Image Deblurring
in the Light of the Cosine Transform

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Three Fundamental Challenges of Image Deblurring

$\longleftrightarrow$

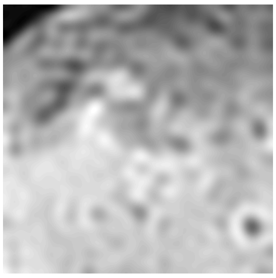
1. Incomplete and noisy data (blurred noisy images).
2. Limited accuracy of the solutions, due to ill-posedness.
3. Memory requirements and computing times.

For example, 3-D tomography reconstructions – if solved by naive methods – will require thousands of Gbytes of memory.

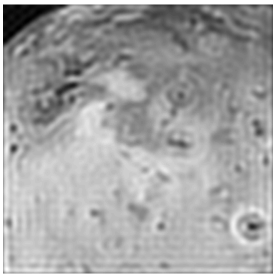
Only *new mathematical techniques* can provide more accurate solutions while, at the same time, achieving the necessary substantial reduction in memory demand and computing time.

We Know How To Do It Right!

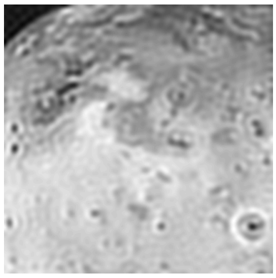
Reasonable computing times and memory requirements were previously achieved via "tricks" – often based on the FFT.



Blurred image



FFT deblurring (artifacts)

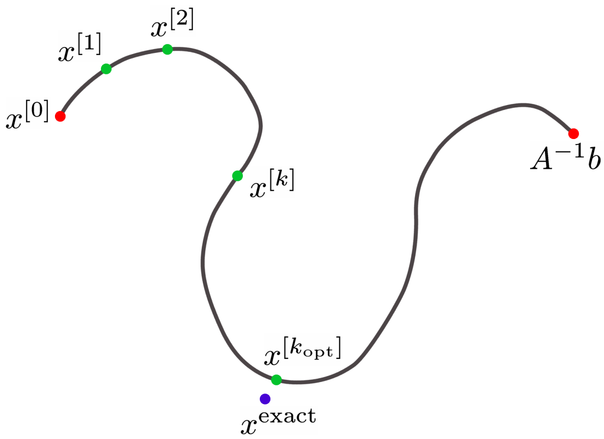


DCT deblurring (no artifacts)

But we can reformulate the problem – e.g., by including natural boundary conditions, and use other algorithms – to obtain a better reconstruction. Pure FFT algorithms belong to the past.

Iterative Methods + Semi-Convergence

Large-scale imaging problems can often be solved by iterative methods, such as (preconditioned) regularizing iterations.



During the first iterations, the iterates $x^{[k]}$ tend to be better and better approximations to the exact solution x^{exact}.

A some stage they start to diverge and converge towards $A^{-1}b$.

Goals (and Non-Goals) of This Talk

Our focus is on regularizing iterations associated with the Krylov subspace methods CGLS, MINRES and GMRES.

▶ The success of these methods is governed by the *initial behavior* of the iterations. We are not concerned with asymptotic analysis.

▶ We are interested in the *visual quality* of the reconstructions. Error norms and residual norms do not provide enough information.

▶ We are mainly concerned with *suppression of noise* (from data) in the reconstructions. We are not concerned with noise $\rightarrow 0$.

> In particular we want to study how image noise propagates through the deblurring algorithm to the reconstruction.

Matlab codes: `www.imm.dtu.dk/~pch/NoisePropagation.html`.

And Now . . . The DCT

We want to do better than just staring at the reconstructions!

Our tool is the **DCT = discrete cosine transformation**, which allows us to study the *spatial frequency contents* in the images.

(DFT could also be used – but we prefer a real transformation.)

