What You Can See in Limited Data Tomography

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3. Determine what features of the body will be easy to reconstruct from limited CT data, and which will be difficult.
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1. Learn a little bit of the history of CT.
2. Learn what limited data tomography is.
3. Determine what features of the body will be easy to reconstruct from limited CT data, and which will be difficult.
4. Understand, geometrically, how this depends on the data.
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Allan won the 1979 Nobel Prize in Medicine! (early AM)...taught!
Modern GE scanner

GE Reconstruction
Modern GE scanner

GE Reconstruction

Cost: DKK 12,000,000
The Mathematical Model of X-ray CT

$f$ a function in the plane representing the density of an object
$L$ a line in the plane over which the photons travel.

The X-ray (Radon) Transform:

\[
\text{Tomographic Data } \sim \mathcal{R}f(L) = \int_{x \in L} f(x) \, ds
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–The 'amount' of material on the line the X-rays traverse.
The Mathematical Model of X-ray CT and the Goal

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**The goal:** Recover a picture of the body (values of \( f(x) \)), from X-ray CT data over a finite number of lines.
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The goal: Recover a picture of the body (values of \( f(x) \)), from X-ray CT data over a finite number of lines.

With complete data (lines throughout the object in fairly evenly spaced directions), good reconstruction methods exist (e.g., Filtered Backprojection [Natterer, Natterer-Wübbling]).
Parallel Beam Scanning Geometry

The angle: $\phi \in [0, 360[,$ \hspace{1em} $\theta(\phi) = (\cos(\phi), \sin(\phi))$

The line over which X-rays travel: $L_{\phi, \rho}$ is the line perpendicular to $\phi$ and $\rho$ units from the origin (in the opposite direction of $\phi$ if $\rho < 0$)
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\[
L(\phi, \rho)
\]

Moral

Each line can be parameterized by a unique 
\((\phi, \rho) \in [0^\circ, 180^\circ[ \times [-R, R].\)

or redundantly by two \((\phi, \rho) \in [0^\circ, 360^\circ[ \times [-R, R].\)
The object: \( f \) is the density function of an object in the plane—inside the disk of radius \( R \) centered at \((0, 0)\).

**Tomographic data:** \( \mathcal{R} f(\phi, \rho) = \int_{x \in L_{\phi, \rho}} f(x) \, ds \) is given when X-rays travel along the line \( L_{\phi, \rho} \). (\sim \text{fan beam but simpler})

**Limited Data Tomography:** When data over some lines are missing.

In practice: a finite number of evenly distributed lines.
Complete X-ray Tomographic Data

**The object:** $f$ is the density function of an object in the plane—inside the disk of radius $R$ centered at $(0, 0)$.

**Tomographic data:** $Rf(\phi, \rho) = \int_{x \in L_{\phi,\rho}} f(x) \, ds$ is given when X-rays travel along the line $L_{\phi,\rho}$. (~fan beam but simpler)

**Complete Tomographic Data:** X-ray data are given over all lines going through the body (e.g., $L_{\phi,\rho}$ for $(\phi, \rho) \in [0^\circ, 180^\circ] \times [-R, R]$).
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**Limited and Complete X-ray Tomographic Data**

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**Limited angle X-ray CT:** the scanner cannot move all the way around the object—it images the object from lines in a limited range of angles:
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**The mask** multiplies the data by 1 on $[a, b] \times [-R, R]$ and 0 off of this set.
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**Example:** The data set includes only horizontal-ish lines—$L_{\phi,\rho}$ with $(\phi, \rho) \in [45^\circ, 135^\circ] \times [-R, R]$
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**Example:** The data set includes only horizontal-ish lines—\( L_{\phi, \rho} \) with \((\phi, \rho) \in [45^\circ, 135^\circ] \times [-R, R]\)

Vertical-ish lines are missing from the data set.
Limited Angle CT in Dental Imaging

Dental Scanner–head goes in “Π” Jaw showing X-ray projection angles

http://www.siltanen-research.net
Limited Angle CT in Luggage Testing

Luggage Scanner

Sample Luggage scan

Scanner moves above and below suitcase

Analogic COBRA carry-on luggage scanner
In many tomography problems, the shapes/boundaries of features in the test object are diagnostically important:

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- Illegal stuff in carry-on luggage
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- Blockages in blood vessels
- Uneven boundaries in some tumors

Goal: learn what object boundaries can be reconstructed from limited data.
In many tomography problems, the shapes/boundaries of features in the test object are diagnostically important:

- Cracks in industrial objects
- Illegal stuff in carry-on luggage
- Cavities in teeth
- Blockages in blood vessels
- Uneven boundaries in some tumors

So, knowing the boundaries of structures in the test object is important.

