k \, .$$

Recall that we solve *noisy* systems $A \, x = b$ with $b = A \, x^* + e$.

The ith component of the error, in the SVD basis, is

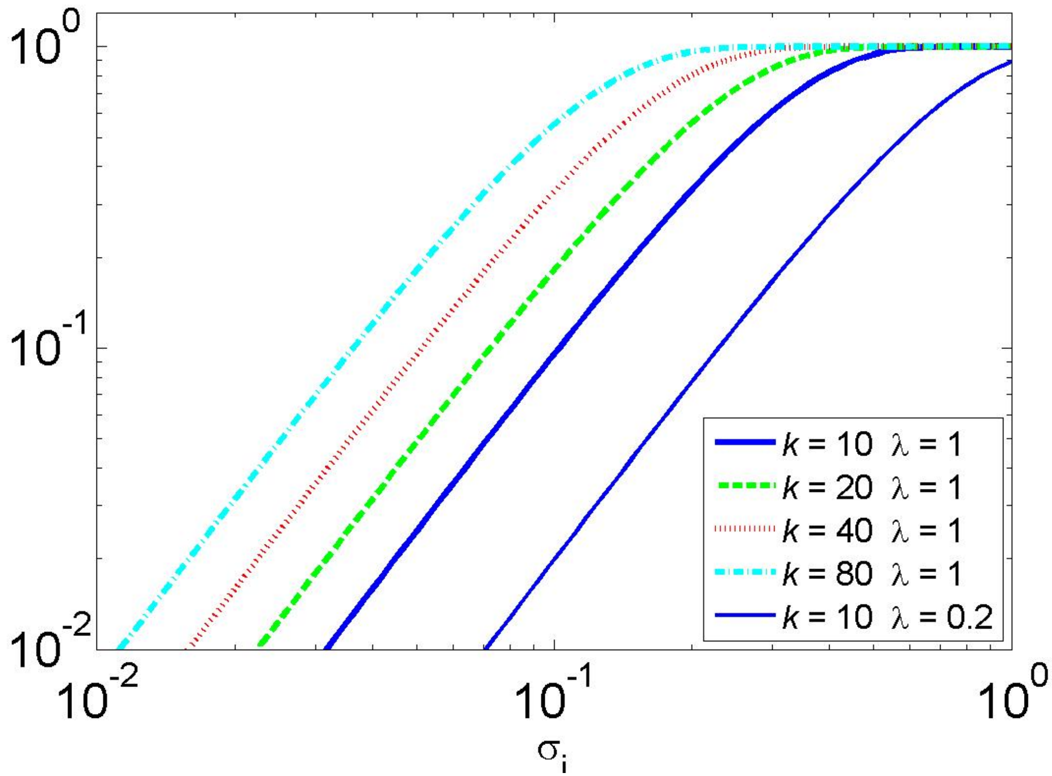
$$v_i^T (x^* - x^k) = (1 - \varphi_i^{[k]}) \, v_i^T x^* - \varphi_i^{[k]} \, \frac{u_i^T (M^{\frac{1}{2}} e)}{\sigma_i} \, .$$

RE: regularization error

NE: noise error

The Behavior of the Filter Factors

Filter factors $\varphi_i^{[k]} = 1 - \left(1 - \lambda\,\sigma_i^2\right)^k$



The filter factors *dampen* the "inverted noise" $u_i^T(M^{\frac{1}{2}}e)/\sigma_i$.

$\lambda\,\sigma_i^2 \ll 1 \Rightarrow \varphi_i^{[k]} \approx k\,\lambda\,\sigma_i^2 \Rightarrow k$ and λ play the same role.

