Generalized Row-Action Methods for Tomographic Imaging

Sparse Tomo Days, Technical University of Denmark, March 27, 2014

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joint work with
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Scientific Computing Section

Outline

1. Algebraic methods for X-ray computed tomography
2. Relaxed incremental proximal methods
3. Numerical experiments

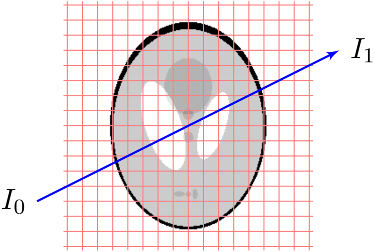
X-ray Computed Tomography

Measurement model

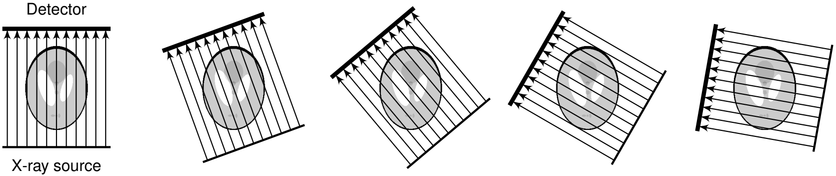
$$I_1 = I_0 \exp\left(- \int_l \mu(u_1, u_2)\, ds \right)$$

$$\log(I_0/I_1) \approx a_i^T x$$

$$b = Ax + e$$



Parallel beam measurement geometry



Algebraic reconstruction technique

Projection on hyperplane $\mathcal{H}_i = \{x \mid a_i^T x = b_i\}$

$$P_{\mathcal{H}_i}(x_k) = \underset{x \in \mathcal{H}_i}{\operatorname{argmin}} \|x - x_k\| = x_k - \frac{a_i(a_i^T x_k - b_i)}{\|a_i\|^2}$$

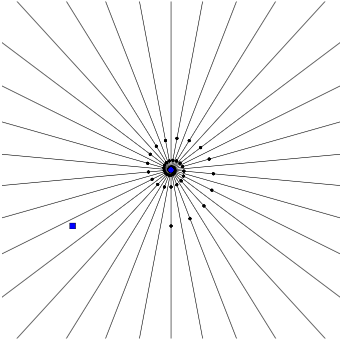
Kaczmarz's method / ART

$$x_{k+1} = \rho P_{\mathcal{H}_{i_k}}(x_k) + (1-\rho)x_k, \qquad i_k \in \{1, \ldots, m\}$$

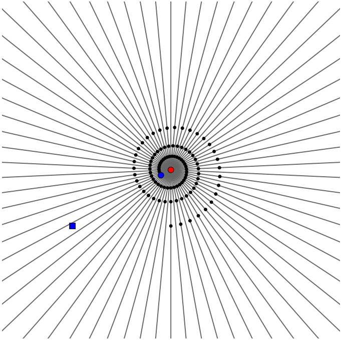
Relaxation parameter $\rho \in (0, 2)$

Example — consistent system

17 equations, $\rho = 1.0$

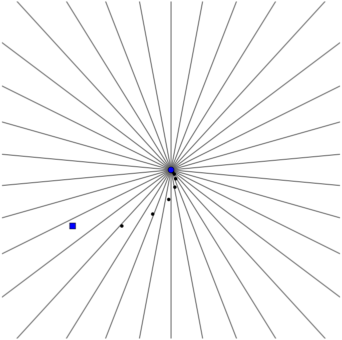


35 equations, $\rho = 1.0$



Example — consistent system

17 equations, $\rho = 0.5$



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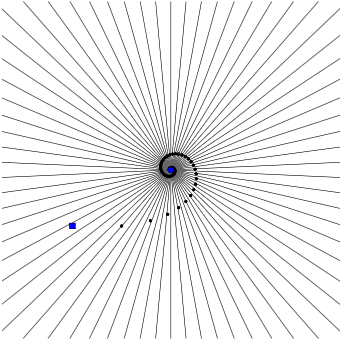

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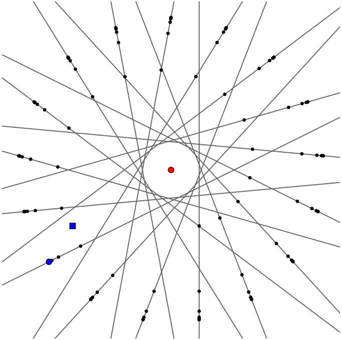