**Goal:** learn what object boundaries can be reconstructed from limited data.
Which features of the body do you see best from this one X-ray image?

My Answer: The bones and their edges (boundaries)!

Answer: The beams tangent to the edges (boundaries) of the bones!
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Which X-ray beams show edges (boundaries) (pic—→)?

**Answer:** The beams tangent to the edges (boundaries) of the bones!

*Now see why mathematically.*
CT data of a disk of radius 1 over vertical lines

The CT data has a “corner” (graph not smooth) at any line tangent to the boundary of the disk.

So the boundary will be easy to see in the data.
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\[ y = Rf \]
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- So the boundary will be easy to see in the data.
In limited data CT, data over some lines are missing.

If a boundary is tangent to a line in the data set, then it will be easy to reconstruct from limited data.

Moral

If a line in a limited data set is tangent to a boundary of the object, that boundary will be easy to see in the reconstruction.

If no line in the data set is tangent to a boundary of the object, that boundary will be difficult to see in the reconstruction.
In limited data CT, data over some lines are missing.

If a boundary is tangent to a line in the data set, then it will be easy to see in the data. easy to reconstruct from limited data.

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If a line in a limited data set is tangent to a boundary of the object, that boundary will be easy to see in the reconstruction. If no line in the data set is tangent to a boundary of the object, that boundary will be difficult to see in the reconstruction.
Example

Limited angle CT data of a disk over lines $L_{\phi,\rho}$ with 
$(\phi, \rho) \in [45^\circ, 135^\circ] \times [-R, R]$ 

[Frikel, Q 2013] *Left:* disk,  *Right:* FBP reconstruction
The data set: all lines with $L_{\phi, \rho}$ with $(\phi, \rho) \in [45^\circ, 135^\circ] \times [-R, R]$.

Which boundaries are visible in the reconstruction?

We claimed that, if a line in the data set is tangent to a boundary, that boundary will be easy to see in the reconstruction from that data. Is that true in this picture? YES!

Moral: A boundary of an object will be visible in the reconstruction from limited data if it is tangent to a line in the data set.
The data set: all lines with $L_{\phi, \rho}$ with $(\phi, \rho) \in [45^\circ, 135^\circ] \times [-R, R]$.

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**Moral**

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The data set: all lines with $L_{\phi, \rho}$ with $(\phi, \rho) \in [45^\circ, 135^\circ] \times [-R, R]$.

Which boundaries of the disk are not visible in the reconstruction?

Yes! We claimed that, if no line in the data set is tangent to a boundary, that boundary will be hard to see in the reconstruction. Is that true in this picture? YES!

Moral: A boundary will be difficult to see in the reconstruction from limited data if no line in the data set is tangent to it.
The data set: all lines with $L_{\phi, \rho}$ with $(\phi, \rho) \in [45^\circ, 135^\circ] \times [-R, R]$.

Which boundaries of the disk are not visible in the reconstruction?

*Answer:* the vertical-ish boundaries.
The data set: all lines with \( L_{\phi, \rho} \) with 
\[(\phi, \rho) \in [45^\circ, 135^\circ[ \times [-R, R].\]

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Moral

A boundary will be difficult to see in the reconstruction from limited data if no line in the data set is tangent to it.
The data set: all lines with $L_{\phi, \rho}$ with $(\phi, \rho) \in [45^\circ, 135^\circ] \times [-R, R]$.

How do the streaks relate to the data set?

They are tangent to the object.

Moral: Streaks in the reconstruction come from lines at the ends of the data set (min. or max. value of $\phi$) that are tangent to the object.
The data set: all lines with $L_{\phi, \rho}$ with $(\phi, \rho) \in [45^\circ, 135^\circ] \times [-R, R]$.

How do the streaks relate to the data set? They are along lines! What values of $\phi$ do they have?

How do the streak lines relate to the object? They are tangent to the object.

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How do the streak lines relate to the object? They are tangent to the object.

Moral

*Streaks in the reconstruction come from lines at the ends of the data set (min. or max. value of $\phi$) that are tangent to the object.*
A boundary of the object is (should be) *visible* in the reconstruction if:

- it is tangent to a line in the data set.

A boundary of the object is (not seen) in the reconstruction if:

- it is not tangent to any line in the data set.

A streak artifact can occur in the reconstruction on a line if:

- it is tangent to the object and at an end of the data set.

This is true by deep mathematics and a precise concept of singularity–microlocal analysis.
A boundary of the object is \textit{(should be) visible} in the reconstruction if:
- it is \textit{tangent} to a line in the data set.

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Exercise

Reconstruction from limited CT data of a brain. **What is the data set, \( \phi \in \mathbb{R} \), \( \rho \in \mathbb{R} \)?**

Brain phantom (left) [radiopedia.org], FBP reconstruction [Frikel, Q 2013]
Exercise

Reconstruction from limited CT data of a brain. What is the data set, $\phi \in \mathbb{R}$, $\rho \in \mathbb{R}$?

