A Frame Theoretic View on Inverse Problems

Jakob Lemvig — Technical University of Denmark (DTU)

joint work with Jürgen Frikel (OTH Regensburg)
Outline

- Frame Theory
  - Frames in abstract Hilbert spaces
  - Structured frames for $L^2(\mathbb{R}^d)$

- Inverse Problems in Abstract Settings: Dual frame based regularizations

- Shearlet Frames in $L^2(\mathbb{R}^2)$

- Inversion of the Radon Transform by Shearlet Frames
Convenient Expansions of Functions

- Expansions of signals $f$ of finite energy, i.e.,
  \[
  f \in L^2(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} |f(x)|^2 \, dx < \infty \} 
  \]
  
  \[
  f(x) = \sum_{j=1}^{\infty} c_k \varphi_k(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x) + \ldots
  \]
  in terms of convenient building blocks $\varphi_k \in L^2(\mathbb{R}^d)$.

- Shearlet analysis: An alternative to Fourier analysis, where the building blocks $\{\varphi_k\}$ are dilations (scales), shears and translations of a single function $\psi \in L^2(\mathbb{R}^2)$:
  \[
  \{\varphi_k\} \sim \left\{2^{3j/4} \psi(2^j x_1 + k2^{j/2} x_2 - m_1, 2^{j/2} x_2 - m_2)\right\}_{j \in \mathbb{Z}, k \in \mathbb{Z}, m \in \mathbb{Z}^2}
  \]

- Frames: A generalization of orthonormal bases (ONB) with more flexibility and freedom.
It is not always possible/desirable to require ONB.

- **Problems:**
  - Non-existence of Gabor ONB with good time-frequency localization
  - Non-existence of ONB sensitive to curvilinear singularities
  - Non-resilience to erasures/noise of expansions in an ONB
It is not always possible/desirable to require ONB.

- **Problems:**
  - Non-existence of Gabor ONB with good time-frequency localization
  - Non-existence of ONB sensitive to curvilinear singularities
  - Non-resilience to erasures/noise of expansions in an ONB

- **Solution:** Frames – a standard tools in applied mathematics and engineering.
It is not always possible/desirable to require ONB.

- **Problems:**
  - Non-existence of Gabor ONB with good time-frequency localization
  - Non-existence of ONB sensitive to curvilinear singularities
  - Non-resilience to erasures/noise of expansions in an ONB

- **Solution:** Frames – a standard tools in applied mathematics and engineering.

- **Key Property** of Frames:

  *Redundancy!*
What is a Frame?

Definition

A sequence \( \{ \varphi_k \}_{k \in \mathbb{N}} \) is a frame for a separable Hilbert space \( X \) if

\[
\exists A, B > 0 : \quad A \| f \|^2 \leq \sum_{k=1}^{\infty} |\langle f, \varphi_k \rangle|^2 \leq B \| f \|^2 \quad \text{for all } f \in X.
\]

If the upper bound holds, then \( \{ \varphi_k \} \) is said to be a Bessel sequence.

Definition

Two Bessel sequences \( \{ \varphi_k \} \) and \( \{ \psi_k \} \) are said to be dual frames if

\[
f = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \psi_k \quad \text{for all } f \in X.
\]
Operators associated with Frames/Bessel sequences

For $\Phi = \{\varphi_k\}_{k \in \mathbb{N}}$, define the Analysis operator:

$$C_{\Phi} : X \rightarrow \ell^2(\mathbb{N}), \quad C_{\Phi}f = \{\langle f, \varphi_k \rangle\}_{k \in \mathbb{N}}$$

and the Synthesis operator:

$$D_{\Phi} : \ell^2(\mathbb{N}) \rightarrow X, \quad D_{\Phi}\{c_k\}_{k \in \mathbb{N}} = \sum_{k=1}^{\infty} c_k \varphi_k$$
Frame Theory

Operators associated with Frames/Bessel sequences

For $\Phi = \{\varphi_k\}_{k \in \mathbb{N}}$, define the Analysis operator:

$$C_\Phi : X \rightarrow \ell^2(\mathbb{N}), \quad C_\Phi f = \{\langle f, \varphi_k \rangle\}_{k \in \mathbb{N}}$$

and the Synthesis operator:

$$D_\Phi : \ell^2(\mathbb{N}) \rightarrow X, \quad D_\Phi \{c_k\}_{k \in \mathbb{N}} = \sum_{k=1}^{\infty} c_k \varphi_k$$

