Discretization of CT Problems

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Some Notation

To represent the scanned object and the projection data in the computer, both must be discretized.

- **Object**: the physical reality is typically continuous, the way of discretization can be chosen freely.

- **Projection data**: the measurements are already discrete (finite set of angles, finite set of detectors).

- **Forward transform**: the Radon transform must be discretized, defining how line integrals are computed or approximated on discrete images.
Image Coordinate Systems

Pixel coordinates — Euclidean coordinates — matrix coordinates

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Pixel Coordinates

- Pixel model is used to store images in computer memory and to display them on the screen.
- Integer coordinates \((v, w)\), where \(v\) denotes the column and \(w\) the row (other conventions exist).
- Coordinates start at 0.
- Vertical: top to bottom.
- Horizontal: left to right.
Euclidean Coordinates

- Euclidean coordinate system is used to perform geometrical operations.
- Real-values coordinates \((a, b)\), where \(a\) denotes the horizontal coordinate and \(b\) denotes the vertical coordinate.
- Coordinates are signed real values.
- Vertical: bottom to top.
- Horizontal: left to right.
Matrix Indices

- Matrix model is used to perform matrix computations (e.g., in MATLAB).
- Integer coordinates $\square_{ij}$ where $i$ denotes the row and $j$ denotes the column.
- Coordinates start at 1.
- Horizontal: left to right.
- Vertical: top to bottom.
A Pixel as a Constant Function

\[ \text{pixel } \pi = (\pi_{x_1}, \pi_{y_1}, \pi_{x_2}, \pi_{y_2}) \]

\[ \text{pixel region } \quad R_\pi = \{(x, y) \in \mathbb{R}^2 : \pi_{x_1} \leq x < \pi_{x_2} \text{ and } \pi_{y_1} \leq y < \pi_{y_2}\} \]

\[ \text{pixel indicator function } \quad f_\pi(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R_\pi \\ 0 & \text{otherwise} \end{cases} \]

Given a set of pixels \( \{\pi_1, \ldots, \pi_n\} \) and a corresponding set of grey levels \( \{x_1, \ldots, x_n\} \) we define the image function

\[ f_i(x, y) = \sum_{i=1}^n x_i f_{\pi_i}(x, y) . \]

This is a piecewise constant function on the pixel grid.
Intersection of Line and Pixel

Recall our definition of the line

\[ L_{\theta, s} = \{(x, y) \in \mathbb{R}^2 : x \cos \theta + y \sin \theta = s\} . \]

For any pixel \( \pi \) and line \( L_{\theta, s} \), the intersection \( L_{\theta, s} \cap R_{\pi} \) is a (possibly empty) line segment, which we call the intersection segment of \( L \) and \( \pi \).

The intersection length \( c(L_{\theta, s}, \pi) \) (for line \( L_{\theta, s} \) and pixel \( \pi \)) is the length of this segment. We then have

\[
c(L_{\theta, s}, \pi) = \int_t f_\pi(s \cos \theta - t \sin \theta, s \sin \theta + t \cos \theta) \, dt = \mathcal{R}\{f_\pi\}(\theta, s) .
\]

This is actually the value of the Radon transform applied to the pixel indicator function \( f_\pi \) sampled at the point \((\theta, s)\) in the sinogram.
Common choice: center grid around the origin, use unit squares as pixels.

This corresponds to the standard representation of $M \times N$ bitmap images in a computer (as a 2D array of pixels).

We number the pixels by two pixel coordinates:

$$(v, w) \quad v = 0, 1, 2, \ldots, N - 1 \quad w = 0, 1, 2, \ldots, M - 1.$$ 

The pixel region is then given by

$$\begin{align*}
\pi_{x_1} &= -N/2 + v \\
\pi_{x_2} &= -N/2 + v + 1 \\
\pi_{y_1} &= M/2 - w - 1 \\
\pi_{y_2} &= M/2 - w
\end{align*}$$

This configuration of pixel regions is the \textit{standard grid} of size $M \times N$. 
Projection Models

- Line model
- Strip model
- Interpolation (or Joseph) model
The Line Model

Using that $f = f_i$ we obtain the line model given by

$$\mathcal{R}\{f_i\}(\theta, s) = \sum_{i=1}^{n} x_i \mathcal{R}\{f_{\pi_i}\}(\theta, s) = \sum_{i=1}^{n} c(L_{\theta,s}, \pi_i) x_i .$$

Hence, the measurement $\mathcal{M}\{f_i\}(\theta, s)$ is a linear combination of the pixel values $x_i$,

$$\mathcal{M}\{f_i\}(\theta, s) = \sum_{i=1}^{n} c(\theta, s, \pi_i) x_i ,$$

where $c(L_{\theta,s}, \pi) = \text{intersection length}$ (see next slide). Sometimes/often referred to as Siddon’s method. It is exact for piecewise constant $f$.

For piecewise constant functions $f_i$, analytical operations with the Radon transform are formulated as a finite series of multiplications and additions.

