Computational Uncertainty Quantification for Inverse problems

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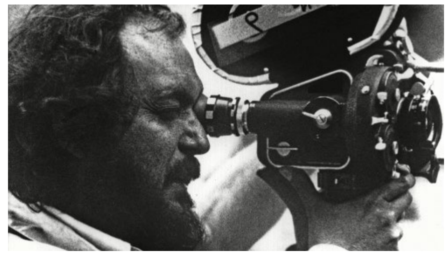
Inverse Problems



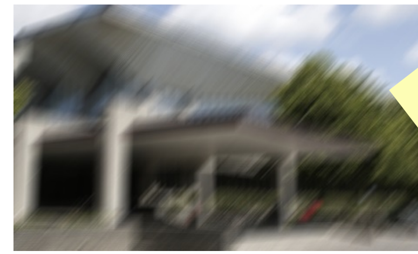
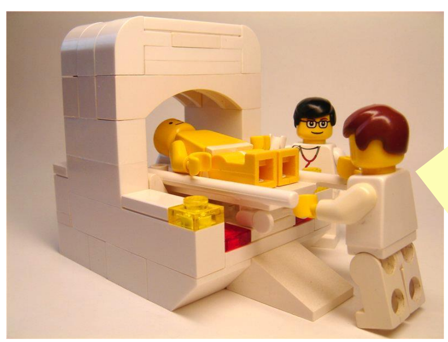


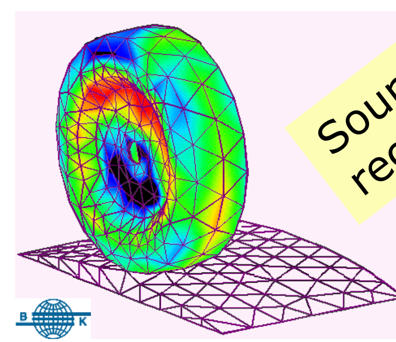
Image deblurring

The inverse problem:
solve millions of equations

Light influx thru windows



Computed tomography

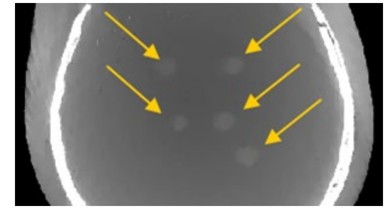


Sound source reconstruction

Uncertainty Quantification (UQ)

X-ray medical imaging (CT)

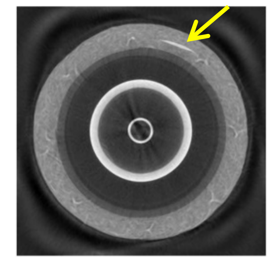
- How reliable are the computed spots?



Tomographic industrial inspection

- How reliable are the computed defects?

FORCE TECHNOLOGY



All kinds of errors have influence on, e.g., a least squares solution:

$$x = \arg\min \| A\, x - b \|_2$$

algorithm-error model-error data-error

UQ: the mathematical study of the impact of all forms of error and uncertainty.

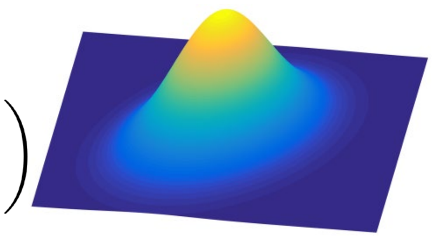
OK – I Know How to Do That …

Statistics 101

Linear model with Gaussian noise and a Gaussian prior for $\boldsymbol{x}$:

$$\boldsymbol{b} = \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{n}\ , \qquad \boldsymbol{n} \sim \mathcal{N}(\boldsymbol{0}, \lambda^{-1}\boldsymbol{I})\ , \qquad \boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, \delta^{-1}\boldsymbol{I})\ .$$

The *posterior* for the solution $\boldsymbol{x}$ is



$$p(\boldsymbol{x}|\boldsymbol{b},\lambda,\delta) \propto \exp\left(-\frac{\lambda}{2}\,\|\boldsymbol{A}\,\boldsymbol{x} - \boldsymbol{b}\|_2^2\right) \cdot \exp\left(-\frac{\delta}{2}\,\|\boldsymbol{x}\|_2^2\right)$$

and it gives a complete statistical quantification of the uncertainty in $\boldsymbol{x}$.