- $\Phi = \{\varphi_k\}_{k \in \mathbb{N}}$ is Bessel $\iff C_\Phi$ is a bounded operator. Here: $(C_\Phi)^* = D_\Phi$. 
Frame Theory

Operators associated with Frames/Bessel sequences

For $\Phi = \{\varphi_k\}_{k \in \mathbb{N}}$, define the Analysis operator:

$$C_\Phi : X \to l^2(\mathbb{N}), \quad C_\Phi f = \{\langle f, \varphi_k \rangle\}_{k \in \mathbb{N}}$$

and the Synthesis operator:

$$D_\Phi : l^2(\mathbb{N}) \to X, \quad D_\Phi \{c_k\}_{k \in \mathbb{N}} = \sum_{k=1}^{\infty} c_k \varphi_k$$

- $\Phi = \{\varphi_k\}_{k \in \mathbb{N}}$ is Bessel $\iff$ $C_\Phi$ is a bounded operator. Here: $(C_\Phi)^* = D_\Phi$.
- The frame inequalities mean $A \|f\|^2 \leq \|C_\Phi f\|^2 \leq B \|f\|^2$ for all $f \in X$. 
Operators associated with Frames/Bessel sequences

For $\Phi = \{\varphi_k\}_{k \in \mathbb{N}}$, define the Analysis operator:

$$C_\Phi : X \rightarrow \ell^2(\mathbb{N}), \quad C_\Phi f = \{\langle f, \varphi_k \rangle\}_{k \in \mathbb{N}}$$

and the Synthesis operator:

$$D_\Phi : \ell^2(\mathbb{N}) \rightarrow X, \quad D_\Phi \{c_k\}_{k \in \mathbb{N}} = \sum_{k=1}^{\infty} c_k \varphi_k$$

- $\Phi = \{\varphi_k\}_{k \in \mathbb{N}}$ is Bessel $\iff$ $C_\Phi$ is a bounded operator. Here: $(C_\Phi)^* = D_\Phi$.
- The frame inequalities mean $A \|f\|^2 \leq \|C_\Phi f\|^2 \leq B \|f\|^2$ for all $f \in X$.
- $\Phi$ and $\Psi$ are dual frames $\iff D_\Psi C_\Phi = I_X$, i.e., $f = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \psi_k \quad \forall f \in X$
Operators associated with Frames/Bessel sequences

For \( \Phi = \{\varphi_k\}_{k \in \mathbb{N}} \), define the Analysis operator:

\[
C_\Phi : X \to \ell^2(\mathbb{N}), \quad C_\Phi f = \{\langle f, \varphi_k \rangle\}_{k \in \mathbb{N}}
\]

and the Synthesis operator:

\[
D_\Phi : \ell^2(\mathbb{N}) \to X, \quad D_\Phi \{c_k\}_{k \in \mathbb{N}} = \sum_{k=1}^{\infty} c_k \varphi_k
\]

- \( \Phi = \{\varphi_k\}_{k \in \mathbb{N}} \) is Bessel \( \Leftrightarrow C_\Phi \) is a bounded operator. Here: \((C_\Phi)^* = D_\Phi\).
- The frame inequalities mean \( A \|f\|^2 \leq \|C_\Phi f\|^2 \leq B \|f\|^2 \) for all \( f \in X \).
- \( \Phi \) and \( \Psi \) are dual frames \( \Leftrightarrow D_\Psi C_\Phi = I_X \), i.e., \( f = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \psi_k \quad \forall f \in X \)
- A frame has at least one dual frame: the canonical dual \( \{ (D_\Phi C_\Phi)^{-1} \varphi_k \}_{k \in \mathbb{N}} \).
The Picture of Frame Expansions:

- **Signal** $f$
- **Coefficients** $c_k = \langle f, \varphi_k \rangle$
- **Signal** $f = \sum_k c_k \psi_k$

Want: Analysis & synthesis to be linear and **continuous** operations, often assuming some structure on $\varphi_k$ and/or $\psi_k$.
Structured Frames in $L^2(\mathbb{R}^d)$

Generalized Shift-invariant (GSI) systems are of the form: \( \{T_{C_{p},k}g_{p}\}_{k \in \mathbb{Z}^d, p \in P} \), where \( g_{p} \in L^2(\mathbb{R}^d) \), \( C_{p} \in \text{GL}(d, \mathbb{R}) \), \( P \) a countable index set.
Generalized Shift-invariant (GSI) systems are of the form: \( \{ T_{C_p} k g_p \}_{k \in \mathbb{Z}^d, p \in P} \), where \( g_p \in L^2(\mathbb{R}^d) \), \( C_p \in \text{GL}(d, \mathbb{R}) \), \( P \) a countable index set.

