LECTURES ON MICROLOCAL CHARACTERIZATIONS IN LIMITED-ANGLE TOMOGRAPHY

Jürgen Frikel
Nov. 11: Introduction to the mathematics of computerized tomography
Nov. 18: Introduction to the basic concepts of microlocal analysis
Today: Microlocal analysis of limited angle reconstructions in tomography
Dec. 02: Wrap up & Discussion (possible Synergies, Projects, Grants) I

References:
4 Lectures

1. Nov. 11: Introduction to the mathematics of computerized tomography
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4. Dec. 02: Wrap up & Discussion (possible Synergies, Projects, Grants) I

References:


In **limited angle tomography**, the projections $g_\theta = R_\theta f$ are known only for certain directions $\theta \in S_\Phi \subset S^1$, for other directions $\theta$ the projections $g_\theta$ are unknown. In other words, in limited angle tomography we are given truncated data

$$g_\Phi(\theta, s) = \chi_{S_\Phi \times \mathbb{R}} \cdot Rf(\theta, s).$$

FBP inversion formula applied to limited angle data

$$R^\dagger g_\Phi(x) = \frac{1}{4\pi} \int_{S_\Phi^1} \psi \ast g_\theta(x \cdot \theta) \, d\theta = ???$$

What do we reconstruct?
In limited angle tomography, the projections \( g_\theta = R_\theta f \) are known only for certain directions \( \theta \in S^1_\Phi \subseteq S^1 \), for other directions \( \theta \) the projections \( g_\theta \) are unknown. In other words, in limited angle tomography we are given truncated data

\[
g_\Phi(\theta, s) = \chi_{S^1_\Phi \times \mathbb{R}} \cdot R f(\theta, s).
\]

FBP inversion formula applied to limited angle data

\[
R^\dagger g_\Phi(x) = \frac{1}{4\pi} \int_{S^1_\Phi} [\psi * g_\theta](x \cdot \theta) \, d\theta = ???
\]

What do we reconstruct?

Reconstructions for an angular range \([0^\circ, 100^\circ]\)
Observations at a first glance:

- Only certain singularities of the original object can be reconstructed
- Artifacts (new singularities) are generated
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- Only certain singularities of the original object can be reconstructed
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Goal: Use microlocal analysis to

- Characterize singularities that can be reliably reconstructed,
- Develop strategy to reduce artifacts.
Which of the original singularities are reliably reconstructed?

Reconstructions for an angular range $[0^\circ, 100^\circ]$
Which of the original singularities are reliably reconstructed?

Reconstructions for an angular range $[0^\circ, 140^\circ]$
Which of the original singularities are reliably reconstructed?

Reconstructions for an angular range $[0^\circ, 100^\circ]$
Today

- A formula for limited angle FBP reconstructions
- Characterization of visible and invisible singularities
- Severe ill-posedness of limited angle tomography
- Characterization & reduction of artifacts
**NOTATION**

We study the restricted or limited angle Radon transform

\[ R_{\Phi} f(\theta, s) = \chi_{S^1_{\Phi} \times \mathbb{R}} \cdot R f(\theta, s), \]

where \( 0 < \Phi < \pi/2 \) and

\[ S^1_{\Phi} = \{ \theta \in S^1 : \theta = \pm (\cos \varphi, \sin \varphi), \ |\varphi| < \Phi \}. \]

Moreover, we define the polar wedge

\[ W_{\Phi} = \mathbb{R} \cdot S^1_{\Phi} = \{ r\theta : r \in \mathbb{R}, \ \theta \in S^1_{\Phi} \}. \]
Let \( f \in S(\mathbb{R}^2) \). Then, the FBP reconstruction formula \( R^\dagger_\Phi g = \frac{1}{4\pi} R^\ast_\Phi \Lambda g \) and the Lambda reconstruction formula \( \Lambda_\Phi g = \frac{1}{4\pi} R^\ast_\Phi \left( -\frac{d^2}{ds^2} g \right) \) satisfy

\[
P_\Phi f = R^\dagger_\Phi (R^\ast f) \quad \text{and} \quad P_\Phi (\Lambda f) = \Lambda (P_\Phi f) = \Lambda (R_\Phi f),
\]