Matrix formulation of DCT (x = signal, $\hat{x}$ = spectrum):

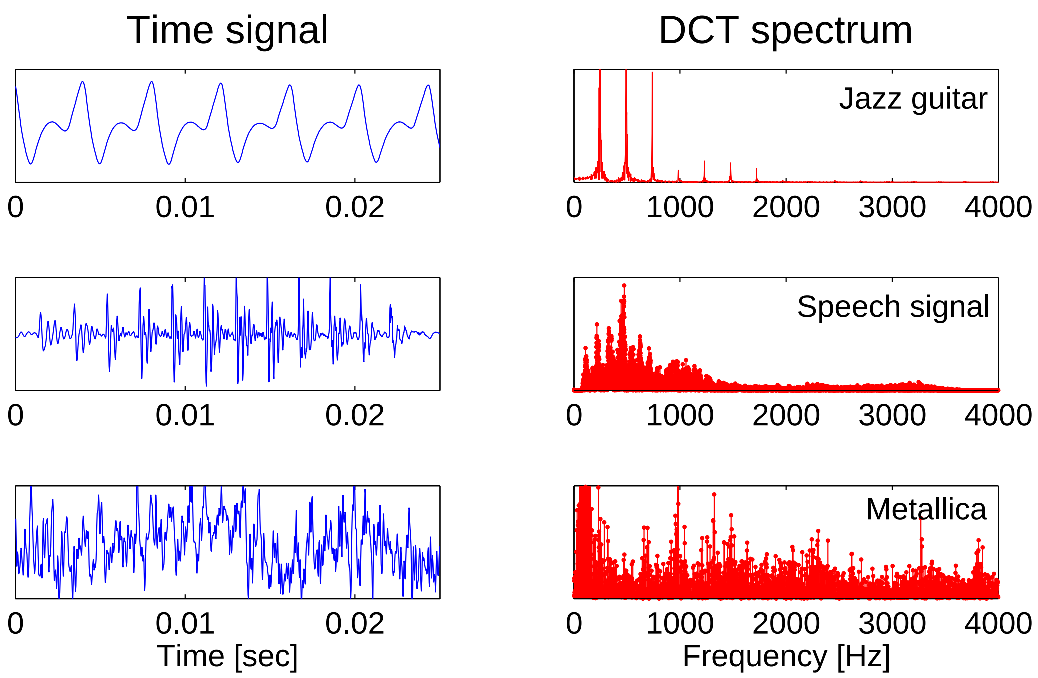
$$\hat{x} = \mathtt{dct}(x) = \mathcal{C}\, x,$$

where $\mathcal{C}$ is an $n \times n$ orthogonal matrix with elements

$$\mathcal{C}_{ij} = \begin{cases} \sqrt{1/n} & i = 1 \\ \sqrt{2/n}\, \cos\bigl(\pi(i-1)(2j-1)/(2n)\bigr), & i > 1. \end{cases}$$

The rows of $\mathcal{C}$ are sampled cosines.

DCT for Spectral Analysis of Time Series

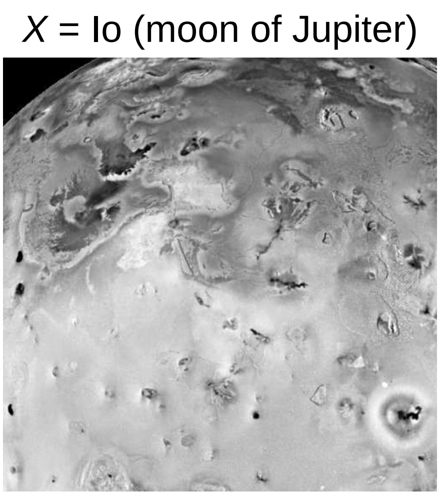


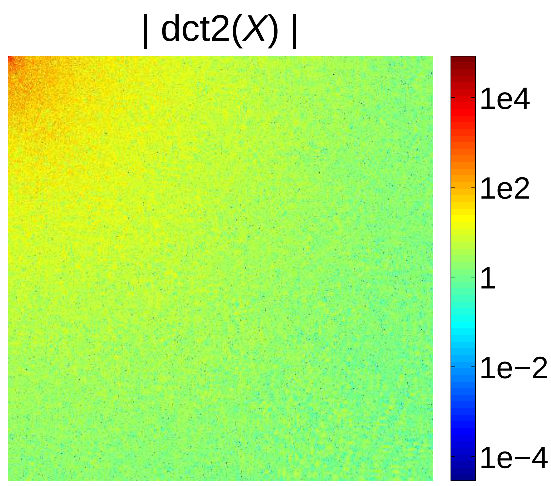
In *white noise* all spectral components have the same expectation.