More About the Noise Error

$$\mathsf{NE}_i = \Psi^{[k]}(\sigma_i, \lambda)\, u_i^T (M^{\frac{1}{2}} e)$$

$$\Psi^{[k]}(\sigma, \lambda) = \frac{1 - (1 - \lambda\, \sigma^2)^k}{\sigma}$$

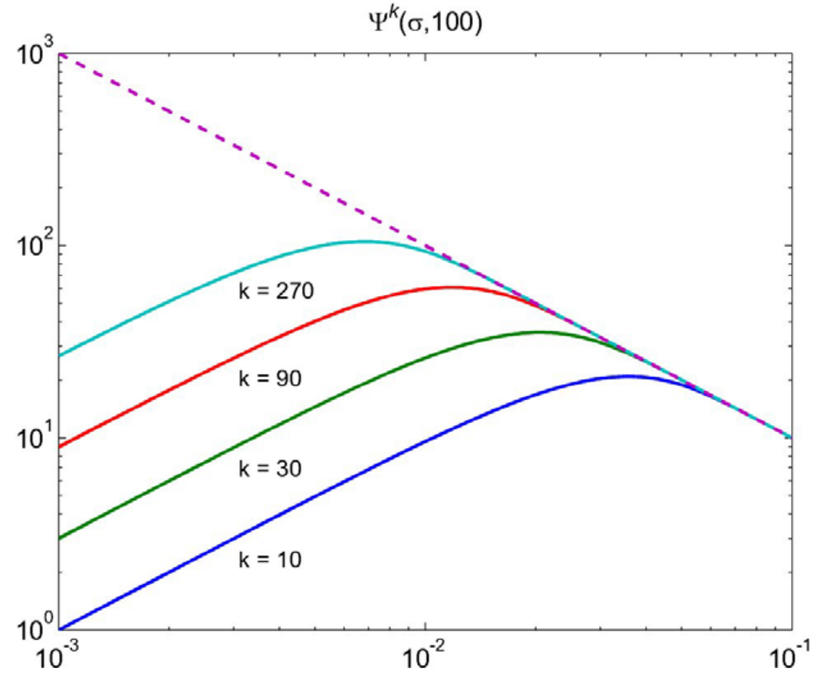
Fix σ and λ: $\Psi^{[k]} \nearrow$ with k.

Fix λ:
max of $\Psi^{[k]}$ is attained for

$$\sigma = \sigma_k^* = \sqrt{\tfrac{1-\zeta_k}{\lambda}}$$

where ζ_k is the unique root in $(0, 1)$ of

$$g_{k-1}(y) = (2k-1)y^{k-1} - (y^{k-2} + \ldots + y + 1) = 0.$$

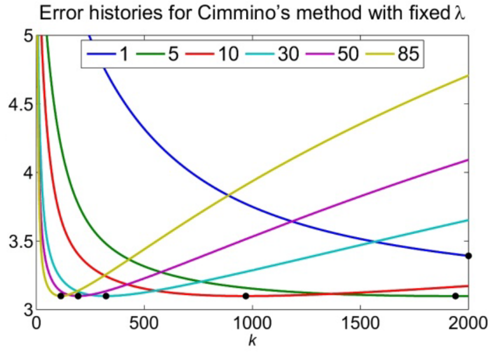


Choosing the SIRT Relaxation Parameter

$$x^{k+1} = x^k + \lambda_k T A^T M (b - A\, x^k), \qquad k = 0, 1, 2, \ldots$$

Goal: fast semi-convergence to the minimum error.

Training. Using a noisy test problem, find the *fixed* $\lambda_k = \lambda$ that gives fastest semi-convergence to the minimum error.



Line search (Dos Santos, Appleby & Smolarski, Dax). Minimize the error $\|x^k - x^*\|_2$ in each iteration – must assume that $A\,x = b$ is consistent. When $T = I$ we get:

$$\lambda_k = (r^k)^T M r^k \,/\, \|A^T M r^k\|_2^2, \qquad r^k = b - A\,x^k.$$

Our New Strategies: Limit the Noise Error

Assume we used a *fixed* λ in steps $1, \ldots, k-1$; then

$$\begin{aligned} \|\mathsf{NE}\|_2 &\leq \max_i \Psi^{[k]}(\sigma_i, \lambda)\, \|M^{\frac{1}{2}} e\|_2 \\ &\leq \Psi^{[k]}(\sigma_k^*, \lambda)\, \|M^{\frac{1}{2}} e\|_2 = \sqrt{\lambda}\, \frac{1-\zeta_k^k}{\sqrt{1-\zeta_k}}\, \|M^{\frac{1}{2}} e\|_2 . \end{aligned}$$

