X-ray Computed Tomography: Forward problem and FBP reconstruction

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02946 Scientific Computing for Computed Tomography

January 2023

SC for CT, Week 1, Mon+Tues: Overview

Monday

- 09.00 Lecture – Lambert-Beer's Law and the Radon Transform
- 10.15 Exercises
- 11.15 Lecture – Reconstruction by Filtered Back-Projection (FBP)
- 12.00 - - - Lunch break - - -
- 13.00 Scan at DTU 3D Imaging Center / FBP exercises
- 15.00 FBP exercises / Scan at DTU 3D Imaging Center

Tuesday

- 09:00 Real data reconstruction 2D (MATLAB)
- 11:00 Real data reconstruction 2D/3D - Core Imaging Library
- 12.00 - - - Lunch break - - -
- 13.00 Real data continued - reconstruct your own data?
- 14:30 Micro project - intro and group formation

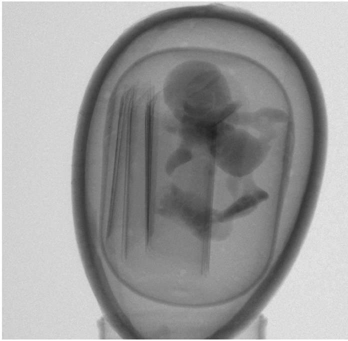
Outline

Tomography

What is tomography?

- From Greek: *Tomos* a section or slice, *Graphos* to describe.
- Imaging of slices of an object – without actually slicing it!
- These days not just slices but 3D images can be obtained.
- To see the inside, need information obtained from the outside.
- Some applications of tomography . . .

Look inside Kinder Surprise Egg without opening it



Tomographic reconstruction: Projections → 3D interior model

Medical Imaging

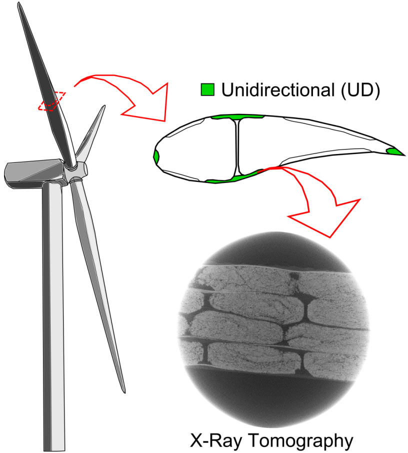
Non-Destructive Inspection and Testing

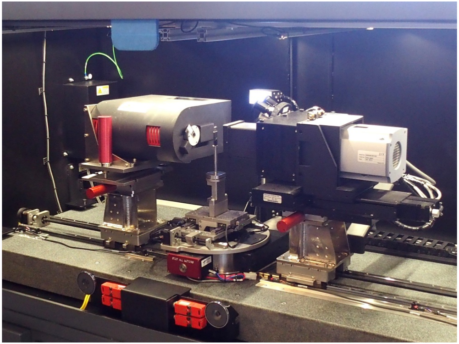
Production, security, metrology, etc.

Example: airport luggage scanner for threat detection.

Materials Science

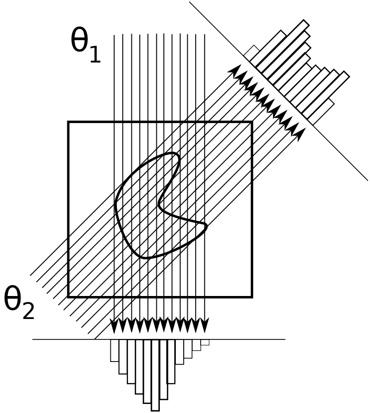
Development of advanced materials requires understanding their properties at the micro and nano scale.
Example: maximize strength of glass fibre for wind turbine blades.
Laboratory micro-CT scanners and large-scale synchrotron facilities.



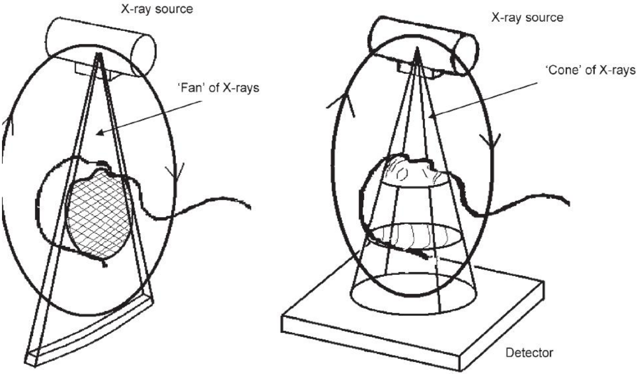


X-Ray Tomography: Imaging From Projections

- Projections are measured around an object using X-rays.
- Goal is to reconstruct the object from the projections.
- Simplest is 2D *parallel beam geometry*, which we focus on.
- Used in early scanners and in large-scale synchrotron facilities.



Cone-Beam Geometry



- Cone-beam (medical CT scanners, lab-based micro-CT, etc.)
- Cone-beam restricted to central slice → fan-beam
- Move source far away *approx* parallel-beam (synchrotron)

Contrast Mechanism: X-Ray Attenuation

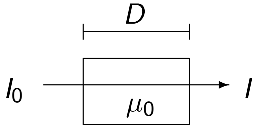
“Heavier” matter attenuate X-rays more: air – tissue – bone – metal.
Quantified by so-called linear attenuation coefficient μ.

Wilhelm Conrad Röntgen and the first X-ray image ever taken showing his wife’s hand (1895).

Lambert-Beer Law of Attenuation

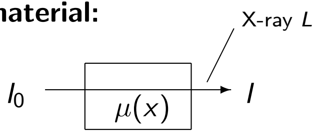
Homogeneous material:

$$I = I_0 \exp\left\{ -\mu_0 D \right\}$$



Non-homogeneous (more interesting) material:

$$I = I_0 \exp\left\{ -\int_L \mu(x) dx \right\}$$

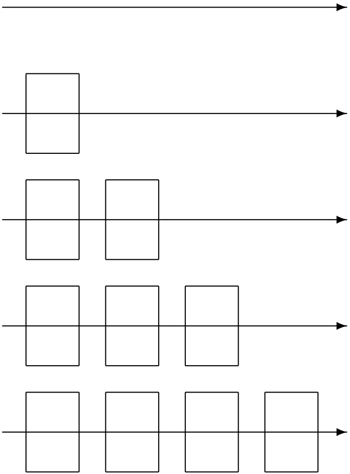


Rearrange to line integral form:

$$-\log \frac{I}{I_0} = \int_L \mu(x) dx$$

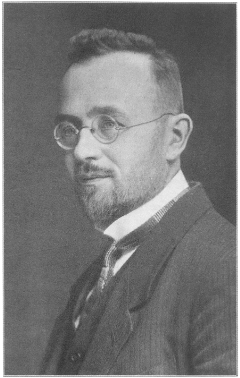
I_0 is the incident flux and $\mu(x)$ is the absorption coefficient..
I is called the *intensity*, I/I_0 is called the *transmission*, while the corresponding $-\log(I/I_0)$ is called the *absorption*.