But this basic result can **only** be used:

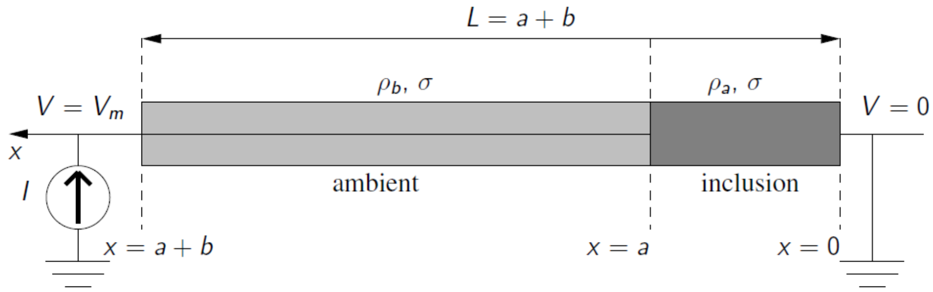
- for least squares problems
- with Gaussian noise, uncertainties, and priors
- and without constraints (e.g. non-negativity).

CUQI

1. Define a common **framework** for general UQ problems.
2. Develop new sampling **methods** and yet unknown **algorithms**.

Case: UQ for Electric Conductivity

Mirza Karamehmedovic
DTU Compute



Unknown: the length *a*.

Cannot use Ohm's law.

Use a stochastic differential equation to model the current.

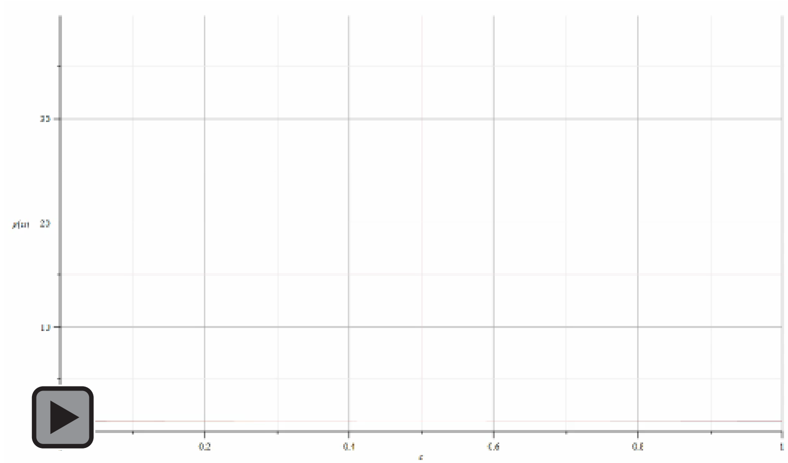
Bayes theorem:

$$p(a|V_m) \propto p(V_m|a) \cdot p(a) \ .$$

solution's posterior

data likelihood
Gaussian

prior for solution
truncated Gaussian



The evolution of the solution's posterior $p(a|V_m)$ as we increase the number of observations.