### Necessary condition for Frame (covering of frequency domain)

If the GSI system is a frame with bounds \( A \) and \( B \), then

\[
A \leq \sum_{p \in P} \frac{1}{|\det C_p|} |\hat{g}_p(\gamma)|^2 \leq B \quad \text{a.e. } \gamma \in \mathbb{R}^d
\]

Here we ignore a technical Local Integrability Condition. Result for \( d = 1 \) due to [Christensen, Hasannasab, L.] and for general \( d \in \mathbb{N} \) by [Führ, Jakobsen, L.]
Structured Frames in $L^2(\mathbb{R}^d)$

Generalized Shift-invariant (GSI) systems are of the form: $\{T_{C_p,k}g_p\}_{k \in \mathbb{Z}^d, p \in P}$, where $g_p \in L^2(\mathbb{R}^d)$, $C_p \in \text{GL}(d, \mathbb{R})$, $P$ a countable index set.

**Necessary condition for Frame (covering of frequency domain)**

If the GSI system is a frame with bounds $A$ and $B$, then

$$A \leq \sum_{p \in P} \frac{1}{|\det C_p|} |\hat{g}_p(\gamma)|^2 \leq B \quad \text{a.e. } \gamma \in \mathbb{R}^d$$

**Sufficient condition for Bessel (not too much overlap)**

If

$$B := \text{ess sup} \sum_{\gamma \in \mathbb{R}^d} \sum_{p \in P} \sum_{\alpha \in \mathbb{Z}^d} \frac{1}{|\det C_p|} |\hat{g}_p(\gamma)\hat{g}_p(\gamma + C_p^\# \alpha)| < \infty$$

then the GSI system is Bessel with bounds $B$. Here $C_p^\#: = (C^T_p)^{-1}$.
Setup: Let $X, Y$ be separable (inf. dim.) Hilbert spaces. Let

$$K : D(K) \rightarrow Y, X = \overline{D(K)}, Y = \overline{R(K)}$$

be an injective, closed operator (typically, it is compact).

**Problem**

Given $g \in Y$ and $\varepsilon > 0$ s.t. $\|Kf - g\|_Y < \varepsilon$, recover $f$. 
An Inverse Problem

Setup: Let $X, Y$ be separable (inf. dim.) Hilbert spaces. Let

$$K : D(K) \to Y, X = \overline{D(K)}, Y = \overline{R(K)}$$

be an injective, closed operator (typically, it is compact).

Problem

Given $g \in Y$ and $\varepsilon > 0$ s.t. $\|Kf - g\|_Y < \varepsilon$, recover $f$.

- It holds $(K^{-1})^* = (K^*)^{-1}$; for short, we write this operator as $K^{-*} : X \to Y$.
- $D(K^{-*}) = X \iff K^{-1}$ bounded. If $K$ is compact, $K^{-1}$ is unbounded.
• Take dual frames \( \{ \varphi_k \} \) and \( \{ \psi_k \} \) for \( X \) s.t. \( \varphi_k \in D(K^{-*}) \):

\[
f = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle_X \psi_k \quad \text{for all } f \in X.
\]

• Note that

\[
\langle f, \varphi_k \rangle_X = \langle f, K^* K^{-*} \varphi_k \rangle_X = \langle Kf, K^{-*} \varphi_k \rangle_Y
\]

• This gives the inversion formula:

\[
f = \sum_{k=1}^{\infty} \langle Kf, K^{-*} \varphi_k \rangle_Y \psi_k \quad \text{for all } f \in X.
\]
Regularization Strategy based on Dual Frames 1

- Set $w_k = \kappa_k K^{-*} \varphi_k$. Pick weights $\kappa_k > 0$ s.t. $W = \{w_k\}_{k \in \mathbb{N}}$ is a Bessel sequence in $X$.