Intensity vs Transmission vs Absorption



$I_0 = 10000$	I	I/I_0	$-\log(I/I_0)$
	10000	1.0000	0.0
	5000	0.5000	0.7
	2500	0.2500	1.4
	1250	0.1250	2.1
	625	0.0625	2.8

The Origin of Tomographic Reconstruction



Original reference:

Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. (Johann Radon, 1917):

An object can be reconstructed **perfectly** from a **full** set of line integrals.

Parameterizing Lines in the Plane

How to describe a line?

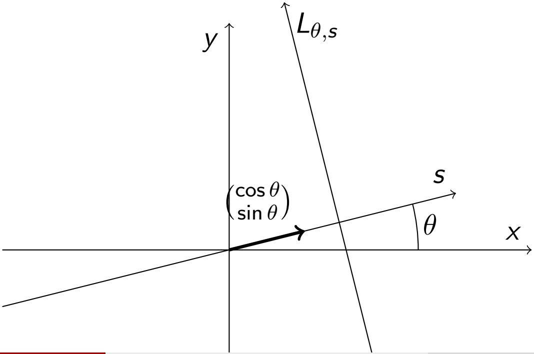
A familiar way: $y = \alpha x + \beta$. Vertical lines excluded.

Alternative: "normal form":

$$L_{\theta,s} = \{(x,y) \mid x\cos\theta + y\sin\theta = s\}$$

s is the signed orthogonal distance of line to origin

θ is the angle between the x-axis and unit normal vector to $L_{\theta,s}$



The Radon Transform for Parallel-Beam Geometry

Object $f(x,y)$:

- contained in a unit disk of radius 1

Line of integration $L_{\theta,s}$ given by:

- θ: angle of line to be projected onto
- s: position on line

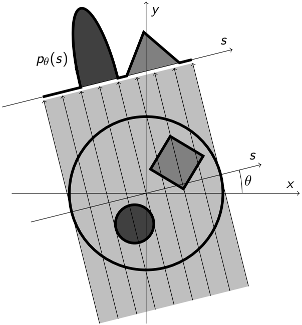
Projection: all line integrals at angle θ:

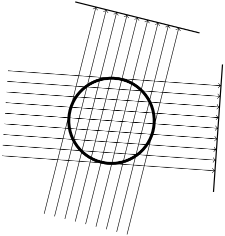
$$p_\theta(s) \;=\; \int_{L_{\theta,s}} f(x,y)\,d\ell \quad \text{for} \quad s \in [-1,1].$$

The Radon transform is:

$$[\mathcal{R}f](\theta,s) \;=\; p_\theta(s) \;=\; \int_{L_{\theta,s}} f(x,y)\,d\ell$$

for $\theta \in [0^\circ, 360^\circ[$ and $s \in [-1,1]$.





Projections all around the object

The Radon transform describes the forward problem of how (ideal) X-ray projection data arises in a parallel-beam scan geometry.

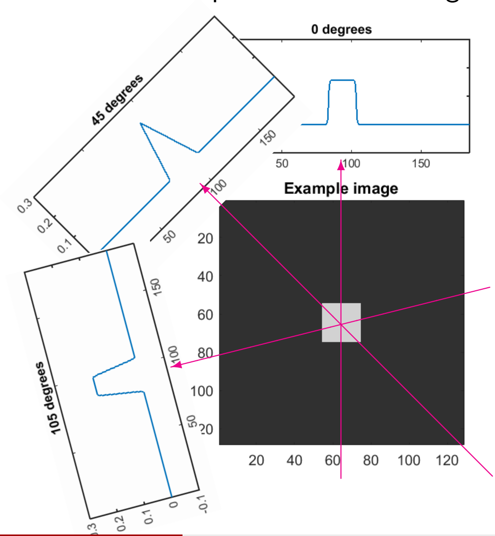
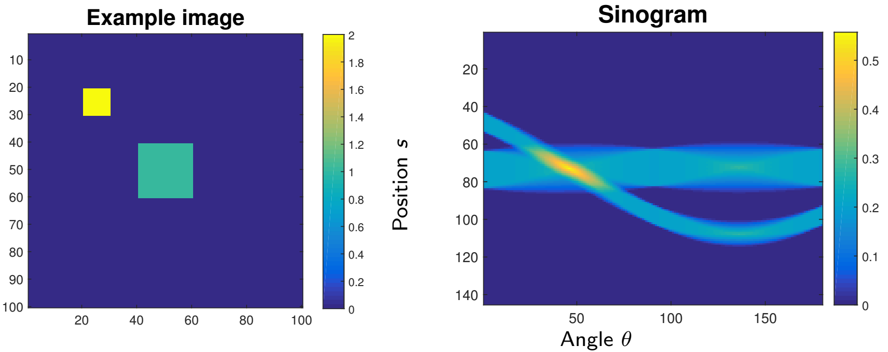


Image and Sinogram

The output of the Radon transform is called a sinogram:



Note that $[0^\circ, 180^\circ]$ captures all necessary projections of the object. The angular range $[180^\circ, 360^\circ]$ gives a "mirror image."

Pen and Paper Exercise: Radon Transform of a Disk

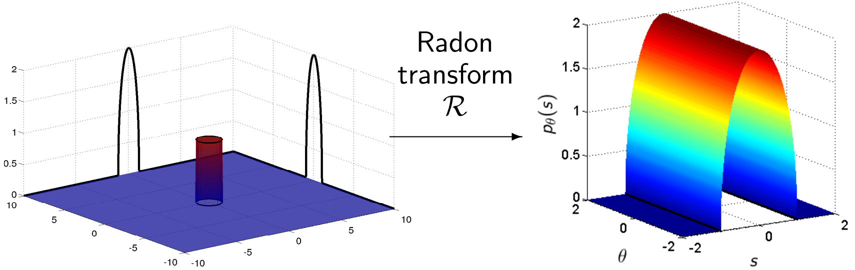
Given an image with a small centered disk of radius $r < 1$:

$$f(x,y) = \begin{cases} 1 & \text{for} \quad x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

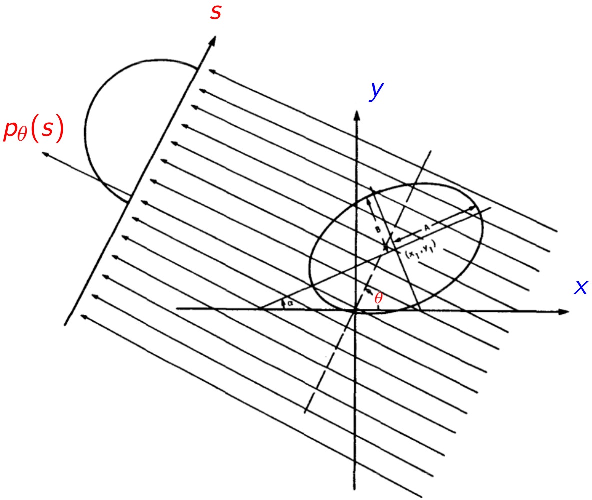
Derive that the Radon transform is:

$$[\mathcal{R}f](\theta, s) = \begin{cases} 2\sqrt{r^2 - s^2} & \text{for} \quad |s| \leq r \\ 0 & \text{otherwise} \end{cases}$$