where \( P_\Phi f = F^{-1}(\chi_{W_\Phi} \hat{f}) \). This formula is also valid for \( f \in \mathcal{E}'(\mathbb{R}^2) \). Furthermore, the maps \( R^\dagger_\Phi R \) and \( \Lambda^\dagger_\Phi R \) are weakly continuous from \( \mathcal{E}'(\mathbb{R}^2) \) to \( S'(\mathbb{R}^2) \).
The theorem shows that a perfect reconstruction of a function $f$ is only possible if

$$\text{supp } \hat{f} \subset W_\Phi$$
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The theorem characterizes the kernel of \( R_\Phi \):
\[
R_\Phi f \equiv 0 \quad \text{for any } f \text{ with } \text{supp} \hat{f} \subset \mathbb{R}^2 \setminus W_\Phi
\]
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The theorem characterizes the kernel of $R_\Phi$:

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Reconstructions for an angular range $[-80^\circ, 80^\circ]$ ($\Phi = 80^\circ$)
REMARKS

› The theorem shows that a perfect reconstruction of a function $f$ is only possible if

\[ \text{supp } \hat{f} \subset W_\Phi \]

› The theorem characterizes the kernel of $\mathcal{R}_\Phi$:

\[ \mathcal{R}_\Phi f \equiv 0 \quad \text{for any } f \text{ with } \text{supp } \hat{f} \subset \mathbb{R}^2 \setminus W_\Phi \]

Reconstructions for an angular range $[-80^\circ, 80^\circ]$ ($\Phi = 80^\circ$)
Corollary (Quinto (1993); F., Quinto (2013))

Let \( f \in \mathcal{E}'(\mathbb{R}^2) \). Then

\[
WF(\Lambda(P\Phi f)) = WF(P\Phi f) \subset \mathbb{R}^2 \times W_\Phi.
\]

Reconstruction at the angular range \([-45^\circ, 45^\circ]\)
Corollary (Quinto (1993); F., Quinto (2013))

Let \( f \in \mathcal{E}'(\mathbb{R}^2) \). Then

\[
WF(\Lambda(P_\Phi f)) = WF(P_\Phi f) \subset \mathbb{R}^2 \times W_\Phi.
\]

We can only expect to reconstruct singularities \((x, \xi)\) where \(\xi \in W_\Phi\)

Visible singularities (red) at the angular range \([-45^\circ, 45^\circ]\)
Visible singularities

\[ \text{WF}_\Phi(f) := \{(x, \xi) \in \text{WF}(f) : \xi \in W_\Phi\} \]

Visible singularities (red) at the angular range \([-45^\circ, 45^\circ]\)
**Visible Singularities**

Visible singularities

\[ \text{WF}_{\Phi}(f) := \{(x, \xi) \in \text{WF}(f) : \xi \in W_{\Phi}\} \]

Invisible singularities, \((x, \xi)\) with \(\xi \in \mathbb{R}^2 \setminus \overline{W}_{\Phi}\), are smeared or distorted

Visible singularities (red) at the angular range \([-45^\circ, 45^\circ]\)

Visible singularities (red) at the angular range \([-45^\circ, 45^\circ]\)

\[ f \]

\[ f_{\text{FBP}} = P_{\Phi}f \]
INVISIBLE SINGULARITIES

Original

Sinogram for $[0^\circ, 70^\circ]$  

Sinogram for $[0^\circ, 120^\circ]$

FBP for $[0^\circ, 70^\circ]$  

FBP for $[0^\circ, 120^\circ]$
Remarks about ill-posedness

Recall: In case of full data we have the Sobolev-space estimates

\[ c \| f \|_{H^\alpha_0} \leq \| \mathcal{R} f \|_{H^{\alpha+1/2}} \leq C \| f \|_{H^\alpha_0} \]

That is, the tomography problem is mildly ill-posed (of order 1/2)
REMARKS ABOUT ILL-POSEDNESS

Recall: In case of full data we have the Sobolev-space estimates

\[ c \| f \|_{H^\alpha_0} \leq \| Rf \|_{H^{\alpha+1/2}} \leq C \| f \|_{H^\alpha_0} \]

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Can such an estimate hold for the limited angle Radon transform?
REMARKS ABOUT ILL-POSEDNESS

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Can such an estimate hold for the limited angle Radon transform?