This is indeed always possible:

**Lemma**

Let $\{\theta_k\}$ be a sequence of positive numbers such that $\sum_{k \in \mathbb{N}} \theta_k < \infty$. Take $\kappa_k = \sqrt{\theta_k} / \|K^{-*} \varphi_k\|_Y$. Then $W = \{w_k\}_{k \in \mathbb{N}}$ is Bessel with bound $B = \sum_{k \in \mathbb{N}} \theta_k$. 
Regularization Strategy based on Dual Frames

- Set $w_k = \kappa_k K^{-*} \varphi_k$. Pick weights $\kappa_k > 0$ s.t. $W = \{w_k\}_{k \in \mathbb{N}}$ is a Bessel sequence in $X$.

This is indeed always possible:

**Lemma**

Let $\{\theta_k\}$ be a sequence of positive numbers such that $\sum_{k \in \mathbb{N}} \theta_k < \infty$. Take $\kappa_k = \sqrt{\theta_k} / \|K^{-*} \varphi_k\|_Y$. Then $W = \{w_k\}_{k \in \mathbb{N}}$ is Bessel with bound $B = \sum_{k \in \mathbb{N}} \theta_k$.

- The inversion formula is now:

$$f = \sum_{k=1}^{\infty} \frac{1}{\kappa_k} \langle Kf, w_k \rangle_Y \psi_k$$

for all $f \in D(K)$.

- Or in terms of analysis, synthesis and multiplication operators:

$$f = D\Psi M_{1/\kappa} C_W Kf$$

where $M_{1/\kappa}$ is a (often unbounded) multiplication operator on $\ell^2(\mathbb{N})$ defined by $M_{1/\kappa} \{c_k\}_{k \in \mathbb{N}} = \{c_k / \kappa_k\}_{k \in \mathbb{N}}$. 
We define the recovery operator $R = D_\Psi M_{1/\kappa} C_W$:

$$R : D(R) \to X, \quad Rg = \sum_{k \in \mathbb{N}} \frac{1}{\kappa_k} \langle g, w_k \rangle_Y \psi_k$$

where the domain is determined by a Picard condition:

$$D(R) = \left\{ g \in Y : \sum_{k \in \mathbb{N}} \frac{|\langle g, w_k \rangle_Y|^2}{\kappa_k^2} < \infty \right\}.$$ 

The recovery strategy from $\|Kf - g\| < \varepsilon$ is:

$$D_\Psi M_{1/\kappa} SC_W g$$

where $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is a threshold procedure.
Shearlet Frames in $L^2(\mathbb{R}^2)$

Shearlet Systems in $L^2(\mathbb{R}^2)$

- **Anisotropic scaling $A$:**
  \[ A = \begin{pmatrix} 2 & 0 \\ 0 & 2^{1/2} \end{pmatrix}, \]

- **Shearing $S_k$ (direction parameter $\leftrightarrow$ rotations):**
  \[ S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \]

- The shearlet system generated by $\psi \in L^2(\mathbb{R}^2)$ is
  \[ \left\{ \psi_{j,k,m} = D_{S_k A^j} T_m \psi = 2^{3j/4} \psi(S_k A^j \cdot -m) : j \in \mathbb{Z}, k \in \mathbb{Z}, m \in \mathbb{Z}^2 \right\} \]

- **Frequency localization of $\psi$:**
  \[ |\hat{\psi}(\gamma_1, \gamma_2)| \leq C \min(1, |2\gamma_1|^{\alpha}) \min(1, |\gamma_1|^{-\delta}) \min(1, |\gamma_2|^{-\delta}) \]
  for some $C > 0$, $\alpha > \delta > 3$
Shearlet Frames in $L^2(\mathbb{R}^2)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D

\[ \psi_{j,k,m} \text{ for } j = 0, k = 0, m = (0,0) \]
Shearlet Frames in $L^2(\mathbb{R}^2)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D

$$\psi_{j,k,m} \text{ for } j = 1, k = 0, m = (0, 0)$$
Shearlet Frames in $L^2(\mathbb{R}^2)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D

\[ \psi_{j,k,m} \text{ for } j = 2, k = 0, m = (0,0) \]
Shearlet Frames in $L^2(\mathbb{R}^2)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D

\[ \psi_{j,k,m} \text{ for } j = 1, k = 0, m = (0, 0) \]
Shearlet Frames in $L^2(\mathbb{R}^2)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D

\[ \psi_{j,k,m} \text{ for } j = 1, k = 0, m = (1, -1) \]
$\psi_{j,k,m}$ for $j = 1, k = 0, m = (0, 0)$
Shearlet Frames in $L^2(\mathbb{R}^2)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D