Image **Radon transformed image**



Radon Transform of an Ellipse (Kak & Slaney)



Radon Transform of an Ellipse (Kak & Slaney)

Given the ellipse with semi-axes A and B centered at the origin

$$f(x,y) = \begin{cases} c & \text{for } \frac{x^2}{A^2} + \frac{y^2}{B^2} \leq 1 \quad \text{(inside the ellipse)} \\ 0 & \text{otherwise} \quad \text{(outside the ellipse.)} \end{cases}$$

The corresponding Radon transform is

$$p_\theta^0(s) = \begin{cases} \frac{2cAB}{a^2(\theta)} \sqrt{a^2(\theta) - s^2} & \text{for} \quad |s| \leq a(\theta) \\ 0 & \text{otherwise} \end{cases}$$

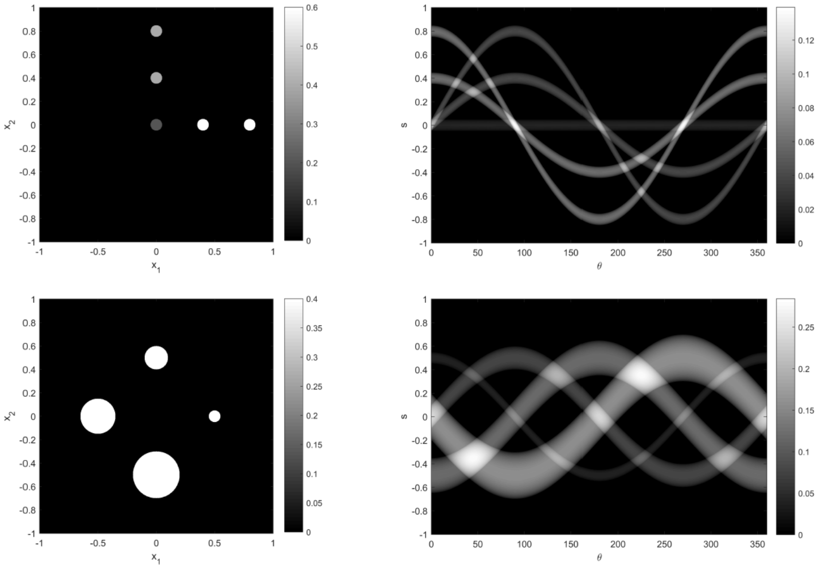
where $a^2(\theta) = A^2 \cos^2\theta + B^2 \sin^2\theta$.

If centered at (x_c, y_c) and rotated by α, then given from $p_\theta^0(s)$ above as

$$p_\theta(s) = p_{\theta-\alpha}^0(s - z_c \cos(\gamma_c - \theta))$$

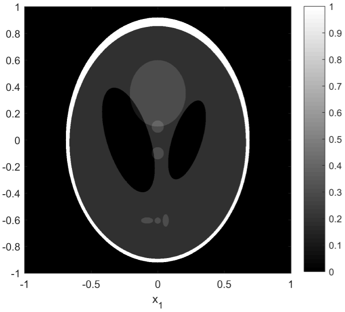
with $z_c = \sqrt{x_c^2 + y_c^2}$ and $\gamma_c = \tan^{-1}(y_c/x_c)$.

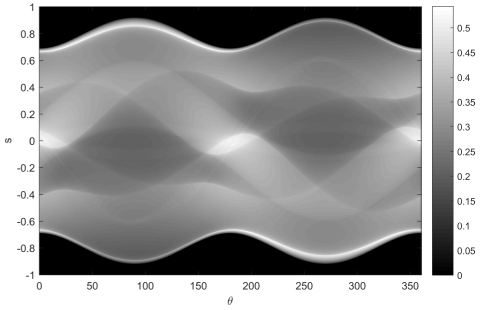
More sinogram examples



Shepp-Logan test phantom

Commonly used test image (phantom): Superposition of ellipses, simulating cross section of human head.





Connection: Radon Transform and Lambert-Beer

The Radon transform

$$p_\theta(s) \;=\; \int_{L_{\theta,s}} f(x,y)\,d\ell$$

and the Lambert-Beer law along the same line $L_{\theta,s}$

$$I_{\theta,s} = I_0 \exp\left(-\int_{L_{\theta,s}} \mu(x,y)\,d\ell\right)$$

are connected through the identifications

$$f(x,y) = \mu(x,y)$$
$$p_\theta(s) = -\log\left(\frac{I_{\theta,s}}{I_0}\right)$$

Parametrized Form of Radon Transform

The Radon transform

$$p_\theta(s) = \int_{L_{\theta,s}} f(x, y)\, d\ell$$

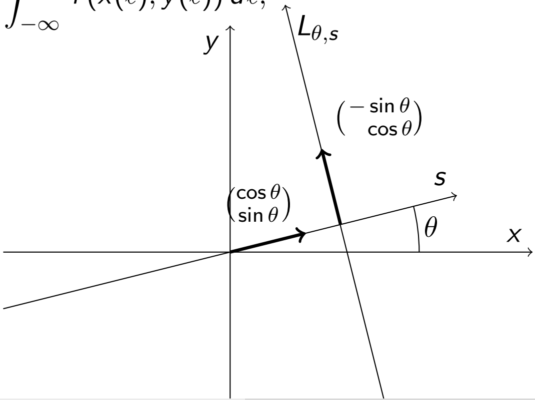
can be written explicitly using a parametrization of the line $L_{\theta,s}$

$$p_\theta(s) = \int_{-\infty}^{\infty} f(x(\ell), y(\ell))\, d\ell,$$

where at fixed θ and s:

$$x(s) = s\cos\theta - \ell\sin\theta$$
$$y(s) = s\sin\theta + \ell\cos\theta$$

The line $L_{\theta,s}$ is traced as ℓ runs from $-\infty$ to ∞.



Dirac Delta Function

A generalized function – or distribution – with heuristic definition:

$$\delta(t) = \begin{cases} +\infty & \text{for} \quad t = 0 \\ 0 & \text{for} \quad t \neq 0 \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t) dt = 1 \; .$$

Important property of Dirac delta function:

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - T)\, dt = f(T) \; .$$

This is called the *sifting property*: the Dirac delta function acts as a sieve and "sifts out" (or "samples") the value of f at $t = T$.

A Convenient Explicit Form of Radon Transform

Useful alternative expression for the Radon transform
we need it later

$$p_\theta(s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, \delta(x\cos\theta + y\sin\theta - s)\, dx\, dy \ .$$

Interpretation:
The line $L_{\theta,s}$ consists of exactly those (x, y) at which

$$x\cos\theta + y\sin\theta - s = 0 \ ,$$

which is the argument of the Dirac delta function δ.

Thus, the integrand is restricted to function values of $f(x, y)$ on $L_{\theta,s}$, which amounts to the corresponding line integral.