<u>Strategy Ψ_1:</u> choose $\lambda_0 = \lambda_1 = \sqrt{2}/\sigma_1^2$ and

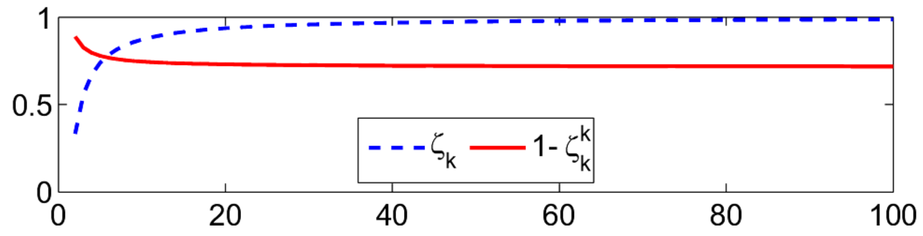
$$\lambda_k = \frac{2}{\sigma_1^2}\,(1-\zeta_k), \qquad k = 2, 3, \ldots$$

<u>Strategy Ψ_2:</u> choose $\lambda_0 = \lambda_1 = \sqrt{2}/\sigma_1^2$ and

$$\lambda_k = \frac{2}{\sigma_1^2}\, \frac{1-\zeta_k}{(1-\zeta_k^k)^2}, \qquad k = 2, 3, \ldots$$

Our New Strategies: What we Achieve

For both strategies we obtain relaxation parameters $\lambda_k > 0$ that lead to a *diminishing step-size strategy* with $\lambda_k \to 0$ such that $\sum_k \lambda_k = \infty$.



As a result:

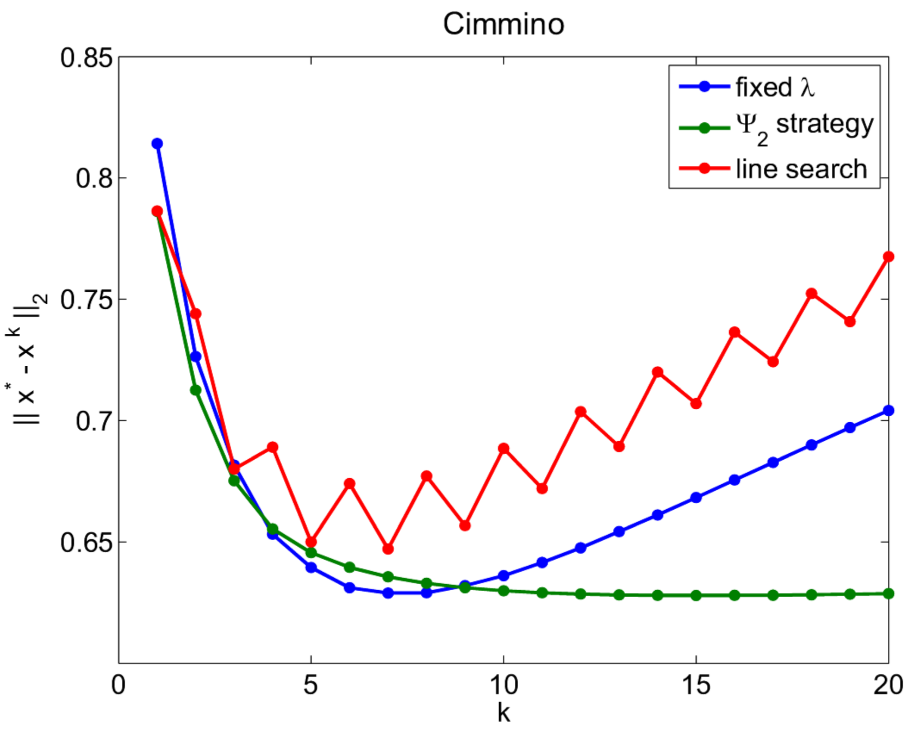
$$\|\mathsf{NE}\|_2 \lesssim \frac{\sqrt{2}}{\sigma_1}\,(1-\zeta_k^k)\,\|M^{\frac{1}{2}}e\|_2 \quad \text{for strategy } \Psi_1$$

$$\|\mathsf{NE}\|_2 \lesssim \frac{\sqrt{2}}{\sigma_1}\,\|M^{\frac{1}{2}}e\|_2 \quad \text{for strategy } \Psi_2$$

Also, for both strategies we have convergence:

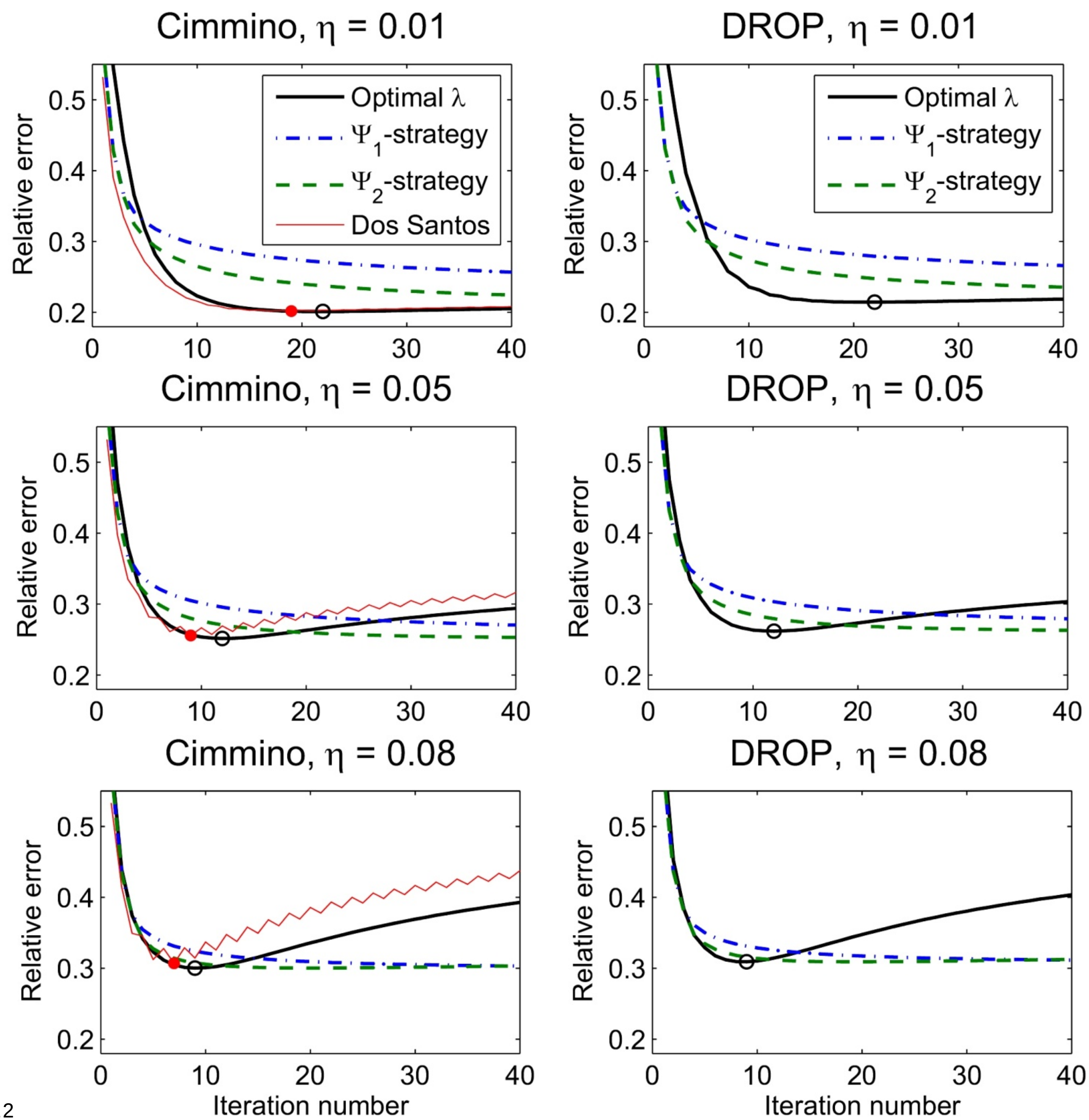
$$x^k \to \operatorname{argmin}\|A\,x - b\|_M \quad \text{as} \quad k \to \infty.$$

Error Histories for Cimmino Example



All three strategies give fast semi-convergence:

- The fixed λ requires training and thus a realistic test problem.
- The Dos Santos line search often gives a 'zig-zag' behavior.
- Our new strategy clearly controls the noise propagation.



For high noise levels, our new strategies ”track” the optimal λ; line search shows zig-zag behavior.

Conclusions

- We proposed two new strategies for choosing λ_k.
- Our strategies control the noise component of the error.
- In case of noise-free data our strategies give convergence to $\mathrm{argmin}_x \; || \, A\,x - b \, ||_M$.
- Our strategies are competitive with:
  - the optimal fixed λ (which requires training), and
  - line search (which requires a consistent system).
- Our strategies also work for SIRT methods with $T \neq I$.