Two-Dimensional DCT Analysis

The *two-dimensional DCT* of a square image X is:

$$\widehat{X} = \texttt{dct2}(X) = \mathcal{C}\, X\, \mathcal{C}^T.$$



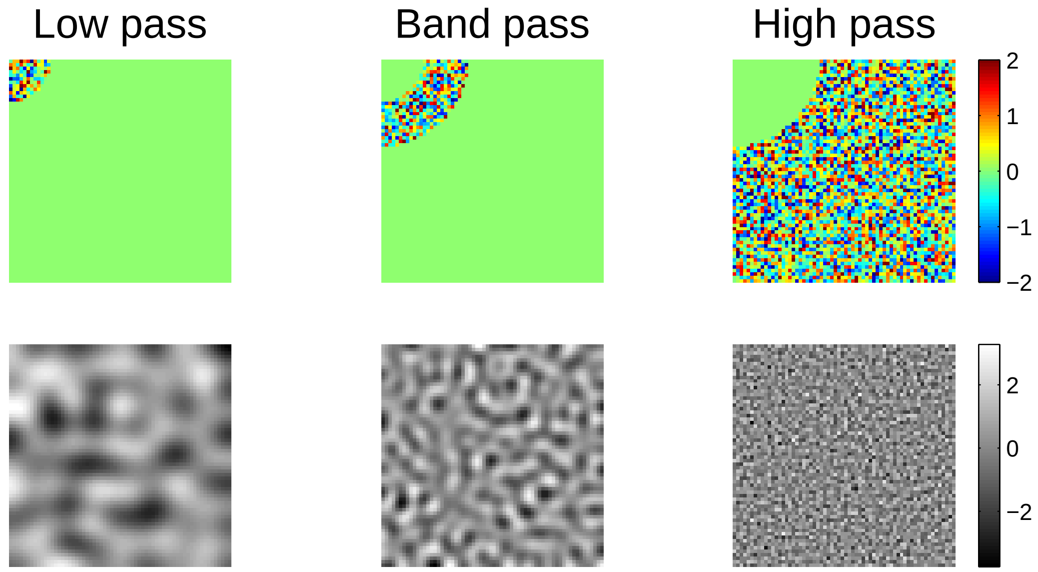


Images are dominated by spatially low-frequent information.

Filtered White Noise → "Freckles"

Top: $\Psi = \mathtt{randn(n)} \odot$ (filter matrix of 0s and 1s)

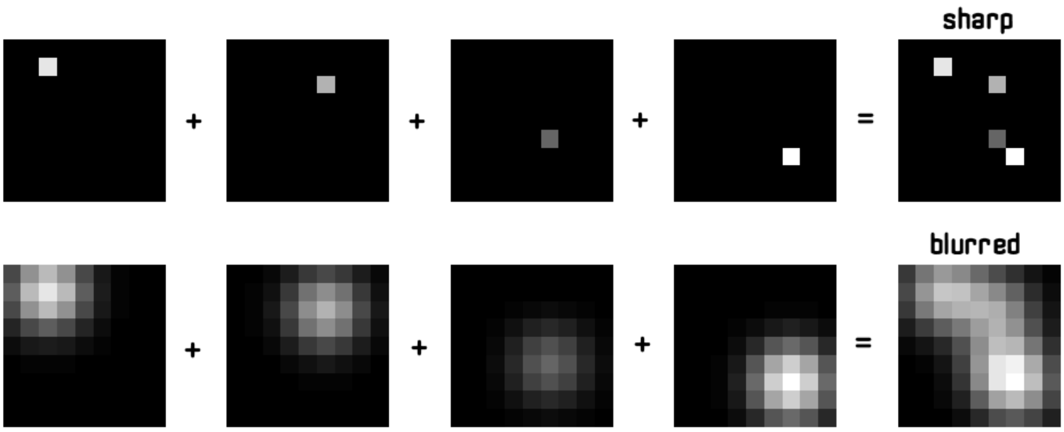
Bottom: $X_{\text{filt}} = \mathtt{idct2}(\Psi)$



The Point Spread Function

The point spread function is the image of a single bright pixel.

The blurred image is the sum of all the blurred pixels.



The Discrete Problem of Image Deblurring

Notation: X = sharp image, B = blurred image.