- NO, such Sobolev space cannot hold (for any \( \alpha \in \mathbb{R}_+ \)) for the limited angle Radon transform \( \mathcal{R}_\Phi \)!
  Therefore, the limited angle tomography is severely ill-posed!
**Remarks about ill-posedness**

Recall: In case of full data we have the Sobolev-space estimates

\[ c \| f \|_{H^0} \leq \| \mathcal{R} f \|_{H^{\alpha+1/2}} \leq C \| f \|_{H^\alpha} \]

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Can such an estimate hold for the limited angle Radon transform?

- **NO**, such Sobolev space cannot hold (for any $\alpha \in \mathbb{R}$) for the limited angle Radon transform $\mathcal{R}_\Phi$!
  Therefore, the limited angle tomography is severely ill-posed!

- On the previous slide we have seen that for $\Phi < \pi/2$ we can always construct a function $f$ that is discontinuous, i.e., $\| f \|_{H^\alpha} = \infty$ ($f \notin H^\alpha$) for $\alpha > 1$, for which however $\mathcal{R}_\Phi f$ is smooth, i.e., $\| \mathcal{R}_\Phi f \|_{H^\alpha} < \infty$ for all $\alpha > 1$. Similar constructions can be made for all $\alpha$. Therefore, the left-hand-side Sobolev-space estimate cannot hold.
Remarks about ill-posedness

Recall: In case of full data we have the Sobolev-space estimates

\[ c \|f\|_{H^\alpha_0} \leq \|\mathcal{R} f\|_{H^\alpha+1/2} \leq C \|f\|_{H^\alpha_0} \]

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Can such an estimate hold for the limited angle Radon transform?

▷ NO, such Sobolev space cannot hold (for any \( \alpha \in \mathbb{R} \)) for the limited angle Radon transform \( \mathcal{R}_\Phi \)!
Therefore, the limited angle tomography is severely ill-posed!

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▷ We don’t have control over the Fourier region outside of \( W_\Phi \)! Here, anything can happen and that’s where the severe instabilities come from.
**Remarks about ill-posedness**

Recall: In case of full data we have the Sobolev-space estimates

\[ c \|f\|_{H_0^\alpha} \leq \|\mathcal{R}f\|_{H_{\alpha+1/2}} \leq C \|f\|_{H_0^\alpha} \]

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Can such an estimate hold for the limited angle Radon transform?

- NO, such Sobolev space cannot hold (for any \( \alpha \in \mathbb{R} \)) for the limited angle Radon transform \( \mathcal{R}_\Phi \)!
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- We don’t have control over the Fourier region outside of \( W_\Phi \)!
  Here, anything can happen and that’s where the severe instabilities come from.

- The existence of invisible singularities makes the problem severely (or exponentially) ill-posed
**Remarks about ill-posedness**

Recall: In case of full data we have the Sobolev-space estimates

\[ c \| f \|_{H^0_0} \leq \| Rf \|_{H^{1/2}} \leq C \| f \|_{H^0} \]

That is, the tomography problem is mildly ill-posed (of order 1/2)

Can such an estimate hold for the limited angle Radon transform?

- NO, such Sobolev space cannot hold (for any \( \alpha \in \mathbb{R} \)) for the limited angle Radon transform \( R_\Phi \)! Therefore, the limited angle tomography is severely ill-posed!

- On the previous slide we have seen that for \( \Phi < \pi/2 \) we can always construct a function \( f \) that is discontinuous, i.e., \( \| f \|_{H^\alpha} = \infty \) (\( f \notin H^\alpha \)) for \( \alpha > 1 \), for which however \( R_\Phi f \) is smooth, i.e., \( \| R_\Phi f \|_{H^\alpha} < \infty \) for all \( \alpha > 1 \). Similar constructions can be made for all \( \alpha \). Therefore, the left-hand-side Sobolev-space estimate cannot hold.

- We don’t have control over the Fourier region outside of \( W_\Phi \)! Here, anything can happen and that’s where the severe instabilities come from.