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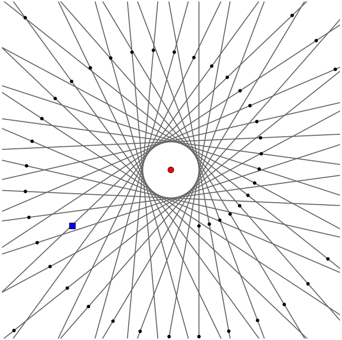

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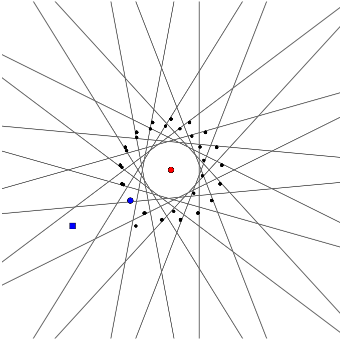

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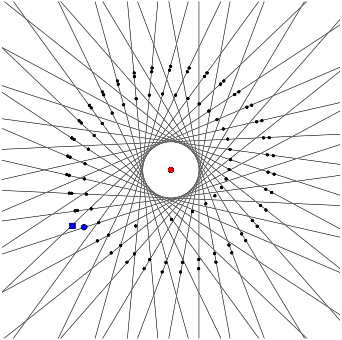
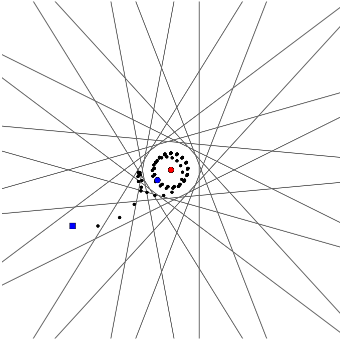


Example — inconsistent system

17 equations, $\rho = 0.2$



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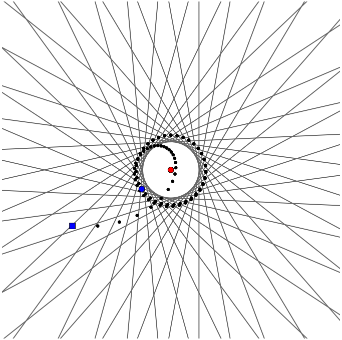
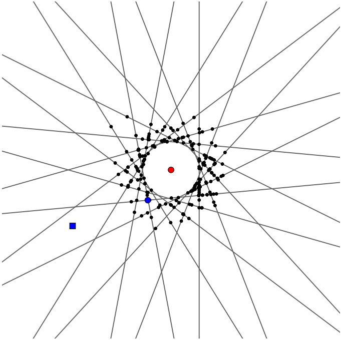


Example — inconsistent system, randomization

17 equations, $\rho = 1.0$



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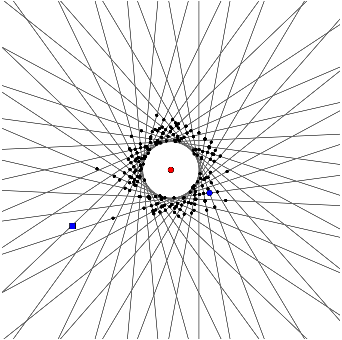

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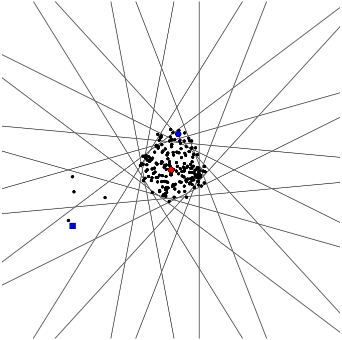

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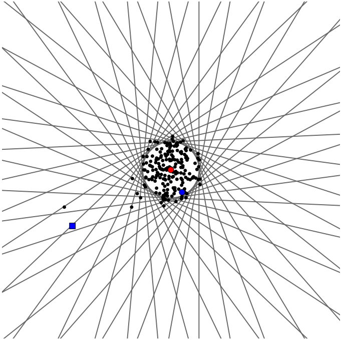

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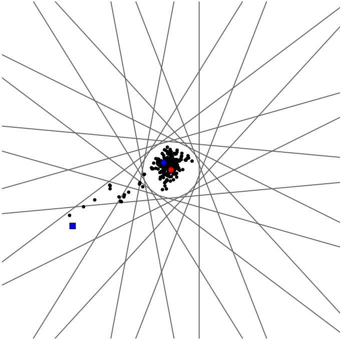