As explained shortly, this links the analytical model of the Radon transform with the field of linear algebra.
The Geometry of the Line Model

\[ x_i = i - \frac{1}{2}, \quad y_j = j - \frac{1}{2} \]

\[ R(\theta, s) = \sum_{i=1}^{n} c(L_{\theta,s}, \pi_i) x_i \]
The Strip Model

Each *detector element* has a finite size and simultaneously measures the intensity for all lines (X-rays) that intersect with it.

Suppose that a set of parallel lines \( \{L_{\theta,s} : s_1 \leq s \leq s_2\} \) intersects with a particular detector element. In this case

\[
\mathcal{M}\{f_i\}(\theta, s) = \sum_{i=1}^{n} a(\theta, s, \pi_i) x_i ,
\]

with the line-pixel coefficient for image pixel \( \pi_i \) given by

\[
a(\theta, s, \pi_i) = \int_{s' = s_1}^{s_2} c(L_{\theta,s'}, \pi_i) \, ds'.
\]

This equals the *area of intersection* between pixel \( \pi_i \) and the area covered by the set of parallel lines surrounding \( L_{\theta,s} \).

This model is also exact for piecewise constant \( f \).
The Interpolation Model (or Joseph Model)

Based on an image representation where the value of the image function is specified in the center points of each pixel.

Key idea: put an artificial pixel over the line $L_{\theta,s}$ with a constant intensity, in such a way that we can use the line model within this artificial pixel.

The intensity value associated with the artificial pixels is found by linear interpolation between to neighbour pixels, either in the same row column.
If the line $L_{\theta,s}$ makes an angle with the $x$-axis within $[-90^\circ, 90^\circ]$ then two neighbouring pixels in the same column $v$ are used, otherwise two neighbouring pixels in the same row $w$ are used.

Consider the latter case (cf. next slide). The pixel midpoints of row $w$ are at the horizontal center line at $\tilde{y} = M/2 - w - \frac{1}{2}$.

To determine the intersection between this horizontal line and the line $L_{\theta,s}$, we insert $\tilde{y}$ into the line’s equation $\tilde{x} \cos \theta + \tilde{y} \sin \theta = s$, yielding

$$\tilde{x} = \frac{s - (M/2 - w - \frac{1}{2}) \sin \theta}{\cos \theta}.$$

This determines the point $(\tilde{x}, \tilde{y})$ where $L_{\theta,s}$ intersects with the horizontal center line of row $w$. 
Inderpolation or Joseph model
\((x(y_j), y_j) = (\tilde{x}, \tilde{y})\)

\[\begin{align*}
  x_i &= i - \frac{1}{2}, \\
  y_j &= j - \frac{1}{2}
\end{align*}\]

\((x_i - 1, y_j), (x_i, y_j)\)

\((x(y_j), y_j) = (\tilde{x}, \tilde{y})\)

\(\cos \mu_j = \) length of line line segment > 1

\(f(x(y_j), y_j)\) linearly interpolated from \(f(x_{i-1}, y_j)\) and \(f(x_i, y_j)\)
Details of the Interpolation Model

Now perform linear interpolation between the pixel values of the two pixels in row \( w \), with column coordinates \( v_{\text{left}} \) and \( v_{\text{right}} = v_{\text{left}} + 1 \)

Recalling that the pixels have unit size, the two interpolation weights are the two distances \( \tilde{x} - v_{\text{left}} \) and \( v_{\text{right}} - \tilde{x} \). Hence, the interpolated contribution from \( L_{\theta,s} \) at row \( w \) is

\[
(\tilde{x} - v_{\text{left}}) f(w, v_{\text{right}}) + (v_{\text{right}} - \tilde{x}) f(w, v_{\text{left}}).
\]

The length of the line segment of \( L_{\theta,s} \) in the artificial pixel is \( a/|\cos \theta| \). Hence, the associated line-pixel coefficients are

\[
(\tilde{x} - v_{\text{left}})/| \cos \theta | \quad \text{and} \quad (v_{\text{right}} - \tilde{x})/| \cos \theta |.
\]

Gives exact data only for the very special case where the image is linear in \( x \) and \( y \), i.e., \( f(x, y) = \alpha x + \beta y + \gamma \) where \( \alpha, \beta \) and \( \gamma \) are constants.
The system matrix $A$ contains one row for each measurement and one column for each pixel in the image. The entries of the system matrix are the line-pixel coefficients $a(\theta, s, \pi)$ associated with the chosen projection model. Specifically, for the line $L_{\theta,s}$ we define the row vector:

$$a_{\theta,s} = (a(\theta, s, \pi_1) \ a(\theta, s, \pi_2) \ \cdots \ a(\theta, s, \pi_n))$$

which contains the line-pixel coefficients of the line $L_{\theta,s}$ for all pixels in $\Pi$. Thus we can express the measurement as an inner product:

$$b_i = M(f_i)(\theta, s) = a_{\theta,s} \cdot x.$$ 

We assemble all the row vectors $a_{\theta,s}$ into the system matrix $A$. Each row corresponds to a single detector element at a given angle. The measurements jointly form the vector $b \in \mathbb{R}^m$, such that

$$b = A x .$$

The CT reconstruction problem amounts to solving this system for $x$. 