Data Statistics

The measured intensity I in a detector pixel is proportional to I_0 and

$$d = \exp\left\{-\int_L \mu(x)\,dx\right\} .$$

Moreover, I is a *photon count* which follows a Poisson distribution, i.e.,

$$I \sim \mathcal{P}(I_0\,d) , \qquad I_0\,d = \text{expected value} = \text{variance}.$$

For large values of I this can be approximated by a Gaussian distribution,

$$I \sim \mathcal{N}(I_0\,d, I_0\,d) \qquad \text{or} \qquad I = I_0\,d + \sqrt{I_0\,d}\,Z , \quad Z \sim \mathcal{N}(0,1) .$$

Hence the *absorption* $b = -\log I/I_0 = -(\log d + \log I/(I_0\,d))$ is

$$\begin{aligned} b &= -\log d - \log\left(1 + \frac{1}{\sqrt{I_0\,d}}\,Z\right) \\ &\approx -\log d - \frac{1}{\sqrt{I_0\,d}}\,Z \sim \mathcal{N}\left(\int_L \mu(x)\,dx , \frac{1}{I_0\,d}\right) . \end{aligned}$$

Other error Sources

In addition to the Poisson noise, data can be affected by numerous other issues.

- Detector noise.
- Scatter (some X-rays do not follow straight line).
- X-rays are not monochromatic, but have full spectrum. Attenuation coefficient depends on energy $\rightarrow$ beam hardening.
- Bad detectors, e.g., void measurements.
- Too dense features in the object, e.g., metal blocking rays completely.
- Object changing during acquisition, e.g., motion.

Exercises

Available from:
`http://www2.compute.dtu.dk/~pcha/HDtomo/SCforCT.html`

File:
ExWeek1Days1and2.pdf

Exercise 1: The Radon Transform.

Outline

Back Projection

Mathematically, *back-projection* is written as integration over all θ,

$$\mathcal{B}[p_\theta(s)](x, y) = \int_0^{2\pi} p_\theta(x \cos\theta + y \sin\theta)\, d\theta.$$

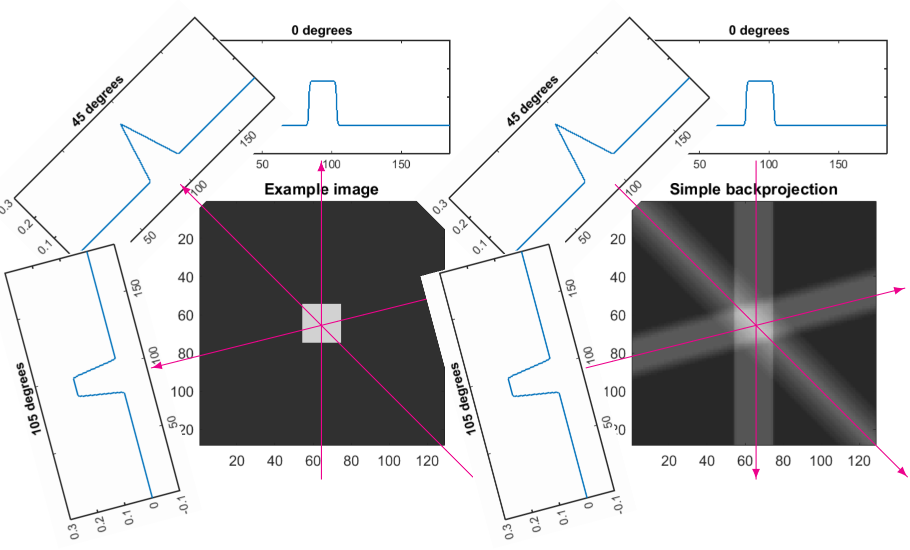
Interpretation: "Smearing and summation"

Each point (x, y) and each angle θ define a unique location (θ, s) in the sinogram, with $s = x\cos\theta + y\sin\theta$.

For a given θ, in the back-projection the image point (x, y) is assigned the sinogram value at s, i.e., the value $p_\theta(s)$. This is "smearing."

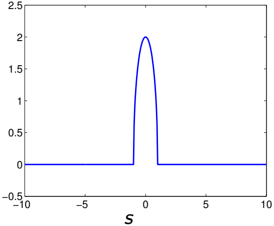
Back-projection then sums all contributions, at each (x, y), by integrating over θ. This is "summation."

Back-projection: Does it invert projection? No



And Now: The Fourier Transform

Fourier transform ("Frequency representation"):

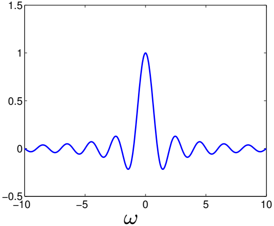


1D Fourier transform $\mathcal{F}_1$

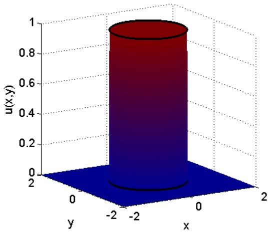
⟶

⟵

1D Inverse Fourier transform $\mathcal{F}_1^{-1}$



Also possible for functions of more than 1 variable:



2D Fourier transform $\mathcal{F}_2$

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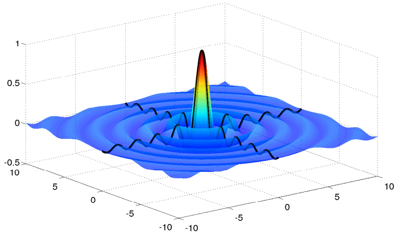
2D Inverse Fourier transform $\mathcal{F}_2^{-1}$



1D Fourier transform and inverse:

$$\widehat{f}(\omega) = \mathcal{F}_1[f(t)](\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\omega t}\, dt$$

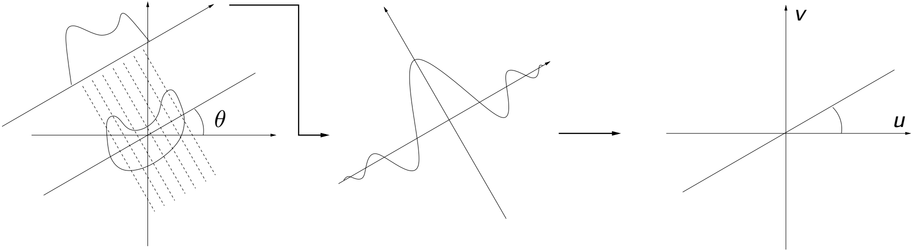
$$f(t) = \mathcal{F}_1^{-1}[\widehat{f}(\omega)](t) = \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{+j2\pi\omega t}\, d\omega$$

2D Fourier transform and inverse:

$$F(u,v) = \mathcal{F}_2[f(x,y)](u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+yv)}\, dx\, dy$$

$$f(x,y) = \mathcal{F}_2^{-1}[F(u,v)](x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{+j2\pi(ux+yv)}\, du\, dv$$

Fourier Slice Theorem: Key to Reconstruction

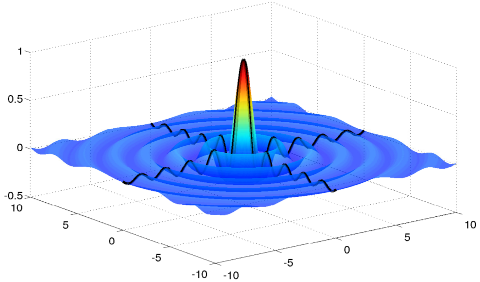


Projection at angle θ 1D Fourier transform At angle θ in Fourier space

All 1D Fourier transformed projections

=

2D Fourier transform



Derivation of Fourier Slice Theorem (1)