Image deblurring is a *discrete inverse problem* of the form

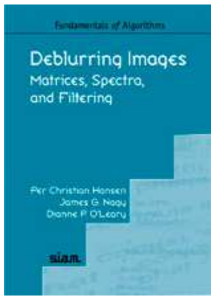
$$\boxed{A\,x = b}$$

where x and b are "stacked" versions of X and B,

$$\boxed{\text{vec notation:} \quad x = \text{vec}\,(X)\,, \quad b = \text{vec}\,(B).}$$

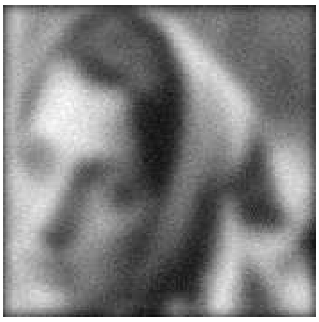
The (huge) PSF matrix A comes from the point spread function and represents the blurring.

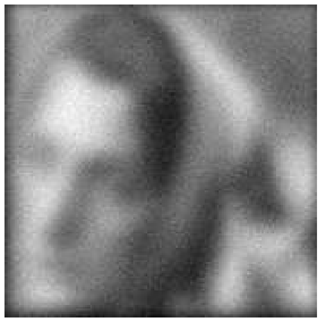
Usually A has structure, e.g., block-Toeplitz with Toeplitz blocks, and the structure depends on the PSF and the boundary conditions.



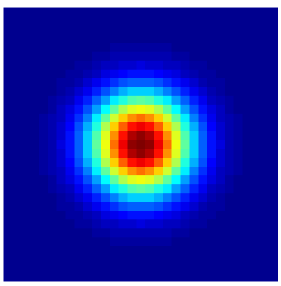
Our Test Problems

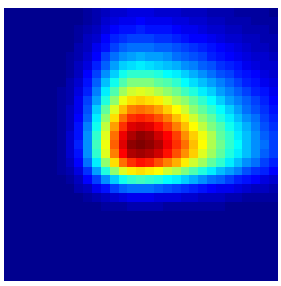
Sharp and blurred images due to symmetric/nonsymmetric A:





Zoom of the corresponding point spread functions:





Regularizing Iterations (Regularization by Projection)

We consider several Krylov subspace methods of the form

$$\min_x \|A\,x - b\|_2 \quad \text{s.t.} \quad x \in \mathcal{S}_k, \qquad \mathcal{S}_k = \text{Krylov subspace.}$$

CGLS:

$$\mathcal{S}_k = \text{span}\left\{A^T b, (A^T A) A^T b, \ldots, (A^T A)^{k-1} A^T b\right\}.$$

GMRES and MINRES:

$$\mathcal{S}_k = \text{span}\left\{b, Ab, A^2 b, \ldots, A^{k-1} b\right\}.$$

RRGMRES and MR-II:

$$\mathcal{S}_k = \text{span}\left\{Ab, A^2 b, \ldots, A^{k-1} b, A^k b\right\}.$$

Implementations of these methods produce (explicitly or implicitly) an orthonormal basis for $\mathcal{S}_k$.

Symmetric A, Iterations k = 5, 10, and 20

Nonsymmetric A, Iterations $k = 5$, 10, and 20

A Closer Look at the Krylov Basis – CGLS

The "power vectors" $(A^T A)^{i-1} A^T b$ of the CGLS Krylov subspace $\mathcal{S}_k$ are just more and more blurred versions of B. Not interesting!

$\looparrowright$ We need an orthonormal basis for $\mathcal{S}_k$.

Recall that CGLS is equivalent to LSQR bidiagonalization:

$$A\,V_k = U_{k+1}\,B_k, \qquad U_{k+1} \text{ and } V_k \text{ have orthonormal columns.}$$

