Bayesian Inference and Variational Methods for X-ray CT

Exercise 1

Run the MATLAB script for generating pseudo-random measurement data from the lecture notes. Inspect the transmission sinogram (the variable `T`) using MATLAB's `imagesc` function. Use the interval $[0, 1.1]$ as the display range and add a colorbar:

```
imagesc(T, [0,1.1]), axis image
colorbar
```

Change some parameters in the script and regenerate the measurement data. What happens to the transmission sinogram? Try to answer the following questions experimentally:

- What happens to the noise (the variable `E`) in the transmission sinogram when you increase/decrease the source photon flux (the variable `N0`)?
- What happens to the transmission sinogram contrast when you increase/decrease the phantom intensity (*i.e.*, change `uscale`)?
- What happens to the noise level in the transmission sinogram when you increase/decrese the phantom intensity (*i.e.*, change `uscale`)?

Exercise 2

1. Show that the function
$$g(u) = N_0 \mathbf{1}^T \exp(-Au) + y^T Au$$
is a convex function of u.

2. Derive the first-order necessary optimality condition for the ML estimation problem
$$\hat{u}_{\mathrm{ml}} = \operatorname*{argmin}_u \{g(u)\}.$$

3. Show that the gradient of $g(u)$ is Lipschitz continuous on $\mathbb{R}^n_+$.

4. Let $b = -\log(y/N_0)$ where the logarithm is applied elementwise (we will assume that y is positive so that the elements of b are finite). Show that if the system of equations $Au = b$ is consistent, then any $u \in \{u \mid Au = b\}$ satisfies the first-order optimality condition. (This implies that solving the ML estimation problem is equivalent to solving the consistent set of equations $Au = b$.)

Exercise 3

Recall that we obtain a MAP estimate if we maximize the posterior PDF

$$f_{\mathbf{u}|\mathbf{y}}(u|\mathbf{y}=y) = \frac{\Pr(\mathbf{y}=y|u)f_{\mathbf{u}}(u)}{\Pr(\mathbf{y}=y)}$$

where $f_{\mathbf{u}}(u)$ is the prior PDF associated with $\mathbf{u}$. If we assume that $\mathbf{u} \sim \mathcal{N}(0, \gamma^{-1}I)$, we get the MAP estimation problem

$$\hat{u}_{\text{map}} = \operatorname*{argmin}_{u} \left\{ g(u) + \frac{\gamma}{2}\|u\|_2^2 \right\},$$

where $g(u) = N_0 \exp(-Au) + y^T Au$. Alternatively, using the weighted least-squares approximation of the likelihood function, we obtain the regularized weighted least-squares estimate

$$\hat{u}_{\text{rwls}} = \operatorname*{argmin}_{u} \left\{ \frac{1}{2}\|Au - b\|_W^2 + \frac{\gamma}{2}\|u\|_2^2 \right\}$$

where $W = \mathbf{diag}(y)$.

1. Implement a gradient descent method for the regularized weighted least-squares problem. Your implementation should

   - allow the user to specify the weights
   - use a constant step size
   - use power iteration to estimate a Lipschitz constant
   - choose a suitable stopping criterion based on the first-order optimality condition
   - as an option, allow the user to specify an initial guess.

2. Even when the measurements are assumed to be samples from a Poisson distribution, a simple least-squares estimate $\hat{x}_{\text{ls}} = (A^T A)^{-1} A^T u$ is very often used in practice due to its simplicity. Set the weight matrix, W, to the identity matrix and compute a reconstruction using the gradient descent method. Compare the reconstruction with one obtained using the weights $W = \mathbf{diag}(y)$, and describe the difference between the two reconstructions.