Strategy: Manipulate 1D Fourier-transformed projection into slice through 2D Fourier-transformed image.

$$
\begin{aligned}
\widehat{p_\theta}(\omega) &= \int_{-\infty}^{\infty} p_\theta(s)\, e^{-j2\pi\omega s}\, ds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(x\cos\theta + y\sin\theta - s)\, e^{-j2\pi\omega s}\, dx\, dy\, ds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \int_{-\infty}^{\infty} \delta((x\cos\theta + y\sin\theta) - s)\, e^{-j2\pi\omega s}\, ds\, dx\, dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \int_{-\infty}^{\infty} \delta(s - (x\cos\theta + y\sin\theta))\, e^{-j2\pi\omega s}\, ds\, dx\, dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi\omega(x\cos\theta + y\sin\theta)}\, dx\, dy \ .
\end{aligned}
$$

We used: the definition of the 1D Fourier transform, the Dirac delta expression of $p_\theta(s)$, reordering, the fact that $\delta(-t) = \delta(t)$, and the sifting property.

Derivation of Fourier Slice Theorem (2)

Now continue by reordering, and recognizing as 2D Fouirer transform

$$\begin{aligned}\widehat{p_\theta}(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(x\omega\cos\theta + y\omega\sin\theta)}\, dx\, dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(xu+yv)}\, dx\, dy\Bigg|_{u=\omega\cos\theta,\, v=\omega\sin\theta} \\ &= F(u,v)\Big|_{u=\omega\cos\theta,\, v=\omega\sin\theta}\end{aligned}$$

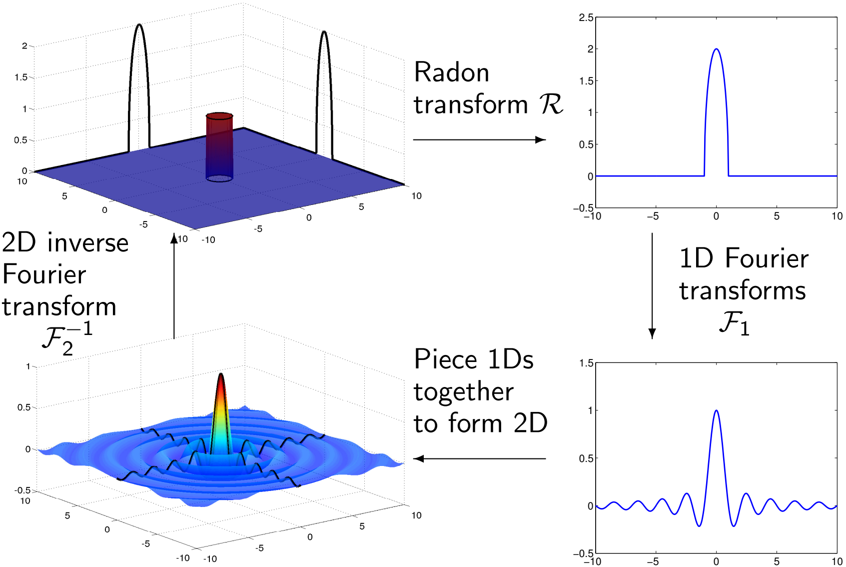
This yields finally the *Fourier slice theorem*:

$$\boxed{\widehat{p_\theta}(\omega) = F(\omega\cos\theta, \omega\sin\theta)}$$

Interpretation: $(u,v) = (\omega\cos\theta, \omega\sin\theta)$ for $\theta \in [0,\pi)$ and $\omega \in (-\infty,\infty)$ specifies a line in 2D Fourier space rotated by θ relative to the positive u axis. This corresponds to the s axis in real space.

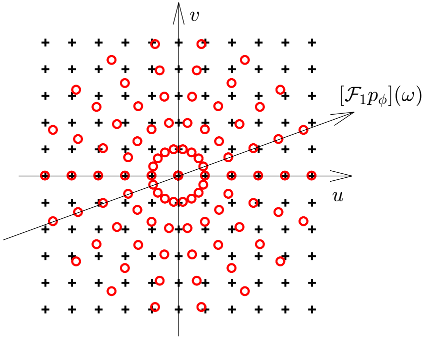
Thus, the 1D Fourer transform of a projection is equivalent to the corresponding *slice*/line through the 2D Fourier transform.

Fourier Reconstruction Method



Problems for the Fourier Reconstruction Method

- In practice we **do not** have all θ and s since we record a finite number of projections with a detector of fixed size.
- Interpolation from polar to Cartesian grid, known as "regridding," is required for 2D inverse Fourier transform:



- Accurate interpolation in the Fourier domain difficult.
- 2D inv. Fourier transform needs all data simultaneously.
- Result: Fourier method rarely used in practice.

Derive Filtered Back-Projection, Part 1

Strategy: Rewrite inverse 2D Fourier transform:

$$f(x,y) = \mathcal{F}_2^{-1}[\mathcal{F}_2 f]$$

[2D Fourier transform definitions]

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{j2\pi(ux+vy)}\, du\, dv$$

[Change to polar coordinates, including Jacobian ω]

$$= \int_0^{2\pi}\int_0^{\infty} F(\omega\cos\theta, \omega\sin\theta)\, e^{j2\pi(\omega\cos\theta x+\omega\sin\theta y)}\, \omega\, d\omega\, d\theta$$

[Split integral over 2π in two:]

$$= \int_0^{\pi}\int_0^{\infty} F(\omega\cos\theta, \omega\sin\theta)\, e^{j2\pi(\omega\cos\theta x+\omega\sin\theta y)}\, \omega\, d\omega\, d\theta\, +$$

$$\int_0^{\pi}\int_0^{\infty} F(\omega\cos(\theta+\pi), \omega\sin(\theta+\pi))\, e^{j2\pi(\omega\cos(\theta+\pi)x+\omega\sin(\theta+\pi)y)}\, \omega\, d\omega\, d\theta$$

Derive Filtered Back-Projection, Part 2

[Using $\sin(\theta + \pi) = -\sin\theta$ and $\cos(\theta + \pi) = -\cos\theta$:]

$$= \int_0^{\pi} \int_0^{\infty} F(\omega\cos\theta, \omega\sin\theta)\, e^{j2\pi(\omega\cos\theta x + \omega\sin\theta y)}\, \omega\, d\omega\, d\theta$$

$$+ \int_0^{\pi} \int_0^{\infty} F(-\omega\cos\theta, -\omega\sin\theta)\, e^{j2\pi(-\omega\cos\theta x - \omega\sin\theta y)}\, \omega\, d\omega\, d\theta$$

[Change sign/bounds in second integral:]

$$= \int_0^{\pi} \int_0^{\infty} F(\omega\cos\theta, \omega\sin\theta)\, e^{j2\pi(\omega\cos\theta x + \omega\sin\theta y)}\, \omega\, d\omega\, d\theta$$

$$+ \int_0^{\pi} \int_{-\infty}^{0} F(\omega\cos\theta, \omega\sin\theta)\, e^{j2\pi(\omega\cos\theta x + \omega\sin\theta y)}\, (-\omega)\, d\omega\, d\theta$$