The CGLS/LSQR solution is

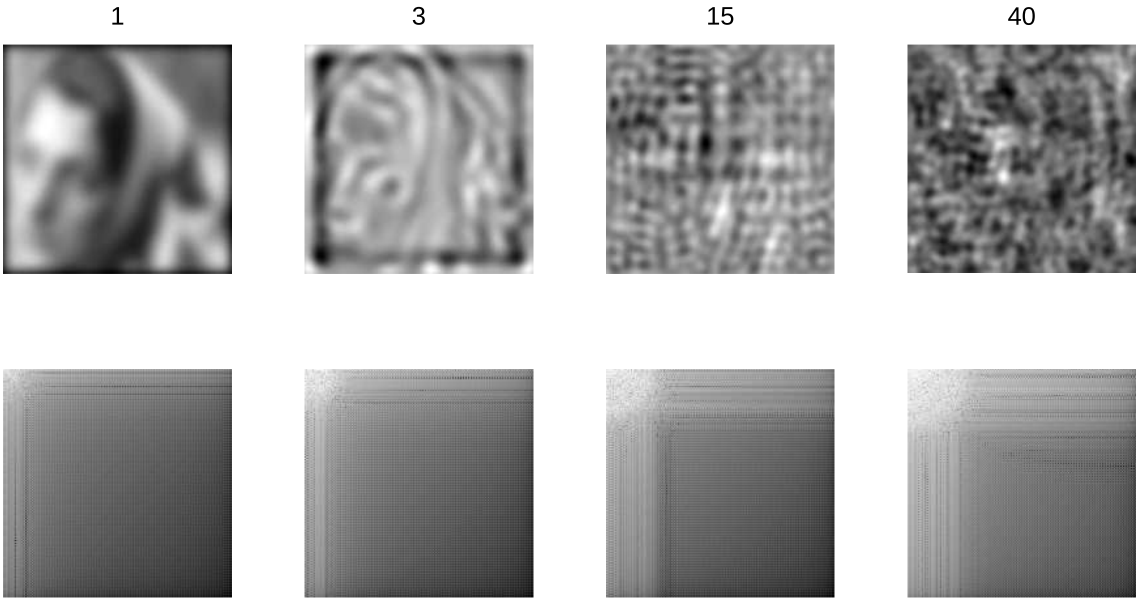
$$\boxed{x^{[k]} = V_k\,\xi_k} \qquad \text{with} \qquad \xi_k = \text{argmin}_\xi \|B_k\,\xi - \beta\,e_1\|_2 .$$

Introducing the SVD, $B_k = P_k \Sigma_k Q_k^T$, we can also write

$$\boxed{x^{[k]} = (V_k\,Q_k)\,\Sigma_k^{-1}(\beta\,P_k^T e_1)}$$

There is useful information in the columns of V_k and $V_k\,Q_k$.

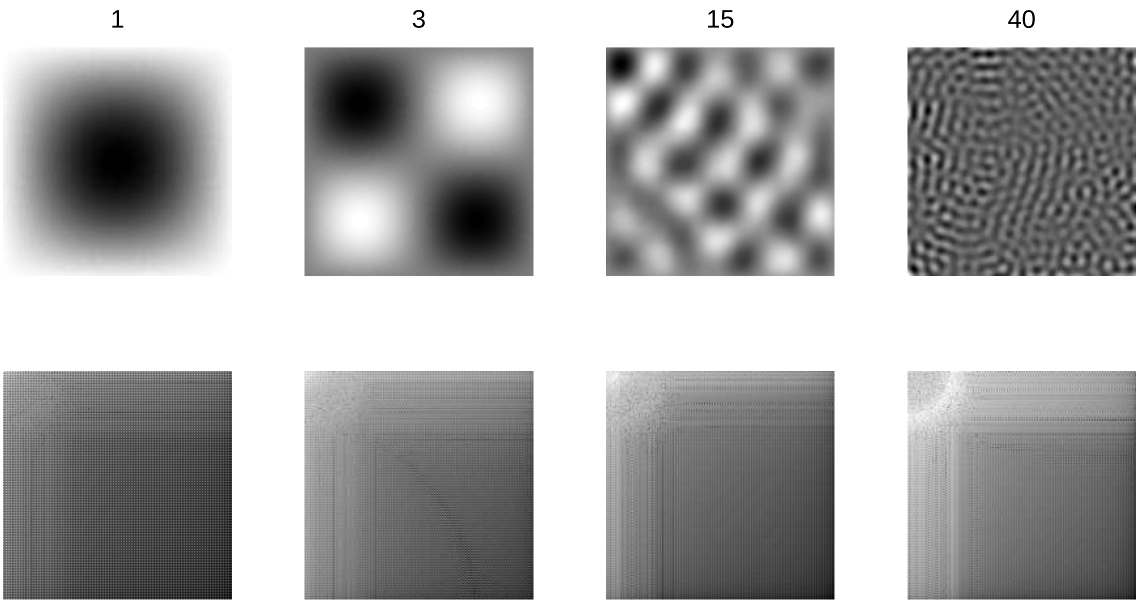
CGLS Krylov Basis (Symm. A) – Columns of V_k



More spectral frequencies are included as k increases.