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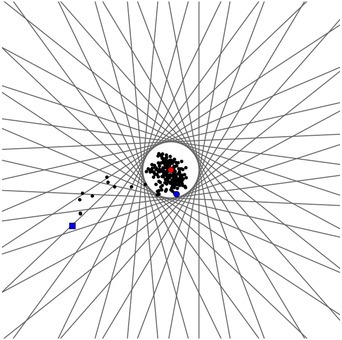
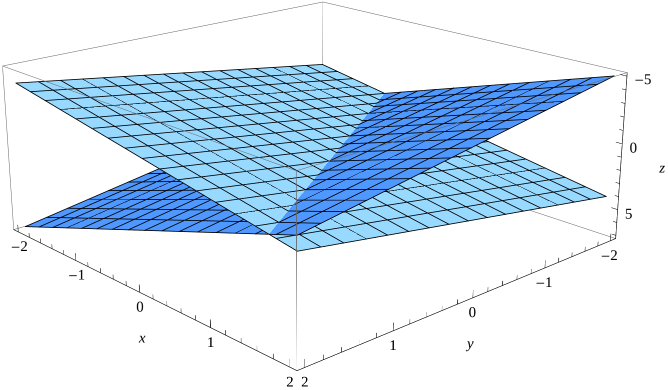
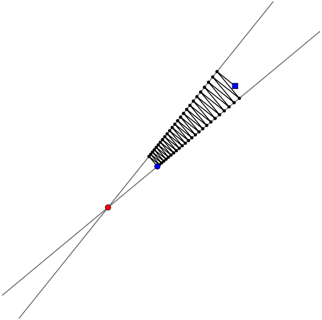


Example — underdetermined system



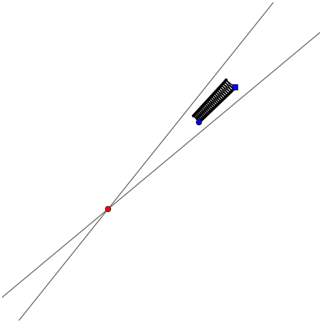
Example — consistent system

$\rho = 1.0$



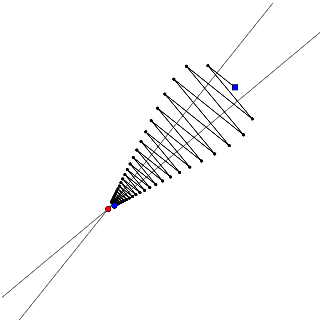
Example — consistent system

$\rho = 0.5$



Example — consistent system

$$\rho = 1.5$$



Tomographic image reconstruction

Ill-posed inverse problem — we need regularization!

Incorporate a priori knowledge in reconstruction problem

- spatial information (smoothness, piecewise constant/affine, ...)
- bounds (nonnegativity, box constraints, ...)
- sparsity
- ...

Large-scale optimization problem

- gradient computation is expensive
- may not be differentiable

Superiorization

Kaczmarz's method is pertubation resilient (Herman et al., 2009)

$$x_{k+1} = \mathcal{P}\big(x_k + t_k v_k\big), \qquad \mathcal{P} = P_{\mathcal{H}_m} \circ \cdots \circ P_{\mathcal{H}_2} \circ P_{\mathcal{H}_1}$$

Converges if $Ax = b$ is consistent and $t_k \to 0$ for $k \to \infty$

Perturbed iteration yields a "superior" solution in some sense

Incremental methods (I)

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} f_i(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

Incremental (sub)gradient iteration:

$$x_{k+1} = P_{\mathcal{C}}\big(x_k - t_k \widetilde{\nabla} f_{i_k}(x_k)\big), \qquad \widetilde{\nabla} f_{i_k}(x_k) \in \partial f_{i_k}(x_k)$$

- sublinear rate of convergence — *initial convergence* often very fast
- diminishing stepsize or “oscillation” that depends on stepsize
- goes back to Kibardin (1980), Litvakov (1966)
- Bertsekas (1996): incremental least-squares and extended Kalman filter

Incremental methods (II)

Incremental proximal iteration:

$$x_{k+1} = \underset{x \in \mathcal{C}}{\operatorname{argmin}} \Big\{ f_{i_k}(x) + \frac{1}{2t_k} \|x - x_k\|^2 \Big\}$$

equivalently

$$x_{k+1} = x_k - t_k g_{k+1}, \quad g_{k+1} \in \partial f_{i_k}(x_{k+1}) + N_{\mathcal{C}}(x_{k+1})$$

Linearized proximal iteration: let $g_k \in \partial f_{i_k}(x_k)$

$$\begin{aligned} x_{k+1} &= P_{\mathcal{C}}\Big(\underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \Big\{ f_{i_k}(x_k) + g_k^T x + \frac{1}{2t_k} \|x - x_k\|^2 \Big\} \Big) \\ &= P_{\mathcal{C}}\big(x_k - t_k g_k \big) \end{aligned}$$

Incremental proximal gradient methods (I)

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m}\big(g_i(x) + h_i(x)\big) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

Algorithm 1 (Bertsekas, 2011)

$$z_k = \underset{u\in\mathbb{R}^n}{\operatorname{argmin}}\Big\{g_{i_k}(u) + \frac{1}{2t_k}\|u - x_k\|^2\Big\}$$

$$x_{k+1} = P_{\mathcal{C}}\big(z_k - t_k\widetilde{\nabla} h_{i_k}(z_k)\big)$$

Interpretation

$$x_{k+1} = P_{\mathcal{C}}\big(x_k - t_k\widetilde{\nabla} g_{i_k}(z_k) - t_k\widetilde{\nabla} h_{i_k}(z_k)\big)$$

Incremental proximal gradient methods (II)

Algorithm 2 (Bertsekas, 2011)

$$z_k = x_k - t_k \widetilde{\nabla} h_{i_k}(x_k)$$
$$x_{k+1} = \operatorname*{argmin}_{u \in \mathcal{C}} \Big\{ g_{i_k}(u) + \frac{1}{2t_k} \|u - z_k\|^2 \Big\}$$

Interpretation

$$\begin{aligned} x_{k+1} &= \operatorname*{argmin}_{u \in \mathcal{C}} \Big\{ g_{i_k}(u) + \widetilde{\nabla} h_{i_k}(x_k)^T u + \frac{1}{2t_k} \|u - x_k\|^2 \Big\} \\ &= P_{\mathcal{C}}\Big(x_k - t_k \widetilde{\nabla} g_{i_k}(x_{k+1}) - t_k \widetilde{\nabla} h_{i_k}(x_k) \Big) \end{aligned}$$

Relaxed incremental proximal gradient methods

Algorithm 1

$$
\begin{aligned}
w_k &= \operatorname*{argmin}_{u \in \mathbb{R}^n} \Big\{ g_{i_k}(u) + \frac{1}{2t_k} \|u - x_k\|^2 \Big\} \\
z_k &= w_k - t_k \widetilde{\nabla} h_{i_k}(w_k) \\
x_{k+1} &= P_{\mathcal{C}}\big(\rho\, z_k + (1-\rho) x_k\big)
\end{aligned}
$$

Algorithm 2

$$
\begin{aligned}
w_k &= w_k - t_k \widetilde{\nabla} h_{i_k}(w_k) \\
z_k &= \operatorname*{argmin}_{u \in \mathbb{R}^n} \Big\{ g_{i_k}(u) + \frac{1}{2t_k} \|u - w_k\|^2 \Big\} \\
x_{k+1} &= P_{\mathcal{C}}\big(\rho\, z_k + (1-\rho) x_k\big)
\end{aligned}
$$

Convergence results

Cyclic order or random order?

- cyclic order yields worst-case performance bounds
- random order yields expected performance bounds

Constant stepsize or diminishing stepize?

- constant stepsize yields convergence within an error bound
- diminishing stepsize yields exact convergence

Randomized cyclic order works well in practice

R-IPG vs. ART

$$\begin{array}{ll} \text{minimize} & (1/2)\sum_{i=1}^{m}(a_i^T x - b_i)^2 \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

Let $g_i(x) = (1/2)(a_i^T x - b_i)^2$ and $h_i(x) = 0$:

$$x_{k+1} = P_{\mathcal{C}}\left(x_k - \rho \frac{a_{i_k}(a_{i_k}^T x_k - b_{i_k})}{t_k^{-1} + \|a_{i_k}\|^2}\right)$$