Collect the two integrals using absolute value:

$$= \int_0^{2\pi} \int_{-\infty}^{\infty} F(\omega\cos\theta, \omega\sin\theta)\, e^{j2\pi\omega(x\cos\theta + y\sin\theta)}\, |\omega|\, d\omega\, d\theta$$

Derive Filtered Back-Projection, Part 3

[Apply Fourier Slice Theorem:]

$$f(x,y) = \int_0^{2\pi} \int_{-\infty}^{\infty} \widehat{p}_\theta(\omega)\, e^{j2\pi\omega(x\cos\theta + y\sin\theta)}\, |\omega|\, d\omega\, d\theta$$

[Use that $x\cos\theta + y\sin\theta$ is constant wrt ω-integration, say s]

$$= \int_0^{2\pi} \left[\int_{-\infty}^{\infty} \widehat{p}_\theta(\omega)\, e^{j2\pi\omega s}\, |\omega|\, d\omega \right]_{s = x\cos\theta + y\sin\theta} d\theta.$$

Recognizing the inner integral as 1D inverse Fourier transform, we define the *filtered projection*:

$$\boldsymbol{q}_\theta(s) = \int_{-\infty}^{\infty} \widehat{p}_\theta(\omega)\, e^{j2\pi\omega s}\, |\omega|\, d\omega$$
$$= \mathcal{F}_1^{-1}\left[\widehat{p}_\theta(\omega)\, |\omega|\right](s) = \mathcal{F}_1^{-1}\left[\mathcal{F}_1[p_\theta](\omega)\, |\omega|\right](s)$$

Projections are filtered by multiplying with the ramp filter $|\omega|$ in the Fourier domain.

Derive Filtered Back-Projection, Part 4

Finally we can write

$$f(x,y) = \int_0^{\pi} \boldsymbol{q}_\theta(x\cos\theta + y\sin\theta)\, d\theta = \mathcal{B}[\boldsymbol{q}_\theta](x,y)$$

by recognizing the back-projection operation.

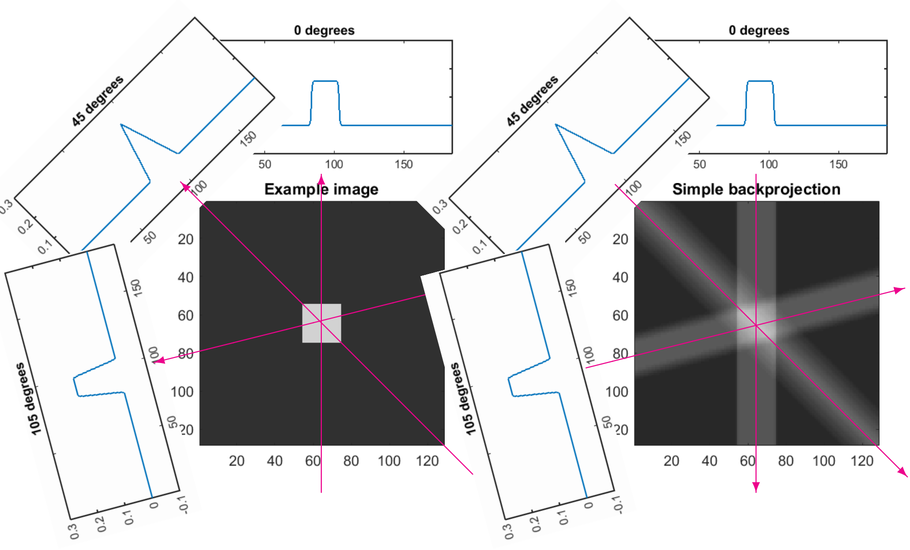
This is the Filtered Back-Projection (FBP) inversion formula for the Radon transform.

Interpretation:

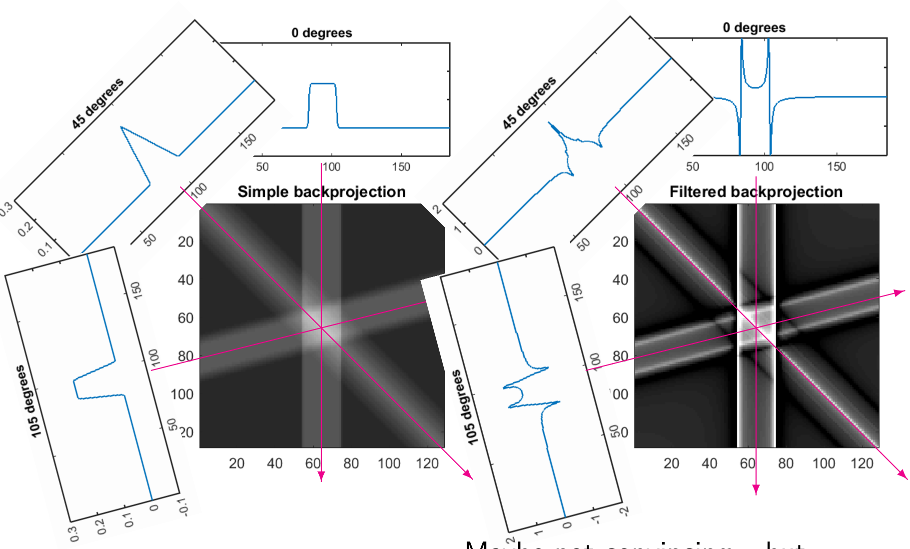
At each point (x, y) in the image f to be reconstructed, each angle θ defines a sinogram location $s = x\cos\theta + y\sin\theta$.

Through back-projection, the point (x, y) is assigned the singram's value at s via the filtered projection, and contributions at all angles θ are summed up.

Recall: Back-Projection Does NOT Invert Projection



Filter Projections by “ramp” Before Back-Projection

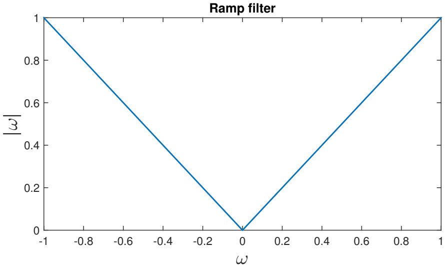


Maybe not convincing – but see next slide!

Back-Projection vs. Filtered Back-Projection

Projections must be filtered with a "ramp" filter before back-projection.

In the Fourier domain: $|\omega|$



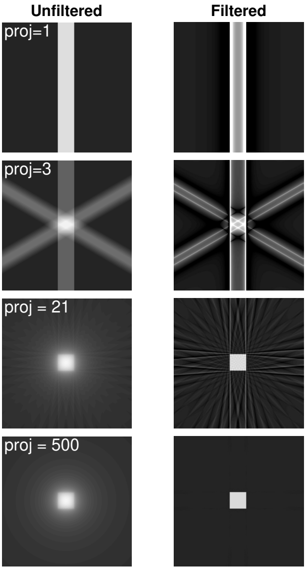
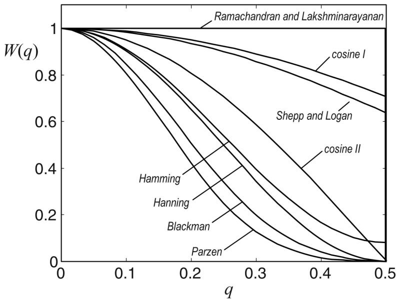


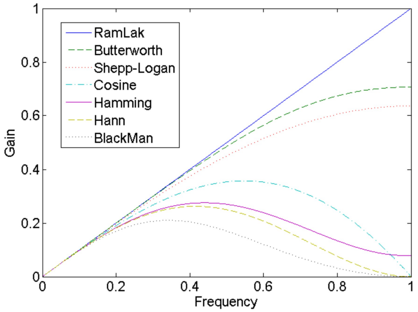
Figure from Buzug.