No high frequencies are present.

CGLS Krylov Basis (Symm. A) – Columns of $V_k \, Q_k$



These vectors first resemble spectral bases, but later some "freckles" = band-pass filtered structures begin to appear.

A Closer Look at the Krylov Basis – GMRES

Again the "power vectors" $A^{i-1}b$ of the GMRES Krylov subspace are just more and more blurred versions of B.

Recall that GMRES produces a "Hessenbergization:"

$$A\,W_k = W_{k+1}\,H_k, \qquad W_{k+1} \text{ has orthonormal columns.}$$

The GMRES solution is

$$\boxed{x^{[k]} = W_k\,\xi_k} \qquad \text{with} \qquad \xi_k = \operatorname{argmin}_\xi \|H_k\,\xi - \beta\,e_1\|_2.$$

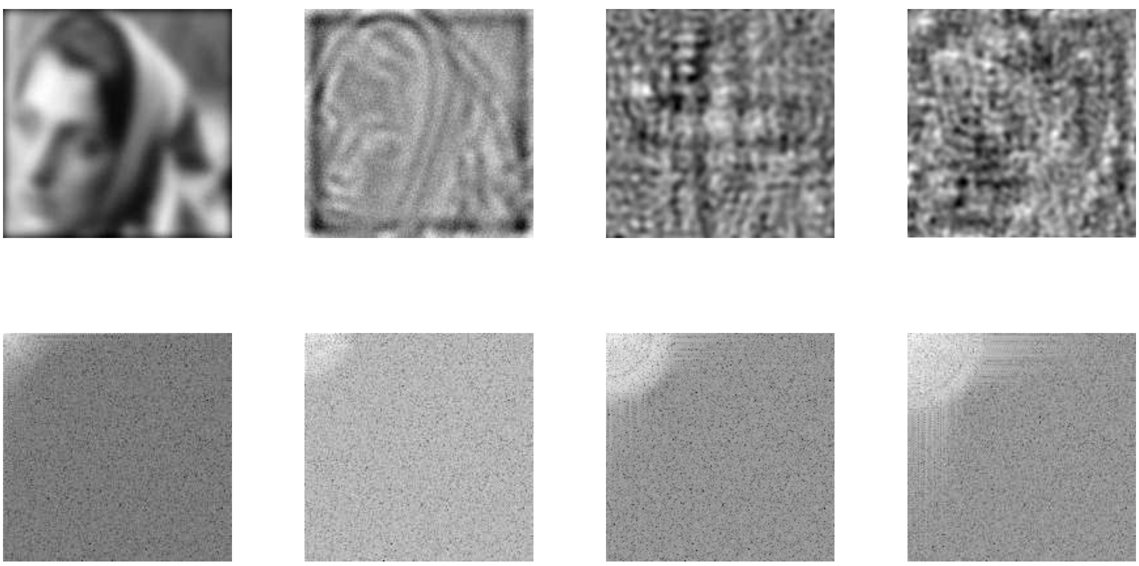
Introducing the SVD $H_k = P_k \Sigma_k Q_k^T$ we can also write

$$\boxed{x^{[k]} = (W_k\,Q_k)\,\Sigma_k^{-1}(\beta\,P_k^T e_1)}$$

There is useful information in the columns of W_k and $W_k\,Q_k$.

The GMRES Krylov Basis – Columns of W_k

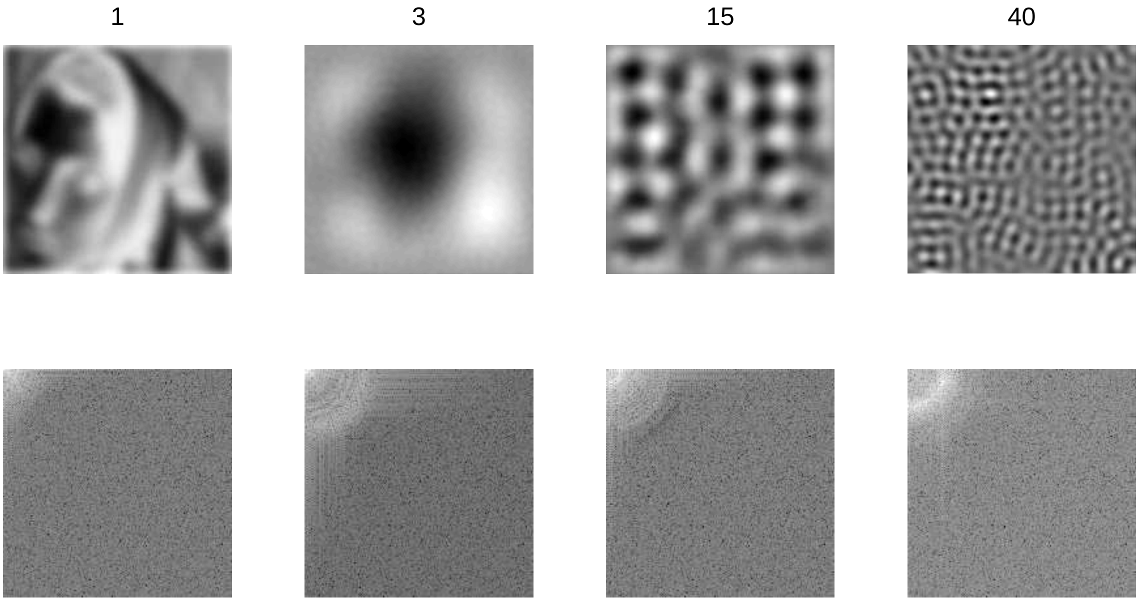
1 3 15 40



More spectral frequencies are included as k increases.

White noise is always present, due to the vector b in the basis!

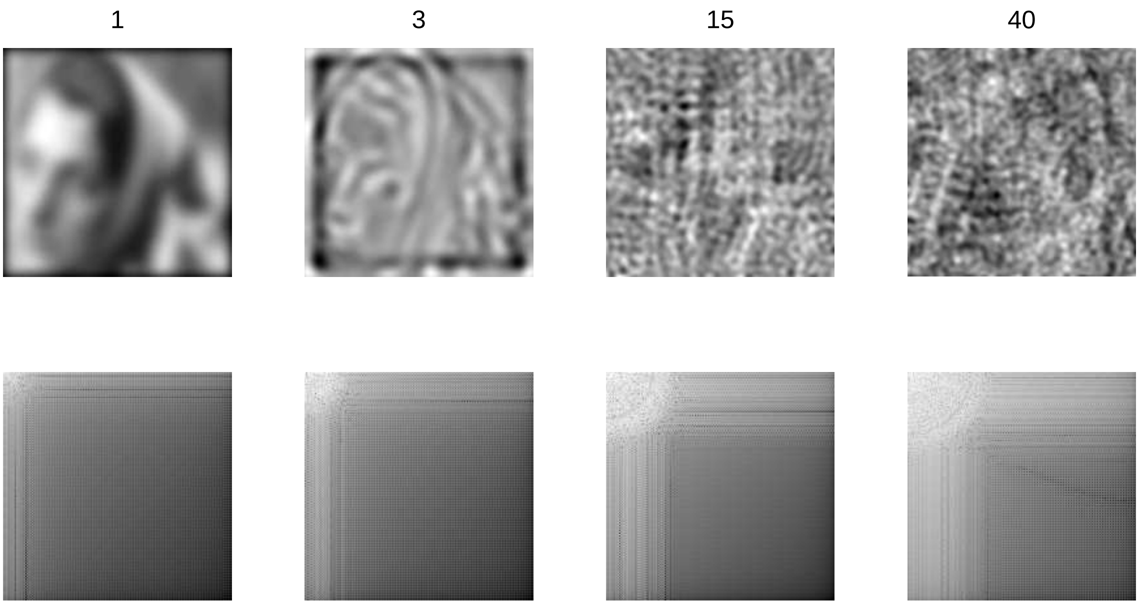
GMRES Krylov Basis – Columns of $W_k Q_k$



These vectors do not resemble spectral bases.

The "freckles" = band-pass filtered structures are still present.

The RRGMRES Krylov Basis – Columns of W_k



Similar to GMRES – but the white noise is gone (thanks to $A\,b$)!

The behavior of RRGMRES resembles that of CGLS.

How the Noise Enters the Picture

A very common assumption is that the noise in the data is *additive:*

$$B = B^{\text{exact}} + E \qquad \Leftrightarrow \qquad b = b^{\text{exact}} + e.$$

This assumption lets us separate the filtered "signal" and "noise" components in the reconstructions.

LSQR:

$$x^{(k)} = V_k B_k^{\dagger} U_{k+1}^T b = \underbrace{V_k B_k^{\dagger} U_{k+1}^T b^{\text{exact}}}_{\text{“signal”}} + \underbrace{V_k B_k^{\dagger} U_{k+1}^T e}_{\text{“noise”}}$$

GMRES:

$$\hat{x}^{(k)} = W_k H_k^{\dagger} W_{k+1}^T b = \underbrace{W_k H_k^{\dagger} W_{k+1}^T b^{\text{exact}}}_{\text{“signal”}} + \underbrace{W_k H_k^{\dagger} W_{k+1}^T e}_{\text{“noise”}}$$

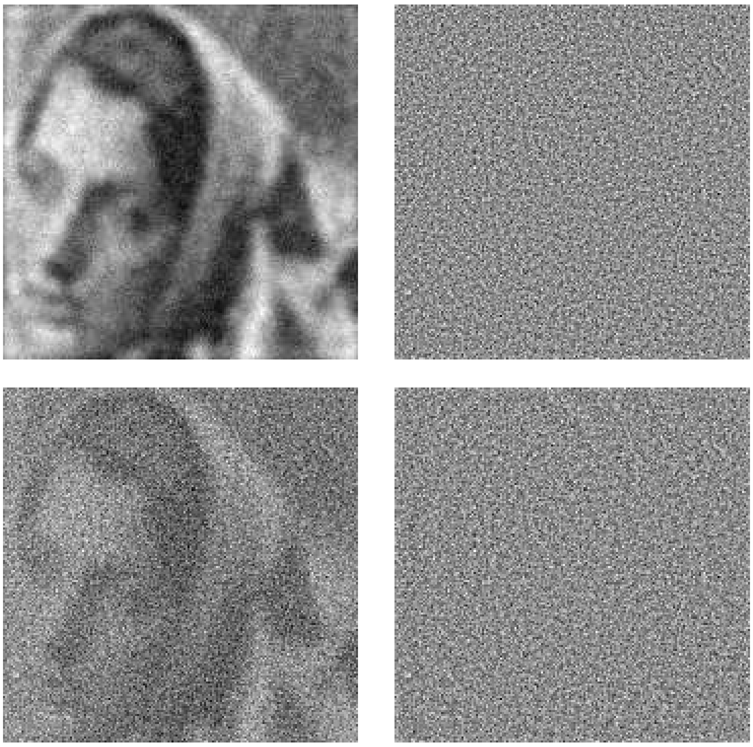
Note: the matrices V_k, B_k, U_{k+1}, W_{k+1}, and H_k are obtained by running CGLS and GMRES on the noisy b.

Signal and Noise Parts in CGLS Iterates ($k = 10$)

Symm. A

Nonsym. A

Signal and Noise Parts in GMRES Iterates ($k = 10$)



Symm. A

Nonsym. A

Signal and Noise Parts in <u>RR</u>GMRES Iterates ($k = 10$)

Symm. A

Nonsym. A

Observations on the Noise Propagation

- **CGLS:**
  - Signal: getting sharper than the blurred data B.
  - Noise: medium-frequency "freckles" and "ringing" that follow the contours in the image.
- **GMRES:**
  - Both signal and noise have a significant white-noise component due to the noise vector in the Krylov subspace.
- **RRGMRES:**
  - Quite similar to CGLS.

Conclusion: it is highly advisable to use RRGMRES (and MR-II) instead of GMRES (and MINRES).

There is no clear winner among CGLS and RRGMRES here.

Conclusions

1. Singular vectors and Krylov subspace vectors $\sim$ spectral bases.
2. Regularization is achieved via projection on Krylov subspace.
3. CGLS and RRGMRES provide useful Krylov subspaces.
4. Don't use GMRES/MINRES (noise component in basis).
5. Propagated noise is correlated with structures in the image (this "masks" some of the noise).
6. Very low noise levels (see paper): the "masking" makes GMRES/RRGRMES solutions appear sharper than they are.
7. Freckles arise in both CGLS and RRGMRES, no clear winner.
8. Stopping criterion – an open problem.
9. Next: edge-preserving methods and nonnegativity constraints.