Brain phantom (left) [radiopedia.org], FBP reconstruction [Frikel, Q 2013]

- Which features of the brain are visible in the reconstruction?
  - Which are invisible?
- Are there added streak artifacts?
FBP for complete data: \( f = \frac{1}{4\pi} \mathcal{R}^* \Lambda (\mathcal{R}f) \)

- \( \mathcal{R}^* \) is the backprojection operator, \( \Lambda \) the filter, \( \mathcal{R}f \) the data.
**Limited Angle FBP**

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- \( \mathcal{R}^* \) is the backprojection operator, \( \Lambda \) the filter, \( \mathcal{R}f \) the data.

The mask for limited angle data: \( \chi(\phi, \rho) = \begin{cases} 1 & \phi \in [a, b] \\ 0 & \text{otherwise} \end{cases} \)
Limited Angle FBP

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Limited Angle FBP

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FBP for limited angle data \( \phi \in [a, b] \): 
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f = \frac{1}{4\pi} \mathcal{R}^* \Lambda (\chi \mathcal{R}f)
\]

- By multiplying by \( \chi \), we restrict to data in \([a, b] \times [-R, R] \).
Limited Angle FBP and Artifacts

FBP for complete data: \( f = \frac{1}{4\pi} \mathcal{R}^* \Lambda (\mathcal{R}f) \)

- \( \mathcal{R}^* \) is the backprojection operator, \( \Lambda \) the filter, \( \mathcal{R}f \) the data.

The mask for limited angle data: \( \chi(\phi, \rho) = \begin{cases} 1 & \phi \in [a, b] \\ 0 & \text{otherwise} \end{cases} \).

FBP for limited angle data \( \phi \in [a, b] \): \( f = \frac{1}{4\pi} \mathcal{R}^* \Lambda (\chi \mathcal{R}f) \)

- By multiplying by \( \chi \), we restrict to data in \([a, b] \times [-R, R]\).

The streaks occur along lines at the ends of the data set.
Limited Angle FBP and Artifacts

FBP for complete data: \( f = \frac{1}{4\pi} R^* \Lambda (Rf) \)

- \( R^* \) is the backprojection operator, \( \Lambda \) the filter, \( Rf \) the data.

The mask for limited angle data: \( \chi(\phi, \rho) = \begin{cases} 1 & \phi \in [a, b] \\ 0 & \text{otherwise} \end{cases} \).

FBP for limited angle data \( \phi \in [a, b] \): \( f = \frac{1}{4\pi} R^* \Lambda (\chi Rf) \)

- By multiplying by \( \chi \), we restrict to data in \([a, b] \times [-R, R]\).

The streaks occur along lines at the ends of the data set. The cause of streaks: the sharp cutoff in the mask at the ends of the data set.
Limited Angle FBP and Artifact Reduction

**FBP for complete data:**
\[ f = \frac{1}{4\pi} \mathcal{R}^* \Lambda (\mathcal{R}f) \]

- \( \mathcal{R}^* \) is the backprojection operator, \( \Lambda \) the filter, \( \mathcal{R}f \) the data.

**The mask for limited angle data:**
\[ \chi(\phi, \rho) = \begin{cases} 1 & \phi \in [a, b] \\ 0 & \text{otherwise} \end{cases} \]

**FBP for limited angle data** \( \phi \in [a, b] \):
\[ f = \frac{1}{4\pi} \mathcal{R}^* \Lambda (\chi \mathcal{R}f) \]

- By multiplying by \( \chi \), we restrict to data in \([a, b] \times [-R, R] \).

*The streaks occur along lines at the ends of the data set.*

*The cause of streaks:* the sharp cutoff in the mask at the ends of the data set

**The solution:** Make a smooth, gradual cutoff in the mask.
有限角度 FBP 和 artifacts 减少

有限角度 FBP: \( f = \frac{1}{4\pi} \mathcal{R}^* \Lambda (\mathcal{R} f) \)

- \( \mathcal{R}^* \) 是背投影操作符，\( \Lambda \) 是滤波器，\( \mathcal{R} f \) 是数据。

有限角度数据的掩模：\( \chi(\phi, \rho) = \begin{cases} 1 & \phi \in [a, b] \\ 0 & \text{otherwise} \end{cases} \)

有限角度 FBP: \( f = \frac{1}{4\pi} \mathcal{R}^* \Lambda (\chi \mathcal{R} f) \)