Additional Filtering Needed in Practice

Ramp filter is part of the inversion formula, but it is a high-pass filter which is problematic in practice when noise is present.

In practice additional low-pass filters $\varphi_{\mathrm{LP}}(\omega)$ are used; they are multiplied with $|\omega|$ in the frequency domain:





Filtered Back-Projection (FBP) Step By Step

To compute FBP reconstructed image from projections:

1. Fourier-transform each projection (1D Fourier transform).
2. Apply ramp filter by multiplication with $|\omega|$.
3. Optionally, apply additional low-pass filter $\varphi_{\mathrm{LP}}(\omega)$ to handle noise.
4. Inverse Fourier-transform to obtain filtered projections.
5. Backproject the filtered projections and sum up.

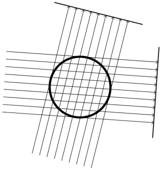
In practice:

- Often done automatically for you by scanner/instrument.
- FBP implementations available.
- Easy to use: main user input is the choice of low-pass filter.
- In MATLAB: `iradon`.
- In AIR Tools II: `fbp`.

Reconstruction for Fan-Beam and Cone-Beam

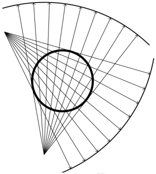
Parallel-beam FBP:

Projections → [Filter] → [Back-project] → Reconstruction



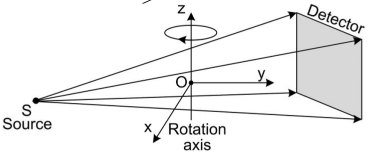
Fan-beam – 2 strategies:

- Rebinning of data, followed by FBP algorithm
- Dedicated reconstruction algorithm



Cone-beam:

- Dedicated reconstruction algorithm
- Feldkamp-Davis-Kress (FDK), an approximate algorithm, is standard



Projections → [Weighting] → [Filter] → [Weighting] → [Back-project] → Reconstruction

Strengths and Weaknesses of FBP

Strengths:

- Fast: Based on FFT and a single back-projection.
- Few parameters to adjust.
- Conceptually easy to understand and implement.
- Reconstruction behavior well understood.
- Typically works very well (for complete & good data)

Weaknesses:

- Large number of projections required.
- Full angular range required.
- Only modest amount of noise in data can be tolerated.
- Fixed scan geometries – others require own inversion formulas.
- Cannot make use of prior knowledge such as non-negativity.

Exercises

Exercise 2: Filtered Back-Projection.

Tuesday

- 09:00 Real data reconstruction 2D (MATLAB)
- 11:00 Real data reconstruction 2D/3D - Core Imaging Library
- 12.00 - - - Lunch break - - -
- 13.00 Real data continued - reconstruct your own data?
- 14:30 Micro project - intro and group formation

Outline

Flat and Dark-Field Correction

Typical data acquired:

- I: Measured intensity images (sample in, source on)
- I_{F}: Measure of the actual flux, called *flat-field* (sample out, source on)
- I_{D}: Background: *dark field* (sample out, source off)

FBP needs line integrals (Radon transform):

- Conversion to linear problem in attenuation coefficient f:

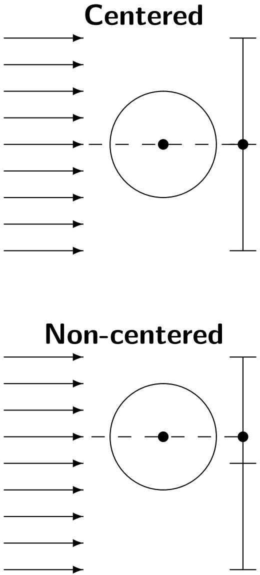
$$\int_L f(x,y)\,d\ell = -\log \frac{I}{I_0}\ .$$

To obtain projection data:

$$Z = -\log Y, \qquad Y = \frac{I - I_{\mathrm{D}}}{I_{\mathrm{F}} - I_{\mathrm{D}}} \quad \leftarrow \text{pixelwise division.}$$

Center-of-Rotation (COR) Correction

- Standard FBP implementations, such as MATLAB's `iradon`, assume a perfectly centered object.
- In other words, the center of rotation should be mapped to the central detector pixel.
- In practice only approximate centering is physically possible.
- Naive reconstruction yields artifacts. Need to perform center-of-rotation correction.
- Can be done by "shifting" projections by padding sinogram with sufficiently many artifical detector pixel values.



Region-of-Interest (ROI) Correction

- In some cases the object to be scanned is too large to fit in the field of view.
- Or we want to focus on some small region-of-interest (ROI).
- Projections are truncated – they do not cover entire object.
- Can we still reconstruct the object? Or just the ROI?

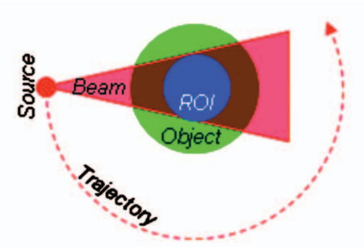


Figure: Wang & Yu, Med. Phys., 36 (2009), 3575–3581.

The Region-of-Interest (ROI) Problem

- Can we exactly reconstruct the object or the ROI? No!
- Interior Radon data are "contaminated" by exterior Radon data.
- Interior reconstruction will also be "contaminated."

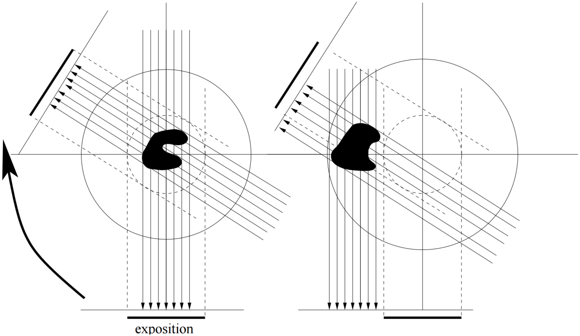
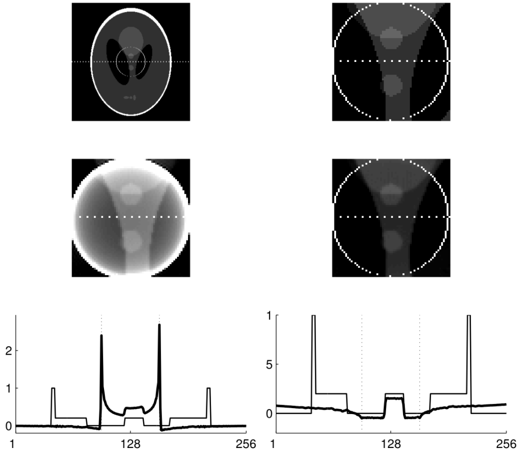


Figure: Bilgot et al., IEEE Nuc. Sci. Symp. Conf. Rec. (2009), 4080–4085.