- The existence of invisible singularities makes the problem severely (or exponentially) ill-posed

- If one would use \( \text{supp} \hat{f} \subset W_\Phi \) or \( \text{WF}(f) \subset \mathbb{R}^2 \times W_\Phi \) as a-priori information, then we could get the same stability as in the case of full data, i.e., we can show that

\[ c \| P_\Phi f \|_{H^0_0} \leq \| R_\Phi f \|_{H^{1/2}} \]
Theorem (F., Quinto (2013); Katsevich (1997))

Let $\Phi \in [0, \pi/2)$ and let $f \in \mathcal{E}'(\mathbb{R}^2)$. Let $\mathcal{R}^\dagger$ be the FBP reconstruction operator. Then

$$\text{WF}_\Phi(f) \subset \text{WF}\left(\mathcal{R}^\dagger(\mathcal{R}_\Phi f)\right) \subset \text{WF}_\Phi(f) \cup \mathcal{A}_\Phi(f),$$

where

$$\mathcal{A}_\Phi = \left\{(x + r\vartheta(\varphi)^\perp, \alpha\vartheta(\varphi)) : (x, \vartheta(\varphi)) \in \text{WF}(f), \ r, \alpha \in \mathbb{R} \setminus \{0\}, \ \varphi = \pm\Phi\right\}$$

is the set of added singularities. Here $\vartheta(\varphi) = (\cos \varphi, \sin \varphi)$ for $\varphi \in [-\pi, \pi)$.

Artifacts are located on straight lines with normal directions $\vartheta(\pm\Phi)$. 
Added singularities

\[ \mathcal{A}_\Phi = \left\{ (x + r\theta(\varphi)^\perp, \alpha\theta(\varphi)) : (x, \theta(\varphi)) \in \text{WF}(f), \ r, \alpha \in \mathbb{R}^*, \ \varphi = \pm \Phi \right\} \]
OUTLINE OF THE PROOF

First note that

\[ \mathcal{R}^\dagger(\mathcal{R}_\Phi f) = P_\Phi f = \mathcal{F}^{-1}(\chi_{W_\Phi} \cdot \hat{f}) = \frac{1}{2\pi} \hat{\check{u}}_\Phi * f, \]

where

\[ \hat{\check{u}}_\Phi = \mathcal{F}^{-1}(\chi_{W_\Phi}). \]

Therefore

\[ \text{WF}(\mathcal{R}^\dagger(\mathcal{R}_\Phi f)) = \text{WF}(\check{u}_\Phi * f) \]
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Therefore

\[ \text{WF}(\mathcal{R}^\dagger(\mathcal{R}_\Phi f)) = \text{WF}(\check{u}_\Phi \ast f) \]

- Then, use the following general result from microlocal analysis: If either \( f \) or \( g \) or both have compact support (as distributions) then

\[ \text{WF}(f \ast g) \subset \left\{ (x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in \text{WF}(f), (y, \xi) \in \text{WF}(g) \right\} \]
**Outline of the Proof**

- First note that
  \[ \mathcal{R}^\dagger(\mathcal{R}_\Phi f) = P\Phi f = \mathcal{F}^{-1}(\chi_{W\Phi} \cdot \hat{f}) = \frac{1}{2\pi} \check{u}_\Phi * f, \]

  where
  \[ \check{u}_\Phi = \mathcal{F}^{-1}(\chi_{W\Phi}). \]

  Therefore
  \[ \text{WF}(\mathcal{R}^\dagger(\mathcal{R}_\Phi f)) = \text{WF}(\check{u}_\Phi * f) \]

- Then, use the following general result from microlocal analysis: If either \( f \) or \( g \) or both have compact support (as distributions) then
  \[ \text{WF}(f * g) \subset \{(x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in \text{WF}(f), \ (y, \xi) \in \text{WF}(g)\} \]

- Applied to our situation, we get
  \[ \text{WF}(\mathcal{R}^\dagger\mathcal{R}_\Phi f) \subset \{(x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in \text{WF}(f), \ (y, \xi) \in \text{WF}(\check{u}_\Phi)\} \]
OUTLINE OF THE PROOF

First note that
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R^\dagger(R_\Phi f) = P_\Phi f = F^{-1}(\chi_{W_\Phi} \cdot \hat{f}) = \frac{1}{2\pi} \hat{u}_\Phi * f,
\]
where
\[
\hat{u}_\Phi = F^{-1}(\chi_{W_\Phi}).
\]
Therefore
\[
WF(R^\dagger(R_\Phi f)) = WF(\hat{u}_\Phi * f)
\]

Then, use the following general result from microlocal analysis: If either \(f\) or \(g\) or both have compact support (as distributions) then
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WF(f * g) \subset \{(x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in WF(f), (y, \xi) \in WF(g)\}
\]

Applied to our situation, we get
\[
WF(R^\dagger R_\Phi f) \subset \{(x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in WF(f), (y, \xi) \in WF(\hat{u}_\Phi)\}
\]

Need to calculate \(WF(\hat{u}_\Phi)\): To that end, note that \(\hat{u}_\Phi\) is a homogeneous distribution as the (inverse) Fourier transform of the homogeneous distribution \(u_\Phi = \chi_{W_\Phi}\). Then, we can use the following general result for homogeneous distributions \(u\):
\[
(x, \xi) \in WF(u) \iff (\xi, -x) \in WF(\hat{u}) \quad \text{for} \ x \neq 0, \ \xi \neq 0
\]
\[
(0, \xi) \in WF(u) \iff \xi \in \text{sing supp}(\hat{u})
\]
OUTLINE OF THE PROOF

▷ First note that
\[ R^\dagger(R_f) = P_f = F^{-1}(\chi_{W} \cdot \hat{f}) = \frac{1}{2\pi} \hat{u} \ast f, \]
where
\[ \hat{u} = F^{-1}(\chi_{W}). \]

Therefore
\[ \WF(R^\dagger(R_f)) = \WF(\hat{u} \ast f) \]

▷ Then, use the following general result from microlocal analysis: If either \( f \) or \( g \) or both have compact support (as distributions) then
\[ \WF(f \ast g) \subset \{ (x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in \WF(f), (y, \xi) \in \WF(g) \} \]

▷ Applied to our situation, we get
\[ \WF(R^\dagger R_f) \subset \{ (x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in \WF(f), (y, \xi) \in \WF(\hat{u}) \} \]

▷ Need to calculate \( \WF(\hat{u}) \): To that end, note that \( \hat{u} \) is a homogeneous distribution as the (inverse) Fourier transform of the homogeneous distribution \( u = \chi_W \). Then, we can use the following general result for homogeneous distributions \( u \):
\[ (x, \xi) \in \WF(u) \iff (\xi, -x) \in \WF(\hat{u}) \quad \text{for } x \neq 0, \xi \neq 0 \]
\[ (0, \xi) \in \WF(u) \iff \xi \in \text{sing supp}(\hat{u}) \]

▷ To calculate \( \WF(\hat{u}) \) we therefore first calculate
\[ \WF(\chi_W) = \{(\alpha \varphi, r \theta^\perp(\varphi)) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : \alpha, r \in \mathbb{R} \setminus 0, \varphi = \pm \Phi \} \cup \{(0) \times \overline{W_{\Phi}}\} \]
To calculate $WF(\tilde{u}_\Phi)$ we first observe that outside of the origin ($x \neq 0$) we have

$$(\alpha \theta(\varphi), r\theta^\perp(\varphi)) \in WF(\chi_{W \Phi}) \quad \text{for } \alpha, r \in \mathbb{R} \setminus 0, \, \varphi = \pm \Phi$$
OUTLINE OF THE PROOF

To calculate $WF(\tilde{u}_\Phi)$ we first observe that outside of the origin ($x \neq 0$) we have

$$(\alpha \theta(\varphi), r \theta^\perp(\varphi)) \in WF(\chi_{\mathcal{W}_\Phi}) \quad \text{for} \quad \alpha, r \in \mathbb{R} \setminus 0, \varphi = \pm \Phi$$

Therefore, by previous result we have

$$WF(\tilde{u}_\Phi) = \{(r \theta^\perp(\varphi), \alpha \theta(\varphi)) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : \alpha, r \in \mathbb{R} \setminus 0, \varphi = \pm \Phi\} \cup \{0\} \times \overline{\mathcal{W}_\Phi}) =: WF_1 \cup WF_2.$$
OUTLINE OF THE PROOF

▷ To calculate $\text{WF}(\mathfrak{u}_\phi)$ we first observe that outside of the origin ($x \neq 0$) we have

$$(\alpha\theta(\varphi), r\theta^\perp(\varphi)) \in \text{WF}(\chi_{W_\phi}) \quad \text{for} \quad \alpha, r \in \mathbb{R} \setminus 0, \varphi = \pm \Phi$$

▷ Therefore, by previous result we have

$$\text{WF}(\mathfrak{u}_\phi) = \left\{ (r\theta^\perp(\varphi), \alpha\theta(\varphi)) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : \alpha, r \in \mathbb{R} \setminus 0, \varphi = \pm \Phi \right\} \cup \{(0) \times \overline{W_\phi}) \}

=: \text{WF}_1 \cup \text{WF}_2$$

▷ We now apply the result about the wavefront set of convolutions (see previous slide) and obtain the assertion

$$\text{WF}(R^\dagger R_\phi f) \subset \left\{ (x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in \text{WF}(f), (y, \xi) \in \text{WF}(\mathfrak{u}_\phi) \right\}

\subset \left\{ (x + r\theta^\perp(\varphi), \alpha\theta(\varphi)) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : \alpha, r \in \mathbb{R} \setminus 0, (x, \alpha\theta(\varphi)) \in \text{WF}(f) \varphi = \pm \Phi \right\}

\quad \cup \left\{ (x, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in \text{WF}(f), \xi \in \overline{W_\phi} \right\}

= \mathcal{A}_\Phi \cup \text{WF}_{\Phi}(f)$$

$\square$
WHAT IS THE CAUSE OF ARTIFACTS?

First observe that if we had
\[ A \Phi = \left\{ (x+y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in \text{WF}(f), (y, \xi) \in \text{WF}(\tilde{u} \Phi), y \neq 0 \right\}, \]

Therefore \[ A \Phi = \emptyset \] only if \[ \text{sing supp} \tilde{\kappa} \Phi = \{0\}. \]

To avoid the generation of additional artifacts, the idea is to develop an FBP type reconstruction formula
\[ R^*P R f = \frac{1}{4\pi} f \ast \tilde{\kappa} \Phi, \]

such that \[ \tilde{\kappa} \Phi \] is a homogeneous distribution with a smooth Fourier transform away from origin (then \[ \text{sing supp} \tilde{\kappa} \Phi = \{0\} \]).

Alternatively, since we know that pseudodifferential operators do not increase wavefront sets, we could formulate the artifact reduction strategy in a more abstract way as follows: Design an FBP reconstruction operator \[ R^*P \] such that \[ R^*P R \] is a standard pseudodifferential operator, then
\[ \text{WF}(R^*P R f) \subset \text{WF}(f). \]
What is the cause of artifacts?

First observe that if we had
\[ \mathcal{A}_\Phi = \{(x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in \text{WF}(f), (y, \xi) \in \text{WF}(\tilde{u}_\Phi), y \neq 0\}, \]

Therefore \( \mathcal{A}_\Phi = \emptyset \) only if \( \text{sing supp} \tilde{\kappa}_\Phi = \{0\} \).

Alternatively, since we know that pseudodifferential operators do not increase wavefront sets, we could formulate the artifact reduction strategy in a more abstract way as follows: Design an FBP reconstruction operator \( R^* \) such that \( R^* \tilde{\kappa}_\Phi \) is a homogeneous distribution with a smooth Fourier transform away from origin (then \( \text{sing supp} \tilde{\kappa}_\Phi = \{0\} \)).
What is the cause of artifacts?

➤ First observe that if we had

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➤ Therefore \( \mathcal{A}_\Phi = \emptyset \) only if \( \text{sing supp } \tilde{u}_\Phi = \{0\} \)
What is the cause of artifacts?

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\[ A_{\Phi} = \left\{ (x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in \text{WF}(f), (y, \xi) \in \text{WF}(\tilde{u}_{\Phi}), y \neq 0 \right\}, \]

Therefore \( A_{\Phi} = \emptyset \) only if \( \text{sing supp} \, \tilde{u}_{\Phi} = \{0\} \).

To avoid the generation of additional artifacts, the idea is to develop an FBP type reconstruction formula

\[ R^* P R f = \frac{1}{4\pi} f * \tilde{\kappa}_{\Phi}, \]

such that \( \tilde{\kappa}_{\Phi} \) is a homogeneous distribution with a smooth Fourier transform away from origin (then \( \text{sing supp} \, \tilde{\kappa}_{\Phi} = \{0\} \)).
**What is the cause of artifacts?**

- First observe that if we had

  $\mathcal{A}_\Phi = \{(x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : (x, \xi) \in \text{WF}(f), (y, \xi) \in \text{WF}(\tilde{u}_\Phi), y \neq 0\}$,

- Therefore $\mathcal{A}_\Phi = \emptyset$ only if $\text{sing supp } \tilde{u}_\Phi = \{0\}$

- To avoid the generation of additional artifacts, the idea is to develop an FBP type reconstruction formula

  $$ \mathcal{R}^* P \mathcal{R} f = \frac{1}{4\pi} f * \check{\kappa}_\Phi, $$

  such that $\check{\kappa}_\Phi$ is a homogeneous distribution with a smooth Fourier transform away from origin (then $\text{sing supp } \check{\kappa}_\Phi = \{0\}$).

- Alternatively, since we know that pseudodifferential operators do not increase wavefront sets, we could formulate the artifact reduction strategy in a more abstract way as follows: Design an FBP reconstruction operator $\mathcal{R}^* P$ such that $\mathcal{R}^* P \mathcal{R}$ is a standard pseudodifferential operator, then $\text{WF}(\mathcal{R}^* P \mathcal{R} f) \subset \text{WF}(f)$. 
Theorem (F., Quinto (2013))

Let $\kappa : S^1 \rightarrow \mathbb{R}$ be a smooth function with $\text{supp}(\kappa) \subset \text{cl}(S_{\Phi}^1)$ and assume $\kappa = 1$ on $S_{\Phi'}^1$ for some $\Phi' \in (0, \Phi)$. Let $\mathcal{K}$ be the operator that multiplies by $\kappa$

$$\mathcal{K}g(\theta, s) = \kappa(\theta)g(\theta, s).$$

Then, the operator

$$\mathcal{R}^\dagger \mathcal{K} \mathcal{R}_\Phi$$

is a standard pseudodifferential operator and for $f \in \mathcal{E}'(\mathbb{R}^2)$,

$$\text{WF}_{\Phi'}(f) \subset \text{WF}(\mathcal{R}^\dagger \mathcal{K}(\mathcal{R}_\Phi f)) \subset \text{WF}_\Phi(f).$$

The reconstruction formula $\mathcal{R}^\dagger \mathcal{K}(\mathcal{R}_\Phi)$ does not produce additional artifacts!
\[ R^\dagger KR_\Phi f = \frac{1}{4\pi} R^* I^{-1} KR_\Phi f \]
Remarks

\[ R^\dagger K R_\Phi f = \frac{1}{4\pi} R^* I^{-1} K R_\Phi f \]

- Preprocessing of limited angle data data \( g_\Phi = R_\Phi f \):
  \[ \tilde{g}_\Phi(\theta, s) = \kappa_\Phi(\theta) \cdot g_\Phi(\theta, s) \]
Remarks

\[
\mathcal{R}^t \mathcal{K} \mathcal{R}_f = \frac{1}{4\pi} \mathcal{R}^* \mathcal{I}^{-1} \mathcal{K} \mathcal{R}_f
\]

- **Preprocessing** of limited angle data data \( g_\Phi = \mathcal{R}_\Phi f \):

  \[
  \tilde{g}_\Phi(\theta, s) = \kappa_\Phi(\theta) \cdot g_\Phi(\theta, s)
  \]

- **Modification of the FBP filter** in the Fourier domain:

  \[
  \hat{\psi}(\theta, r) = |r| \quad \mapsto \quad \hat{\psi}_\Phi(\theta, r) = \kappa_\Phi(\theta) |r|
  \]
Remarks

\[ \mathcal{R}^\dagger \mathcal{K} \mathcal{R}_\Phi f = \frac{1}{4\pi} \mathcal{R}^* \mathcal{I}^{-1} \mathcal{K} \mathcal{R}_\Phi f \]

- **Preprocessing** of limited angle data data \( g_\Phi = \mathcal{R}_\Phi f \):
  \[ \tilde{g}_\Phi(\theta, s) = \kappa_\Phi(\theta) \cdot g_\Phi(\theta, s) \]

- **Modification of the FBP filter** in the Fourier domain:
  \[ \hat{\psi}(\theta, r) = |r| \quad \mapsto \quad \hat{\psi}_\Phi(\theta, r) = \kappa_\Phi(\theta) |r| \]

- **Preconditioner** for the limited angle Radon transform:
  \[ \mathcal{R}_\Phi \quad \mapsto \quad \mathcal{K} \mathcal{R}_\Phi \]
NUMERICAL EXAMPLES

Original

FBP

artifact reduced FBP
NUMERICAL EXAMPLES

Original

FBP

artifact reduced FBP
NUMERICAL EXAMPLES

FBP

artifact reduced FBP

Difference
NUMERICAL EXAMPLES

Lambda

artifact reduced Lambda

Difference
Thanks!