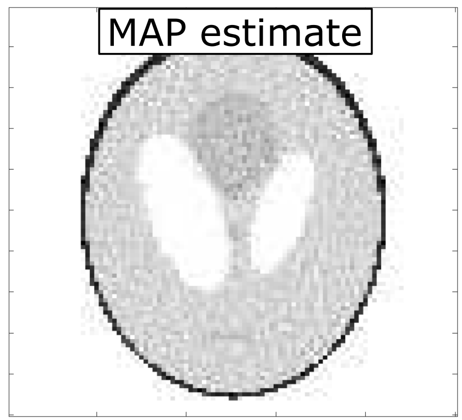
Case: UQ for Non-Negative Prior

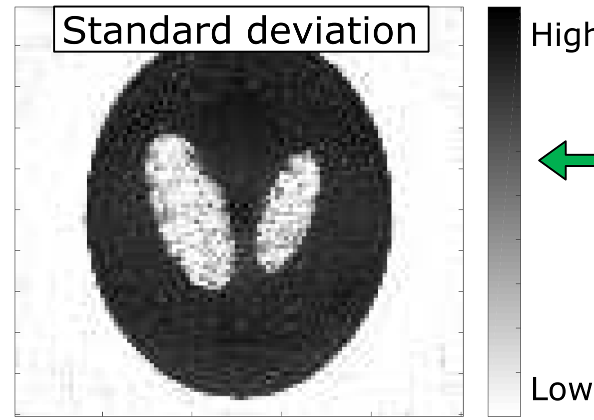
Johnathan M. Bardsley
Univ. of Montana

PCH, DTU Compute

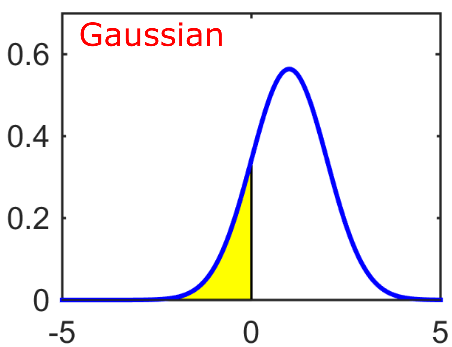
Positron emission tomography

- How reliable is the image?

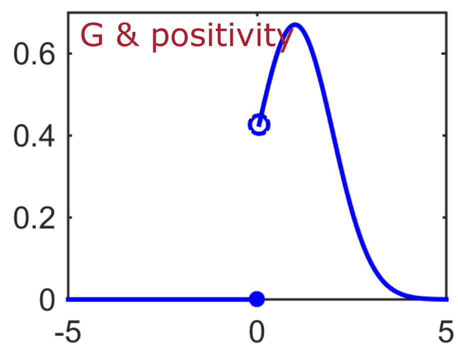




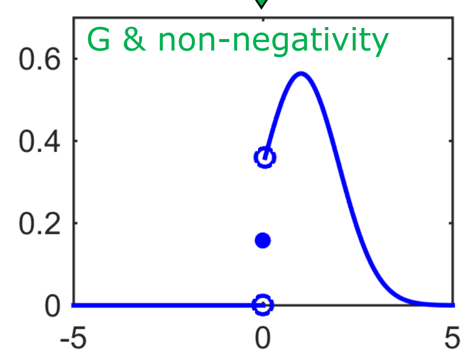
$$\min_{\boldsymbol{x}} \|\boldsymbol{A}\,\boldsymbol{x} - \boldsymbol{b}\|_2 \quad \text{s.t.} \quad \boldsymbol{x} \geq \boldsymbol{0}\ , \quad \boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, \delta^{-1}\boldsymbol{I})$$



Analytical solution



Variable transform
$x = \exp(z)$



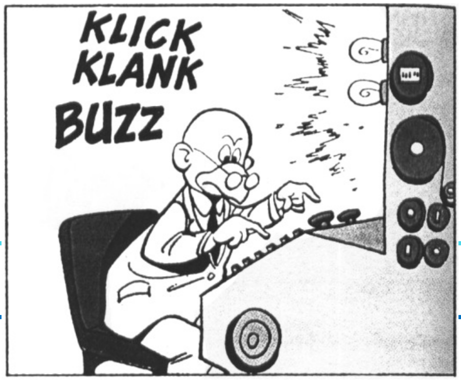
New MCMC method
Nonnegative Hierarchical Gibbs Sampler.

Scientific Computing

Case: UQ for compressed sensing

$$\min_{\boldsymbol{x}} \|\boldsymbol{A}\,\boldsymbol{x} - \boldsymbol{b}\|_2 \quad \text{s.t.} \quad \|\boldsymbol{x}\|_1 \leq \delta$$

- Today: 500 lines of code



Future: 5 lines of model description

```
variable x(n,1)                                % Define unknown vector.
parameter delta:Gauss(mean=0.1,std=0.02)       % Parameter with Gauss distrib.
UQ_data_model(b,Poisson,mean=A*x_exact)        % Data with Poisson noise
UQ_minimize misfit(A*x,b)                      % Solution that fits data
subject_to UQ_prior(x,sparse,delta)            % ... with a sparsity prior.
```

CUQI

1. **Mathematical models** for non-Gaussian cases.
2. **Algorithms** for sampling large-scale UQ problems.
3. **Modeling platforms** with software to aid non-experts.

More Examples and Stuff

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A research initiative sponsored by Villum Fonden (the Villum Foundation)

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