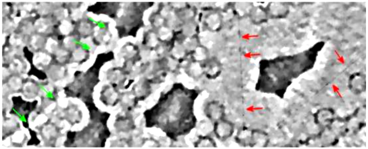
ROI Correction in Practice

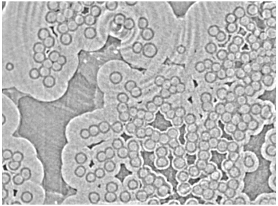
- Boundaries are correct.
- Large artifact apparent.
- Padding of the sinogram yields large improvement, but the intensities are still off.



- The exercise data set is ROI data!

Many Other Possible Artifacts and Corrections





Figures: Aditya Mohan et al., ICASSP (2014), 6909–6913.

- Ring artifacts
- Beam hardening
- Misalignment
- Metal artifacts
- Motion
- ...

Exercises

Exercise 3: Reconstruction of a real data set (2D parallel-beam).

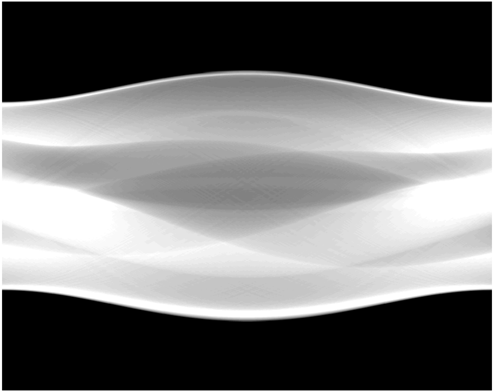
Outline

Micro Project – Exterior Tomography

- In deriving the FBP reconstruction formula, we assume full data is available, i.e., $\theta \in [0^\circ, 180^\circ]$ and the sample is fully contained within the field of view.
- In many cases full data is *not* available. Examples: limited angular range, few projections, or the region-of-interest/interior problem (all encountered in the exercises).
- In *the ROI/interior problem*, only rays passing through the central part of the object are available – because the object is too large or we have zoomed in on a small region.
- In the *exterior problem*, only rays through outer annulus of object are measured, not the rays through the center.
- This can happen, e.g., in non-destructive testing of objects with dense parts where the X-rays do not penetrate sufficiently.

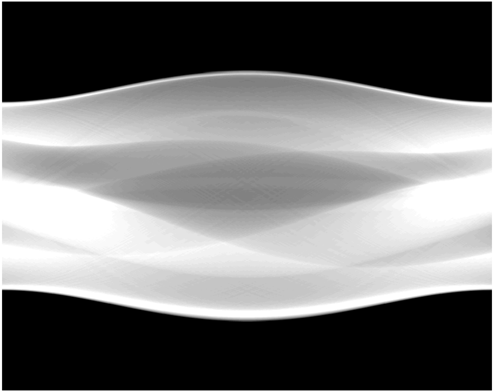
Shepp-Logan, Full Data 0°–180°

Original phantom and the corresponding sinogram.



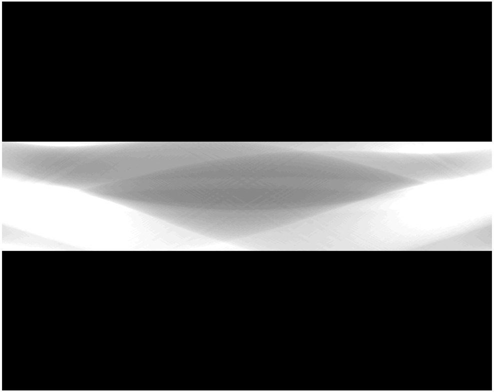
Shepp-Logan, Reconstruction from Full Data 0°–180°

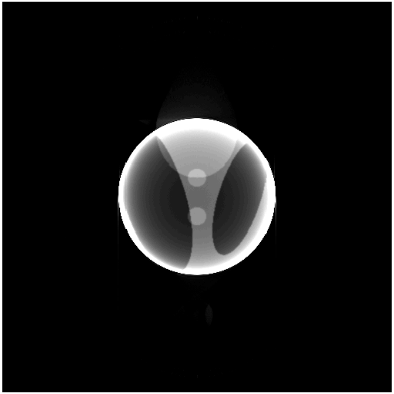
Full sinogram and FBP reconstruction.



Shepp-Logan, Reconstruction from Region-of-Interest Data

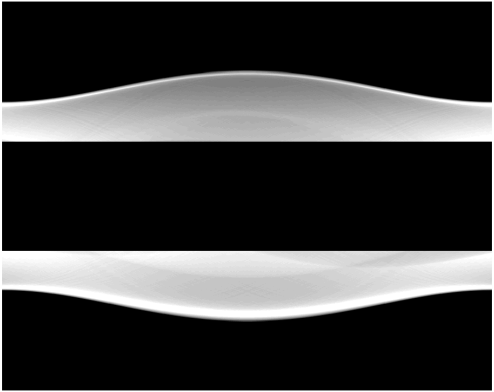
Region-of-interest/interior sinogram and reconstruction.





Shepp-Logan, Reconstruction from Exterior Data

Exterior sinogram.



Reconstruction?

The Micro Project

This is where all the theory comes together.

Use course material and computer simulations to investigate the exterior problem.

Try to establish what can and cannot be reconstructed.

- The micro project is quite open-ended.
- In the lectures and exercises you are introduced to theory, techniques and tools for tomographic reconstruction.
- In the project it is up to you to choose from this material and apply to the exterior problem in order to understand the problem and the limitations of reconstruction from exterior data.

Some Ideas

- Design phantoms with features to illustrate how reconstruction quality depends on the size of features, closeness to boundary, radius of missing interior region, etc.
- Apply FBP including practical corrections such as padding as used for ROI data (Mon+Tues).
- Apply SVD analysis and compare the singular values and vectors with the full data case (Wed+Thurs).
- Apply ideas from micro-local analysis to assess which features can be reconstructed (Wed+Thurs).
- Investigate the effect of having exterior data on one or more of the real data sets provided by us or one that we acquired on Monday. Try to improve reconstruction e.g. using padding.

You are free to pick from this list or pursue your own ideas within the course material and exterior tomography – please do not hesitate to discuss ideas with us.

Work on the Micro Project

- Get together 3–4 persons in each group.
- Each group presents their work Friday afternoon - **everyone must contribute!**
- Start experimenting and familiarizing yourselves with the exterior problem. For example:
  - Create code to simulate missing data in the sinogram caused by the exterior problem.
  - Create interesting phantoms and do FBP reconstruction from simulated exterior data.
  - ...
- Suggestions:
  - Take notes/keep a log of the experiments you do and what you learn from each, etc.
  - Make separate scripts for each study you do, to make it easy for yourself to remember and reproduce later as well as to create figures for the oral presentation on Friday.