$\psi_{j,k,m}$ for $j = 1$, $k = -1$, $m = (0,0)$
Shearlet Frames in $L^2(\mathbb{R}^2)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D

$\psi_{j,k,m}$ for $j = 1, k = -2, m = (0,0)$
Shearlet Frames in $L^2(\mathbb{R}^2)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D

\[ \psi_{j,k,m} \text{ for } j = 1, k = -3, m = (0,0) \]
Shearlet Frames in $L^2(\mathbb{R}^2)$

Cone-adapted Shearlet systems

- Problem: “Length” of $\text{supp}\, \hat{\psi}_{j,k,m}$ goes to $\infty$ as $|k| \to \infty$. 
Shearlet Frames in $L^2(\mathbb{R}^2)$

Cone-adapted Shearlet systems

- Problem: “Length” of $\text{supp} \hat{\psi}_{j,k,m}$ goes to $\infty$ as $|k| \to \infty$.
- Solution: Cone-adapted shearlet system

**Definition**

For $\phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$, the cone-adapted shearlet system $\text{SH}(\phi, \psi, \tilde{\psi})$ is the union:

$$\{T_k\phi\}_{k \in \mathbb{Z}^2} \cup \{D S_k A_j T_m \psi\}_{j \geq 0, |k| \leq [2^{j/2}], m \in \mathbb{Z}^2} \cup \{D \tilde{S}_k \tilde{A}_j T_m \tilde{\psi}\}_{j \geq 0, |k| \leq [2^{j/2}], m \in \mathbb{Z}^2}$$
Shearlet Frames in $L^2(\mathbb{R}^2)$

Shearlet systems as GSI systems

By the commutator relations

$$D_{A^j S_k} T_m = T_{S_{-k} A^{-j} m} D_{A^j S_k},$$

it follows that

$$\psi_{j,k,m} = D_{S_k A^j} T_m \psi = T_{S_{-k} A^{-j} m} D_{A^j S_k} \psi$$

Hence, the shearlet system on horizontal is a GSI system $\{T_{C_p g_p}\}_{k \in \mathbb{Z}^d, p \in P}$ with

$$C_p = C_{(j,k)} = S_{-k} A^{-j} \quad \text{and} \quad g_p = g_{(j,k)} = D_{A^j S_k} \psi$$

where

$$P = \bigcup_{j \in \mathbb{N}_0} \{j\} \times \left[-\left\lfloor 2^{j/2} \right\rfloor, \left\lceil 2^{j/2} \right\rceil\right].$$

Similar for the vertical cones and the central box.
Shearlet Frames in \(L^2(\mathbb{R}^2)\)

**Necessary and Sufficient Conditions**

**Necessary condition for Frame (covering of frequency domain)**

If the shearlet system \(SH(\phi, \psi, \tilde{\psi})\) is a frame with bounds \(A\) and \(B\), then

\[
A \leq |\hat{\phi}(\gamma)|^2 + \sum_{j=0}^{\infty} \sum_{|k| \leq [2j/2]} |\hat{\psi}(S^T_{-k} A^j \gamma)|^2 + \sum_{j=0}^{\infty} \sum_{|k| \leq [2j/2]} |\hat{\psi}(\tilde{S}^T_{-k} \tilde{A}^j \gamma)|^2 \leq B
\]

for a.e. \(\gamma \in \mathbb{R}^2\).

**Sufficient condition for Bessel (not too much overlap)**

If

\[
B := \text{ess sup}_{\gamma \in \mathbb{R}^2} \left( \sum_{\alpha \in \mathbb{Z}^2} |\hat{\phi}(\gamma)\hat{\phi}(\gamma + \alpha)| + \sum_{j=0}^{\infty} \sum_{|k| \leq [2j/2]} |\hat{\psi}(S^T_{-k} A^j \gamma)\hat{\psi}(S^T_{-k} A^j \gamma + \alpha)| \\
+ \sum_{j=0}^{\infty} \sum_{|k| \leq [2j/2]} |\hat{\psi}(\tilde{S}^T_{-k} \tilde{A}^j \gamma)\hat{\psi}(\tilde{S}^T_{-k} \tilde{A}^j \gamma + \alpha)| \right) < \infty,
\]

then \(SH(\phi, \psi, \tilde{\psi})\) is a Bessel system with bounds \(B\). Here \(C_p^\# := (C_p^T)^{-1}\).
What is the weighted system $W$?

