02157 Functional Programming

Lecture 8: Tail recursive (iterative) functions

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Overview

- Memory management: the stack and the heap
- Iterative (tail-recursive) functions is a simple technique to deal with efficiency in certain situations, e.g.
  - to avoid evaluations with a huge amount of pending operations, e.g.

$$7 + (6 + (5 \cdots + f\ 2 \cdots))$$

  - to avoid inadequate use of @ in recursive declarations.
- Iterative functions with accumulating parameters correspond to while-loops
- The notion: continuations, provides a general applicable approach

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An example: Factorial function (I)

Consider the following declaration:

```
let rec fact = function
  | 0 -> 1
  | n -> n * fact(n-1);;
val fact : int -> int
```

- What resources are needed to compute `fact`(N)?

Considerations:

- Computation time: number of individual computation steps.
- Space: the maximal memory needed during the computation to represent expressions and bindings.

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Evaluation:

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\begin{array}{ll}
 & \texttt{fact}(N) \\
\leadsto & (\texttt{n * fact(n-1)}\ ,\ [\texttt{n} \mapsto N]) \\
\leadsto & N * \texttt{fact}(N-1) \\
\leadsto & N * (\texttt{n * fact(n-1)}\ ,\ [\texttt{n} \mapsto N-1]) \\
\leadsto & N * ((N-1) * \texttt{fact}(N-2)) \\
\vdots & \\
\leadsto & N * ((N-1) * ((N-2) * (\cdots (4 * (3 * (2 * 1))) \cdots))) \\
\leadsto & N * ((N-1) * ((N-2) * (\cdots (4 * (3 * 2)) \cdots))) \\
\vdots & \\
\leadsto & N!
\end{array}
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Time and space demands: proportional to N Is this satisfactory?

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Another example: Naive reversal (I)

```
let rec naiveRev = function
  | []    -> []
  | x::xs -> naiveRev xs @ [x];;
val naiveRev : 'a list -> 'a list
```

Evaluation of naiveRev $[x_1, x_2, \ldots, x_n]$:

$$
\begin{array}{ll}
 & \texttt{naiveRev}\,[x_1, x_2, \ldots, x_n] \\
\rightsquigarrow & \texttt{naiveRev}\,[x_2, \ldots, x_n]@[x_1] \\
\rightsquigarrow & (\texttt{naiveRev}\,[x_3, \ldots, x_n]@[x_2])@[x_1] \\
\vdots & \\
\rightsquigarrow & ((\cdots(([\,]@[x_n])@[x_{n-1}])@\cdots@[x_2])@[x_1])
\end{array}
$$

Space demands: proportional to n — satisfactory

Time demands: proportional to n^2 — not satisfactory

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Examples: Accumulating parameters

Efficient solutions are obtained by using *more general functions*:

$$\begin{aligned} \texttt{factA}(n, m) &= n! \cdot m, \text{ for } n \geq 0 \\ \texttt{revA}([x_1, \ldots, x_n], ys) &= [x_n, \ldots, x_1]\,@ys \end{aligned}$$

We have:

$$\begin{aligned} n! &= \texttt{factA}(n, 1) \\ \texttt{rev}\,[x_1, \ldots, x_n] &= \texttt{revA}([x_1, \ldots, x_n], [\,]) \end{aligned}$$

m and *ys* are called *accumulating parameters*. They are used to hold the temporary result during the evaluation.

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Declaration of `factA`

```
let rec factA = function
  | (0,m) -> m
  | (n,m) -> factA(n-1,n*m) ;;
```

An evaluation:

$$
\begin{array}{ll}
 & \texttt{factA(5,1)} \\
\rightsquigarrow & (\texttt{factA(n-1,n*m)}, [\texttt{n} \mapsto 5, \texttt{m} \mapsto 1]) \\
\rightsquigarrow & \texttt{factA(4,5)} \\
\rightsquigarrow & (\texttt{factA(n-1,n*m)}, [\texttt{n} \mapsto 4, \texttt{m} \mapsto 5]) \\
\rightsquigarrow & \texttt{factA(3,20)} \\
\rightsquigarrow & \texttt{...} \\
\rightsquigarrow & \texttt{factA(0,120)} \rightsquigarrow (\texttt{m}, [\texttt{m} \mapsto 120]) \rightsquigarrow 120
\end{array}
$$

Space demand: constant.

Time demands: proportional to *n*

Declaration of `revA`

```
let rec revA = function
  | ([], ys)    -> ys
  | (x::xs, ys) -> revA(xs, x::ys) ;;
```

An evaluation:

```
     revA([1,2,3],[])
⇝   revA([2,3],1::[])
⇝   revA([3],2::[1])
⇝   revA([3],[2,1])
⇝   revA([],3::[2,1])
⇝   revA([],[3,2,1])
⇝   [3,2,1]
```

Space and time demands:

proportional to *n* (the length of the first list)

Iterative (tail-recursive) functions (I)

The declarations of `factA` and `revA` are *tail-recursive functions*

- the recursive call is the *last function application* to be evaluated in the body of the declaration e.g. $itfac(3, 20)$ and $revA([3], [2, 1])$
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Example

```
let rec factA = function
  | (0,m) -> m
  | (n,m) -> factA(n-1,n*m)
             (* recursive "tail-call" *)
```

- only one set of bindings for argument identifiers is needed during the evaluation

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Concrete resource measurements: factorial functions

```
let xs16 = List.init 1000000 (fun i -> 16);;
val xs16 : int list = [16; 16; 16; 16; 16; ...]

#time;; // a toggle in the interactive environment

for i in xs16 do let _ = fact i in ();;
Real: 00:00:00.051, CPU: 00:00:00.046, ...

for i in xs16 do let _ = factA(i,1) in ();;
Real: 00:00:00.024, CPU: 00:00:00.031, ...
```

The performance gain of `factA` is much better than the indicated factor 2 because the `for` construct alone uses about 12 ms:

```
for i in xs16 do let _ = () in ();;
Real: 00:00:00.012, CPU: 00:00:00.015, ...
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Real: time elapsed by the execution. CPU: time spent by all cores.

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Concrete resource measurements: reverse functions

```
let xs20000 = [1 .. 20000];;

naiveRev xs20000;;
Real: 00:00:07.624, CPU: 00:00:07.597,
GC gen0: 825, gen1: 253, gen2: 0
val it : int list = [20000; 19999; 19998; ...]

revA(xs20000,[]);;
Real: 00:00:00.001, CPU: 00:00:00.000,
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```

- The naive version takes 7.624 seconds - the iterative just 1 ms.
- The use of append (@) has been reduced to a use of cons (`::`). This has a dramatic effect of the garbage collection:
  - No object is reclaimed when `revA` is used
  - 825+253 obsolete objects were reclaimed using the naive version

Let's look at memory management

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Memory management: stack and heap

- Primitive values are allocated on the stack
- Composite values are allocated on the heap

```
let xs = [5;6;7];;
let ys = 3::4::xs;;
let zs = xs @ ys;;
let n = 27;;
```

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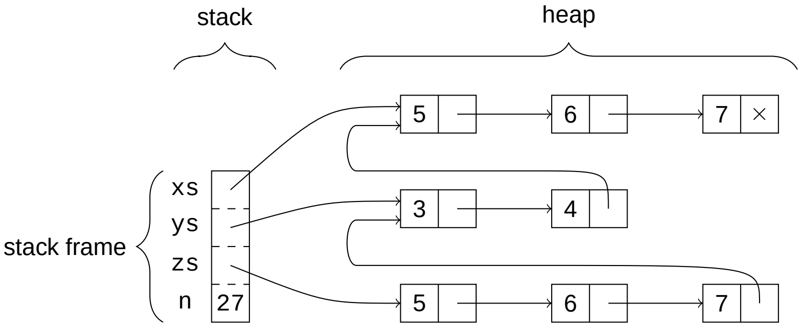

Observations

No unnecessary copying is done:

1. The linked lists for *ys* is not copied when building a linked list for *y* :: *ys*.
2. Fresh cons cells are made for the elements of *xs* only when building a linked list for *xs* @ *ys*.

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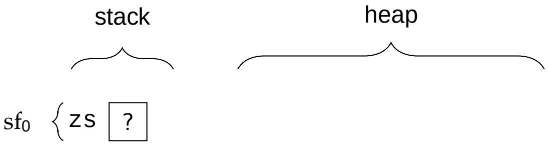
Operations on stack and heap

Example:

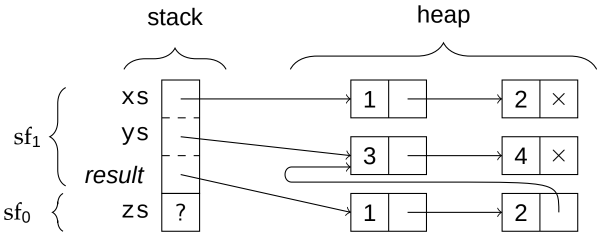
```
let zs = let xs = [1;2]
         let ys = [3;4]
         xs@ys;;
```

Initial stack and heap prior to the evaluation of the local declarations:



Operations on stack: Push

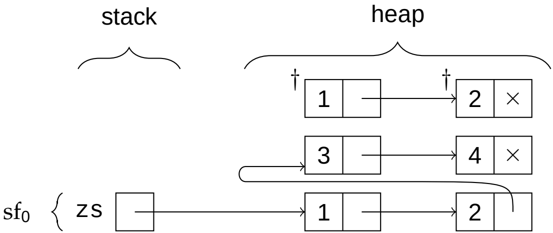
Evaluation of the local declarations initiated by pushing a new stack frame onto the stack:



The auxiliary entry result refers to the value of the `let`-expression.

Operations on stack: Pop

The top stack frame is popped from the stack when the evaluation of the `let`-expression is completed:



The resulting heap contains two obsolete cells marked with '†'

Operations on the heap: Garbage collection

The memory management system uses a *garbage collector* to reclaim obsolete cells in the heap behind the scene.

The garbage collector manages the heap as partitioned into three groups or *generations*: `gen0`, `gen1` and `gen2`, according to their age. The objects in `gen0` are the youngest while the objects in `gen2` are the oldest.

The typical situation is that objects die young and the garbage collector is designed for that situation.

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The limits of the stack and the heap

The stack is big:

```
let rec bigList n = if n=0 then [] else 1::bigList(n-1);;
bigList 120000;;
val it : int list = [1; 1; 1; 1; 1; 1; 1; 1;...]
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Process is terminated due to StackOverflowException.
```

More than $1.2 \cdot 10^5$ stack frames are pushed in recursive calls.

The heap is much bigger:

```
let rec bigListA n xs = if n=0 then xs
                        else bigListA (n-1) (1::xs);;
let xsVeryBig = bigListA 12000000 [];;
val xsVeryBig : int list = [1; 1; 1; 1; 1; 1;...]
let xsTooBig = bigListA 13000000 [];;
System.OutOfMemoryException: ...
```

A list with more than $1.2 \cdot 10^7$ elements can be created.

The iterative `bigListA` function does not exhaust the stack. WHY?

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Iterative (tail-recursive) functions (II)

Tail-recursive functions are also called *iterative functions*.

- The function $f(n, m) = (n - 1, n * m)$ is iterated during evaluations for `factA`.
- The function $g(x :: xs, ys) = (xs, x :: ys)$ is iterated during evaluations for `revA`.

The correspondence between tail-recursive functions and while loops is established in the textbook.

An example:

```
let factW n =
    let ni = ref n
    let r  = ref 1
    while !ni>0 do
          r := !r * !ni ; ni := !ni-1
    !r;;
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Iterative functions (III)

A function $g : \tau$ `->` τ' is an *iteration of f* $: \tau$ `->` τ if it is an instance of:

```
let rec g z = if p z then g(f z) else h z
```

for suitable predicate $p : \tau$ `-> bool` and function $h : \tau$ `->` τ'.

The function g is called an *iterative (or tail-recursive) function*.

Examples: `factA` and `revA` are easily declared in the above form:

```
let rec factA(n,m) =
   if n<>0 then factA(n-1,n*m) else m;;
```

```
let rec revA(xs,ys) =
     if not (List.isEmpty xs)
     then revA(List.tail xs, (List.head xs)::ys)
     else ys;;
```

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for suitable predicate $p : \tau$ `-> bool` and function $h : \tau$ `->` τ'.

The function g is called an *iterative (or tail-recursive) function*.

Examples: `factA` and `revA` are easily declared in the above form:

```
let rec factA(n,m) =
   if n<>0 then factA(n-1,n*m) else m;;
```

```
let rec revA(xs,ys) =
     if not (List.isEmpty xs)
     then revA(List.tail xs, (List.head xs)::ys)
     else ys;;
```

Iterative functions (III)

A function $g : \tau$ `->` τ' is an *iteration of f* $: \tau$ `->` τ if it is an instance of:

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Iterative functions: evaluations (I)

Consider: `let rec g z = if p z then g(f z) else h z`

Evaluation of the $g\ v$:

$$
\begin{array}{ll}
 & g\ v \\
\rightsquigarrow & (\texttt{if } p\ \texttt{z then } g(f\ \texttt{z}) \texttt{ else } h\ \texttt{z}\ ,\ [\texttt{z} \mapsto v]) \\
\rightsquigarrow & (g(f\ \texttt{z}), [\texttt{z} \mapsto v]) \\
\rightsquigarrow & g(f^1 v) \\
\rightsquigarrow & (\texttt{if } p\ \texttt{z then } g(f\ \texttt{z}) \texttt{ else } h\ \texttt{z}\ ,\ [\texttt{z} \mapsto f^1 v]) \\
\rightsquigarrow & (g(f\ \texttt{z}), [\texttt{z} \mapsto f^1 v]) \\
\rightsquigarrow & g(f^2 v) \\
\rightsquigarrow & \ldots \\
\rightsquigarrow & (\texttt{if } p\ \texttt{z then } g(f\ \texttt{z}) \texttt{ else } h\ \texttt{z}\ ,\ [\texttt{z} \mapsto f^n v]) \\
\rightsquigarrow & (h\ \texttt{z}, [\texttt{z} \mapsto f^n v]) \qquad \text{suppose } p(f^n v) \rightsquigarrow \text{false} \\
\rightsquigarrow & h(f^n v)
\end{array}
$$

Iterative functions: evaluations (II)

Observe two desirable properties:

- there are *n* recursive calls of *g*,
- at most *one binding* for the argument pattern z is ‘active’ at any stage in the evaluation, and
- the iterative functions require one stack frame only.

Iteration vs While loops

Iterative functions are executed efficiently:

```
#time;;

for i in 1 .. 1000000 do let _ = factA(16,1) in ();;
Real: 00:00:00.024, CPU: 00:00:00.031,
GC gen0: 0, gen1: 0, gen2: 0
val it : unit = ()

for i in 1 .. 1000000 do let _ = factW 16 in ();;
Real: 00:00:00.048, CPU: 00:00:00.046,
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Example: Fibonacci numbers (I)

A declaration based directly on the mathematical definition:

```
let rec fib = function
  | 0 -> 0
  | 1 -> 1
  | n -> fib(n-1) + fib(n-2);;
val fib : int -> int
```

is highly inefficient. For example:

```
    fib 4
⇝  fib 3 + fib 2
⇝  (fib 2 + fib 1) + fib 2
⇝  ((fib 1 + fib 0) + fib 1) + fib 2
⇝  ··· ⇝ 2 + (fib 1 + fib 0)
⇝  ···
```

Ex: `fib 44` requires around 10^9 evaluations of base cases.

Example: Fibonacci numbers (II)

An iterative solution gives high efficiency:

```
fun recitfib(n,a,b) = if n <> 0
                      then itfib(n-1,a+b,a)
                      else a;;
```

The expression $\texttt{itfib}(n, 0, 1)$ evaluates to F_n, for any $n \geq 0$:

- Case $n = 0$: $\texttt{itfib}(0, 0, 1) \rightsquigarrow 0\ (= F_0)$
- Case $n > 0$:

$$
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 & \texttt{itfib}(n, 0, 1) \\
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\rightsquigarrow & \texttt{itfib}(n-2,\ F_1 + F_0,\ F_1) \\
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Limits of accumulating parameters

Accumulating parameters are not sufficient to achieve a tail-recursive version for arbitrary recursive functions.

Consider for example:

```
type BinTree<'a> =
     | Leaf
     | Node of BinTree<'a> * 'a * BinTree<'a>;;

let rec count = function
    | Leaf            -> 0
    | Node(tl,n,tr) -> count tl + count tr + 1;;
```

A counting function:

```
countA: int -> BinTree<'a> -> int
```

using an accumulating parameter will not be tail-recursive due to the expression containing recursive calls on the left and right sub-trees. (Ex. 9.8)

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Continuations

Continuation: A function for the “rest” of the computation.

The continuation-based version of `bigList` has a continuation

```
c: int list -> int list
```

as argument:

```
let rec bigListC n c =
    if n=0 then c []
    else bigListC (n-1) (fun res -> c(1::res));;
val bigListC : int -> (int list -> 'a) -> 'a
```

- Base case: “feed” the result of `bigList` into the continuation `c`.
- Recursive case: let `res` denote the value of `bigList(n-1)`:
  - The rest of the computation of `bigList n` is `1::res`.
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Observations

- `bigListC` is a tail-recursive function, and
- the calls of `c` are tail calls in the base case of `bigListC` and in the continuation: `fun res -> c(1::res)`.

The stack will hence neither grow due to the evaluation of recursive calls of `bigListC` nor due to calls of the continuations that have been built in the heap:

```
bigListC 16000000 id;;
Real: 00:00:08.586, CPU: 00:00:08.314,
GC gen0: 80, gen1: 60, gen2: 3
val it : int list = [1; 1; 1; 1; 1;...]
```

- Slower than `bigList`
- Can generate longer lists than `bigList`

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Example: Tail-recursive count

```
let rec countC t c =
  match t with
  | Leaf         -> c 0
  | Node(tl,n,tr) ->
    countC tl (fun vl -> countC tr (fun vr -> c(vl+vr+1)))
val countC : BinTree<'a> -> (int -> 'b) -> 'b

countC (Node(Node(Leaf,1,Leaf),2,Node(Leaf,3,Leaf))) id;;
val it : int = 3
```

- Both calls of `countC` are tail calls
- The calls of the `c` is tail call

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- Loops in imperative languages corresponds to a *special case* of recursive function called tail recursive functions.
- Have iterative functions in mind when dealing with efficiency, e.g.
  - to avoid evaluations with a huge amount of pending operations
  - to avoid inadequate use of @ in recursive declarations.
- Memory management: stack, heap, garbage collection
- Continuations – provide a technique to turn arbitrary recursive functions into tail-recursive ones.

  trades stack for heap

Note: Iterative function does not replace algorithmic idea and the use of good algorithms and datastructure.

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