Consider a $2 \times 2$ image, a detector with 4 elements, and a single projection angle $\theta = 30^\circ$. Discretization by the line model gives a system matrix $A$ of full rank, and hence we can reconstruct the image from a single projection.

While this seems to contradict Radon’s fundamental results, such a single-projection reconstruction is possible due to the assumption $f = f_i$. Below is a larger example:
**Image Discretization Issues**

**Pixel size:** It is common practice to choose the pixel size of the image roughly equal to the width of a detector element, to avoid degenerate numerical behavior in the system of equations.

**Image size (i.e., size of image grid):**
- Choosing this size much larger than the object will just reduced the image quality in the relevant domain of the reconstruction.
- On the other hand, choosing the image size smaller than the support of the object will lead to reconstruction artifacts because the actual object that gave rise to the measured data cannot be represented on the chosen pixel grid. An alternative way to say this is that the data vector $b$ is not in the range of the system matrix $A$.
- As a rule of thumb, it is recommended to choose the size of the pixel grid as tight as possible round the object.
The Rows and Columns of the System Matrix

We write the system matrix as

\[
A = \begin{pmatrix}
\vdots & r_1 & \vdots \\
\vdots & \ddots & \vdots \\
\vdots & r_m & \vdots
\end{pmatrix} = \begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\]

The matrix \( A \) maps the discretized absorption coefficients (the vector \( x \)) to the data in the detector pixels (the elements of the vector \( b \)) via

\[
b = \begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{pmatrix} = A x = x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + \cdots + x_n \mathbf{c}_n = \begin{pmatrix}
r_1 \cdot x \\
r_2 \cdot x \\
\vdots \\
r_m \cdot x
\end{pmatrix}
\]

linear combination of columns
The Columns

If the image consists of a single white pixel on a black background, then the corresponding vector $\mathbf{x}$ is all zeros except of a single element $x_j = 1$ in position $j$.

The corresponding right-hand side is

$$
\mathbf{b} = 0 \mathbf{c}_1 + \cdots + 0 \mathbf{c}_{j-1} + 1 \mathbf{c}_j + 0 \mathbf{c}_{j+1} \cdots + 0 \mathbf{c}_n = \mathbf{c}_j.
$$

This means that the $j$th column of $\mathbf{A}$ is the image of a single pixel.
The Rows

The $i$th row of $\mathbf{A}$ maps the image $\mathbf{x}$ to the $i$th detector element,

$$b_i = \mathbf{r}_i \cdot \mathbf{x} = \sum_{j=1}^{n} a_{ij} x_j , \quad i = 1, 2, \ldots, m .$$

This inner product approximates the line integral along ray $i$ in the Radon transform (see example next slide).

The nonzeros of $\mathbf{r}_i$ correspond to those image pixels that are intersected by the $i$th X-ray. Hence, if we reshape $\mathbf{r}_i$ and plot it as a 2D image then we get a picture of the $i$th ray’s path through the object.
Interpretation of a Row in the System Matrix

In this small example:

\[ a_{ij} = \text{length of ray } i \text{ in pixel } j \]

\[ \mathbf{r}_i = [ a_{i1} \ a_{i2} \ 0 \ a_{i4} \ 0 \ 0 \ a_{i7} \ 0 \ 0 ] \]

\[ b_i = \mathbf{r}_i \cdot \mathbf{x} = a_{i1}x + a_{i2}y + a_{i4}x_4 + a_{i7}x_7 \]
Back Projection

Recall that the back projection,

\[ f(\xi) = \int_{0}^{2\pi} g(x \cos \theta + y \sin \theta, \theta) \, d\theta, \]

means that we integrate the sinogram \( g \) along a sinusoidal curve.

Multiplication with the matrix transpose performs this operation:

\[
\mathbf{x} = \mathbf{A}^T \mathbf{b} = \begin{pmatrix} | & | & | \\
\mathbf{c}_1 & \cdots & \mathbf{c}_n \\
| & | & |
\end{pmatrix}^T \begin{pmatrix} | & | & | \\
\mathbf{c}_1^T & \cdots & \mathbf{c}_n^T \\
| & | & |
\end{pmatrix} \mathbf{b} = \begin{pmatrix} | & | \end{pmatrix} \mathbf{b} = \begin{pmatrix} \mathbf{c}_1 \cdot \mathbf{b} \\
\vdots \\
\mathbf{c}_n \cdot \mathbf{b} \end{pmatrix},
\]

where each inner product \( \mathbf{c}_j \cdot \mathbf{b} \) corresponds to the above integration.