- Setup: $X = L^2(\mathbb{R}^2)$, $Y = L^2(S^1 \times \mathbb{R})$, and

$$Rf(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\theta^\perp)dt$$
What is the weighted system $W$?

- Setup: $X = L^2(\mathbb{R}^2)$, $Y = L^2(S^1 \times \mathbb{R})$, and

$$Rf(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\theta^\perp)dt$$

- It can be shown that $R^{-*} = \Lambda_s R$, where $\Lambda f = \mathcal{F}^{-1}(|\gamma| \hat{f}(\gamma))$ is a Riesz potential (here on the $s$-variable)
What is the weighted system $W$?

- **Setup:** $X = L^2(\mathbb{R}^2)$, $Y = L^2(S^1 \times \mathbb{R})$, and

$$Rf(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\theta^\perp)dt$$

- It can be shown that $R^{-*} = \Lambda_s R$, where $\Lambda f = \mathcal{F}^{-1}(|\gamma| \hat{f}(\gamma))$ is a Riesz potential (here on the $s$-variable)

- A intertwining relation gives $R^{-*} = RA$
What is the weighted system $W$?

- Setup: $X = L^2(\mathbb{R}^2)$, $Y = L^2(S^1 \times \mathbb{R})$, and

$$Rf(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\theta^\perp) dt$$

- It can be shown that $R^{-*} = \Lambda_s R$, where $\Lambda f = \mathcal{F}^{-1}(|\gamma| \hat{f}(\gamma))$ is a Riesz potential (here on the $s$-variable)

- A intertwining relation gives $R^{-*} = RA$

- Since $R$ is bounded, we only have to make

$$\{\kappa_{j,k,m} \Lambda \psi_{j,k,m}\}_{j \geq 0, |k| \leq 2^{j/2}} \in \mathbb{Z}^2$$

a Bessel system for some choice of $\kappa_{j,k,m} > 0$. And similar for the vertical cones and the central box.
Since $\Lambda$ is a Fourier multiplier, it commutes with translation:

$$\Lambda T_{C_p k} g_p = T_{C_p k} \Lambda g_p$$

Hence, $\Lambda$ maps the shearlet system to a GSI system (that is not a shearlet system!) with generators $g_p = \Lambda D_{A^j S_k} \psi$. 
Inversion of the Radon Transform by Shearlet Frames

Covering of the Horizontal Cone

Plot of

$$\sum_{j=0}^{\infty} \sum_{|k| \leq 2^{j/2}} |\hat{\psi}(S_{-k}^{T}A^{j}y)|^2$$
Plot of

\[
\sum_{j=0}^{\infty} \sum_{|k| \leq \lfloor 2^{j/2} \rfloor} \kappa_j^2 |\gamma|^2 |\hat{\psi}(S^T_{-k} A^j \gamma)|^2, \quad \kappa_j = 2^{-j},
\]
Inversion of the Radon Transform by Shearlet Frames

Covering of the Horizontal Cone

Plot of

\[ \sum_{j=0}^{\infty} \sum_{|k| \leq 2^{j/2}} \kappa_j^2 |\gamma|^2 |\hat{\psi}(S_k^T A^j \gamma)|^2, \quad \kappa_j = 2^{-5j/4}, \]
Covering of the Horizontal Cone

Plot of

\[
\sum_{j=0}^{\infty} \sum_{|k| \leq [2^{j/2}]} \kappa_j^2 |\gamma|^2 |\hat{\psi}(S^T_{-k} A^j \gamma)|^2, \quad \kappa_j = 2^{-3j/4},
\]
Future tasks

- We really want to solve the inverse problem under the prior information that $f \in C \subset X$ for some image class $C$, e.g., cartoon-like images.

- If $\{\langle f, \phi_k \rangle\}_{k \in \mathbb{N}}$ belongs (after reordering descending in absolute value) to a weak $\ell_p$ space for some small $p > 0$, then so does $\{\langle Kf, K^{-*}\phi_k \rangle\}_{k \in \mathbb{N}}$.

- Understand the role of the weights $\kappa_k$. 
E. J. Candès and D. L. Donoho.
Recovering edges in ill-posed inverse problems: optimality of curvelet frames.
Thank You!

Jakob Lemvig – Technical University of Denmark (DTU)
Group: Harmonic Analysis – Theory and Applications
Email: jakle@dtu.dk.
Preprints: www.compute.dtu.dk/people/J.Lemvig