- 通过乘以 \( \chi \)，我们将数据限制在 \( [a, b] \times [-R, R] \) 内。

条纹位于数据集的两端。

条纹的原因：在数据集的两端的掩模的尖锐裁剪。

解决方案：在掩模中制作平滑的渐变裁剪。

替换 \( \chi \) 为平滑函数 \( \psi(\phi) \) 使得它在 \( [a, b] \) 上为 1，而在 \( a, b \) 处为 0。

 artifacts 减少有限角度 FBP: \( f = \frac{1}{4\pi} \mathcal{R}^* \Lambda (\psi \mathcal{R} f) \)
Sample reconstruction [Frikel Q 2013]

FBP reconstruction: \( \epsilon = 0^\circ \)

FBP reconstruction: \( \epsilon = 20^\circ \)

Artifact reduced reconstruction
Exterior Tomography: tomography using only lines that are outside a disk to reconstruct the region outside the disk.
Exterior Tomography: tomography using only lines that are outside a disk to reconstruct the region outside the disk.

Exterior CT is used to evaluate of rockets because industrial X-ray CT scanners can’t penetrate the thick central part of the rocket, but they can penetrate the outside annulus. Often scientists are interested in cracks, etc., in the rocket shell, anyway.
Exterior Tomography: tomography using only lines that are outside a disk to reconstruct the region outside the disk.

Exterior CT is used to evaluate of rockets because industrial X-ray CT scanners can’t penetrate the thick central part of the rocket, but they can penetrate the outside annulus. Often scientists are interested in cracks, etc., in the rocket shell, anyway.

The mask for exterior CT: \([0^\circ, 180^\circ] \times ([-R, -r] \cup [r, R])\)

where \(r < R\) is the radius of the inner annulus.
Exercise

What boundaries will be easy to see in an exterior reconstruction of the phantom on the left?
Exercise

What boundaries will be easy to see in an exterior reconstruction of the phantom on the left?
Exercise

What boundaries will be easy to see in an exterior reconstruction of the phantom on the left? [Q1988]

?????? See reconstruction in class!
Exercise

*Defects in rocket shells are generally along the circumference direction of the shell.*
Defects in rocket shells are generally along the circumference direction of the shell.

1. Would exterior CT be a good modality for such defects?
2. According to the theory, what types of defects would be easy to see from exterior CT?
3. According to the theory, what types of defects would be easy to see from exterior CT?
4. Were there added artifacts in the reconstruction(s) I showed?
5. Do you think there could be added artifacts in reconstructions from exterior data? Why or why not?
**ROI Tomography:** tomography using only lines that are inside a small region (disk) to reconstruct the object inside the disk.

**Figure:** Skyscan Micro-CT Scanner

**Figure:** Object in scanner
**Region of Interest (ROI) or Interior Tomography**

**ROI Tomography:** tomography using only lines that are inside a small region (disk) to reconstruct the object inside the disk.

**Figure:** Skyscan Micro-CT Scanner

ROI CT is used for nondestructive evaluation of parts of small objects.

**The mask for ROI CT:** $[0^\circ, 180^\circ] \times [-r, r]$ where $r < R$ and $r$ is the radius of the ROI.

**Figure:** Object in scanner
Exercise

Let’s say you have a ROI data set of an object.

1. According to the theory, what types of object boundaries would be easy to reconstruct from the ROI data?

2. According to the theory, what types of object boundaries would be difficult to see from the ROI data?

3. Did you observe this in the reconstructions you did in exercises using iradon?
Visible boundary: boundary tangent to a line in the data set.

Invisible boundary: boundary tangent to no line in the data set.

Added Artifacts: streaks on lines at the ends of the data set that are tangent to the object.
Visible boundary: boundary tangent to a line in the data set.

Invisible boundary: boundary tangent to no line in the data set.

Added Artifacts: streaks on lines at the ends of the data set that are tangent to the object.

Artifact reduction: smooth the mask at the ends of the data set.
Visible boundary: boundary tangent to a line in the data set.

Invisible boundary: boundary tangent to no line in the data set.

Added Artifacts: streaks on lines at the ends of the data set that are tangent to the object.

Artifact reduction: smooth the mask at the ends of the data set.

We make this mathematically precise using the Fourier transform and microlocal analysis.
Visible boundary: boundary tangent to a line in the data set.

Invisible boundary: boundary tangent to *no* line in the data set.

Added Artifacts: streaks on lines at the ends of the data set that are tangent to the object.

Artifact reduction: smooth the mask at the ends of the data set.

We make this mathematically precise using the Fourier transform and microlocal analysis.

It applies to a broad range of limited data tomography problems in X-ray CT, sonar, radar, seismic imaging, . . .
For Further Reading I

General references:


Introductory

For Further Reading II


*Local and Lambda CT*


Microlocal references:

www.springer.com/978-1-4939-0789-2


References to the work in the talk:


