ON THE USE OF HIGHLY DIRECTIONAL REPRESENTATIONS IN INCOMPLETE DATA TOMOGRAPHY

Jürgen Frikel

Insights and algorithms for incomplete data tomography
DTU Compute

Incomplete data in tomography

Microlocal Analysis

Microlocal characterization of incomplete data reconstructions

Use of directional representations in incomplete data tomography
**X-ray tomography:** Classical Radon transform

\[
\mathcal{R} f(\theta, p) = \int_{\mathbb{R}} f(p\theta + t\theta^\perp) \, dt = \ln \left( \frac{I_0}{I(\theta, p)} \right)
\]

**Notation:** \( p \in \mathbb{R}, \, \theta = (\theta_1, \theta_2) \in S^1 \) and \( \theta^\perp = (-\theta_2, \theta_1) \)
Photoacoustic tomography: Spherical Radon transform

\[ Mf(\xi, r) = \int_{S^1} f(\xi + r\zeta) \, d\zeta \]

Notation: \( r > 0, \theta = (\theta_1, \theta_2) \in S^1 \)
INCOMPLETE DATA IN TOMOGRAPHY

full data
sparse angle data
limited angle data

Breast tomosynthesis
Dental CT
Electron microscopy
... and many more
LIMITED ANGLE LAMBDA RECONSTRUCTIONS

Original

X-ray tomography, [0°, 140°]

Photoacoustic tomography, [−45°, 225°]
LIMITED ANGLE LAMBDA RECONSTRUCTIONS

X-ray tomography\textsuperscript{1}, $[0^\circ, 140^\circ]$  

Photoacoustic tomography\textsuperscript{2}, $[-45^\circ, 225^\circ]$

Data by courtesy of \textsuperscript{1}Department of Diagnostic and Interventional Radiology, TUM and \textsuperscript{2}Helmholtz Zentrum München
Observations:

- Only certain features of the original object can be reconstructed,
- Artifacts are generated.
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Need to characterize visible singularities and artifacts (microlocal analysis)

- Facilitate better interpretation of reconstructions,
- Design new improved reconstruction methods (reduction artifacts, design priors, etc.).
Challenges in Limited View Tomography

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Need mathematical tools to implement these insights into algorithms

- Applied harmonic analysis provides highly directional and numerically efficient representations.
Microlocal analysis in tomography


*Microlocal analysis, visible singularities and artifacts in other tomography problems:*
WHAT ARE SINGULARITIES?

Practically: Density jumps, boundaries between regions

Mathematically: Where the function is not smooth...

Paradigm: Fourier transform of $f$ decays rapidly at $\infty$ iff $f$ is smooth.
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Paradigm: Fourier transform of $f$ decays rapidly at $\infty$ iff $f$ is smooth.

Singularities are local and oriented! $\rightsquigarrow$ Wavefront set: localize & microlocalize
Definition (Wavefront set)
A tuple \((x_0, \xi_0) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\}\) is not in the wavefront set \(WF(f)\) of \(f \in \mathcal{D}'(\mathbb{R}^2)\) iff

- there is a cut-off function \(\varphi \in \mathcal{D}(\mathbb{R}^2), \varphi(x_0) \neq 0\), (Localize at \(x_0\))
- there is a conic neighborhood \(\mathcal{N}(\xi_0)\), (Microlocalize at \(\xi_0\))

such that \(\mathcal{F}(\varphi f)\) decays rapidly in \(\mathcal{N}(\xi_0)\).

(Hörmander '90)
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WF simultaneously describes locations and directions of a singularity
**Microlocal Analysis**

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\(\text{WF}\) simultaneously describes locations and directions of a singularity

**Example:**

\(\Omega \subset \mathbb{R}^2\) such that the boundary \(\partial \Omega\) is a smooth manifold:

\[(x, \xi) \in \text{WF}(\chi_\Omega) \iff x \in \partial \Omega, \text{ and } \xi \in N_x,\]

where \(N_x\) is the normal space to \(\partial \Omega\) at \(x \in \partial \Omega\).
**MICROLOCAL ANALYSIS**

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- Singularities of \(f\)
- Directional representations for incomplete data tomography
General Setup

**Forward operator** is a Fourier Integral operator (FIO)

\[ T : \mathcal{E}'(\Omega) \rightarrow \mathcal{E}'(\Xi), \]

where \( \Omega \) is the object space and \( \Xi \) is the data space.
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**Reconstruction problem:** Recover \( f \) (or singularities \( f \)) from the data \( g = T f \)

- Limited data: \( g(y) \) known only for \( y \in A \subset \Xi \) (\( \chi_A \) = characteristic function of \( A \))
- Limited data forward operator: \( T_A f = \chi_A T f \)
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**Reconstruction operators (FBP type):**

\[ B g_A = T^* P g_A, \quad g_A = T_A f \]

\( P \) is a pseudodifferential operator and \( T^* \) dual (or backprojection) operator.
Theorem (Quinto, JF 2013-2015)

Let $T \in \{R, M\}$, $f \in \mathcal{E}'(\Omega)$, and let $P$ be a pseudodifferential operator on $\mathcal{D}'(\Xi)$. Then,

$$\text{WF}(T^* PT_A f) \subset \text{WF}_{[a,b]}(f) \cup \mathcal{A}_{[a,b]}(f).$$
**Visible Singularities and Added Artifacts**

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Visible singularities for $R$:

$$\text{WF}_{[a,b]}(f) := \{(x, \xi) \in \text{WF}(f) : \xi = \alpha \theta(\phi), \alpha \neq 0, \phi \in [a, b]\}.$$
VISIBLE SINGULARITIES AND ADDED ARTIFACTS

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Added singularities for $R$:

$$\mathcal{A}_{[a,b]}(f) = \{(x + t \theta^\perp(\varphi), \alpha \theta(\varphi) \, dx) : \varphi \in \{a, b\}, \alpha, t \neq 0, x \in L(\varphi, s), (x, \alpha \theta(\varphi)) \in \text{WF}(f)\}$$
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**Theorem (Quinto, JF)**

Under additional assumptions on \( P \) we have

\[
\text{WF}_{(a,b)}(f) \subset \text{WF}(T^* P T_A f) \subset \text{WF}_{[a,b]}(f) \cup \mathcal{A}_{[a,b]}(f).
\]
Visible singularities are characterized in terms of their orientation.

Only singularities \((x, \theta(\varphi)) \in WF(f)\) can be reconstructed for which \(\varphi \in [a, b]\).
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Only singularities \((x, \theta(\varphi)) \in WF(f)\) can be reconstructed for which \(\varphi \in [a, b]\)

Artifacts are spread along lines having orientations corresponding to the boundary of the angular range, \(\theta(a)\) or \(\theta(b)\), respectively

Streaks are added at location \(x\) whenever \((x, \theta(a)) \in WF(f)\) or \((x, \theta(b)) \in WF(f)\)
WHAT DO WE LEARN?

- Microlocal characterisations provide insight into the information content of incomplete data.

- X-ray tomography:
  - Reliably reconstructed singularities are \((x, \theta(\varphi)) \in \text{WF}(f_{\text{rec}})\) with \(\varphi \in (a, b)\),
  - Any singularity \((x, \theta(\varphi)) \in \text{WF}(f_{\text{rec}})\) with \(\varphi \notin \{a, b\}\) can be an added streak artifact.

- To avoid generation of artifacts we need to make sure that
  \[
  \text{WF}(f_{\text{rec}}) \subset \mathbb{R}^2 \times (a, b)
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How can we integrate this directional a priori information into the reconstruction?

\[\sim\] Need directional transforms that can simultaneously localize and microlocalize

\[\sim\] Shearlets, curvelets, or similar transforms
The shearlet / curvelet dictionary

\[ \{ \psi_{a,b,\theta} \}_{(a,b,\theta) \in I} \]

simultaneously localize at location \( a \) and along direction \( \theta \).

\( a = \text{scale}, \quad b = \text{location}, \quad \theta = \text{orientation} \).

(Tight frame property: For each \( f \in L^2(\mathbb{R}^2) \) we have

\[ f = \sum_{(a,b,\theta) \in I} \langle f, \psi_{a,b,\theta} \rangle \psi_{a,b,\theta}, \quad \| f \|_2^2 = \sum_{(a,b,\theta) \in I} \| \langle f, \psi_{a,b,\theta} \rangle \|_2^2 \]

Optimally sparse representation of edges (cartoon images)

Theorem (Resolution of the wavefront set)

\( (b,\theta) < \text{WF}(f) \Leftrightarrow \langle f, \psi_{a,b,\theta} \rangle \text{ decays rapidly as } a \to 0 \)

(Candes, Donoho, Kutyniok, Lemvig, Lim, Grohs, Guo, Labate, Easley, ... )
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Optimally sparse representation of edges (cartoon images)

**Theorem (Resolution of the wavefront set)**

\( (b, \theta) \notin WF(f) \iff \langle f, \psi_{a,b,\theta} \rangle \text{ decays rapidly as } a \to 0 \)

(Candes, Donoho, Kutyniok, Lemvig, Lim, Grohs, Guo, Labate, Easley, ... )
Definition (Visible coefficients)
We define the index set of visible coefficients at limited angular range \([a, b]\) as

\[ I_{[a,b]} = \{(a, b, \theta) \in I : \theta \in [a, b]\}. \]

Coefficients with \((a, b, \theta) \in I \setminus I_{[a,b]}\) are called invisible at limited angular range \([a, b]\).
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Decomposition into a visible and an invisible part

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f = \sum_{(a, b, \theta) \in I_{[a,b]}} \langle f, \psi_{(a,b,\theta)} \rangle \psi_{(a,b,\theta)} + \sum_{(a, b, \theta) \in I \setminus I_{[a,b]}} \langle f, \psi_{(a,b,\theta)} \rangle \psi_{(a,b,\theta)}
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\]

\[= f_{\text{visible}} + f_{\text{invisible}}.\]

**Dimensionality reduction:** reconstruct only the visible part

(works with any reconstruction algorithm)
Dimensions of the reconstruction problem in the curvelet domain for an image of size $256 \times 256$. The plot shows the dependence of the full dimension $-$ $-$ $-$ and reduced (adapted) dimension $\text{red}$ on the available angular range $[0, \Phi]$. 
**Numerical Experiments**

Sparse regularization

\[
\hat{c} = \arg \min_c \left\{ \| R T^* c - y^\delta \|_2^2 + \| c \|_{\ell^1_w} \right\}, \quad \hat{f} = T^* \hat{c} = \sum_{\gamma} \hat{c}_\gamma \psi_\gamma.
\]

(JF, 2013; Vandeghinste et al., 2013; Wieczorek et al., 2015)
Reconstruction of the Brainstem image of size $300 \times 300$ using curvelet sparse regularisation (CSR) and adapted curvelet sparse regularisation (ACSR):

Angular range $[0^\circ, 160^\circ]$, $\Delta \theta = 1^\circ$, Noise level 2%.
REAL DATA RECONSTRUCTIONS

No artifact reduction

CT data\(^1\) of an abdomen examination; limited angular range \(\sim 140^\circ\).

\(^{1}\) Data by courtesy of Dr. Peter Noël (Department of Diagnostic and Interventional Radiology, TUM).
REAL DATA RECONSTRUCTIONS

With artifact reduction

CT data\(^1\) of an abdomen examination; limited angular range \(\sim 140^\circ\).

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SUMMARY

- Microlocal is a powerful framework for characterisation of incomplete data reconstructions in tomography
  - Visible singularities
  - Added artifacts

- Harmonic analysis provides tools and makes microlocal insights accessible algorithmically
  - Shearlets, curvelets or similar dictionaries
  - Dimensionality reduction and artifact reduction in limited angle x-ray tomography
Thank you!