Interpretation: ART with damped step size

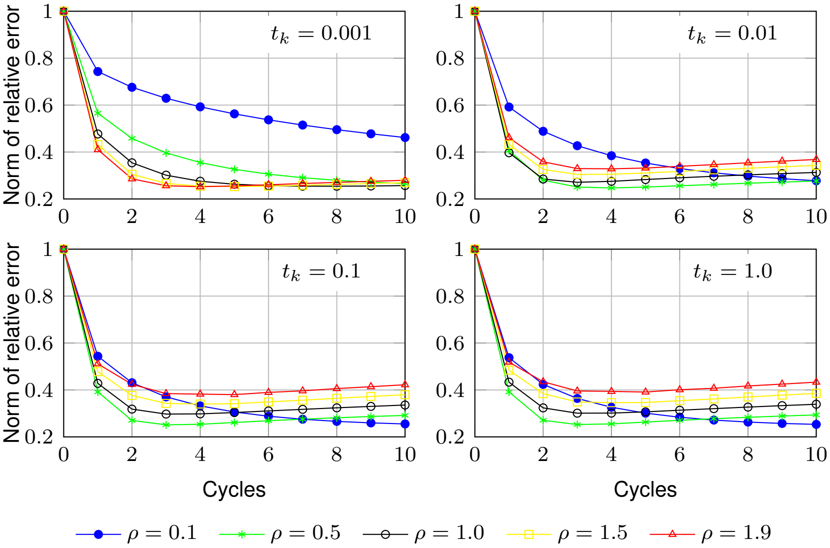
R-IPG vs. ART — numerical example

2D tomography problem

- 256×256 Shepp–Logan phantom
- $n = 256^2 = 65536$ variables
- $p = 120$ uniformly spaced angles
- $r = 362$ parallel rays
- $m = pr = 43440$ measurements
- Gaussian noise: $e_i \sim \mathcal{N}(0, 0.02 \cdot \|b\|_\infty)$

Algorithm: R-IPG with constant step size

R-IPG vs. ART — numerical example



Regularized data fitting

$$\begin{array}{ll} \text{minimize} & f(x) \equiv (1/2)\|Ax - b\|^2 + \lambda\, h(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

Example: express $f(x)$ as

$$f(x) = \sum_{i=1}^{m} \underbrace{(1/2)(a_i^T x - b_i)^2}_{g_i(x)} + \sum_{i=1}^{m} \underbrace{\frac{\lambda}{m} h(x)}_{h_i(x)}$$

Total-variation regularization

$$\begin{array}{ll} \text{minimize} & (1/2)\|Ax-b\|_2^2 + \lambda \sum_{i=1}^n \|D_i x\|_2 \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

Let $g_i(x) = \|A_i x - b_i\|_2^2$ and $h_i(x) = (\lambda/p) \sum_{i=1}^n \|D_i x\|_2$

$$\begin{aligned} z_k &= \operatorname*{argmin}_{u \in \mathbb{R}^n} \Big\{ g_{i_k}(u) + \frac{1}{2t_k} \|u - x_k\|^2 \Big\} \\ &= x_k - A_{i_k}^T (A_{i_k} A_{i_k}^T + t_k^{-1} I)^{-1} (A_{i_k} x_k - b_{i_k}) \\ x_{k+1} &= P_{\mathcal{C}}\big(\rho(z_k - t_k \widetilde{\nabla} h_{i_k}(z_k)) + (1-\rho) x_k\big) \end{aligned}$$

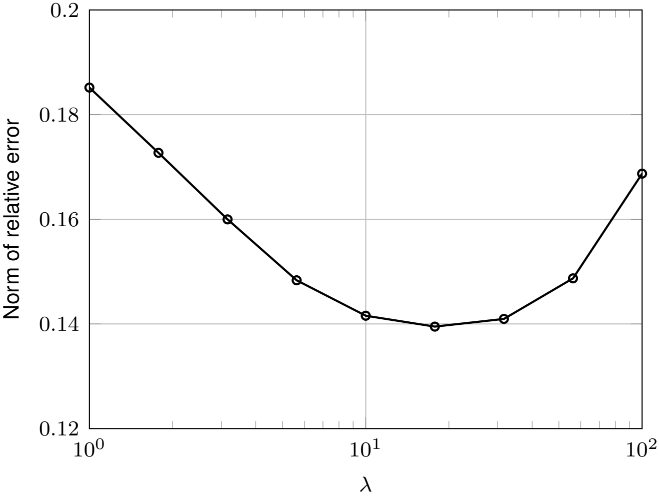
Total-variation regularization — numerical example

2D tomography problem

- 512×512 Shepp–Logan phantom
- $n = 512^2 = 262144$ variables
- $p = 60$ uniformly spaced angles
- $r = 724$ parallel rays
- $m = pr = 43440$ measurements
- Gaussian noise: $e_i \sim \mathcal{N}(0, 0.01 \cdot \|b\|_\infty)$

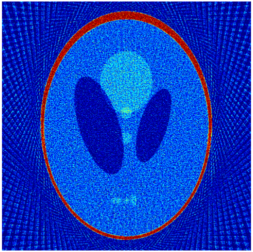
Algorithm: R-IPG1 with diminishing step-size sequence

Regularization curve



Reference

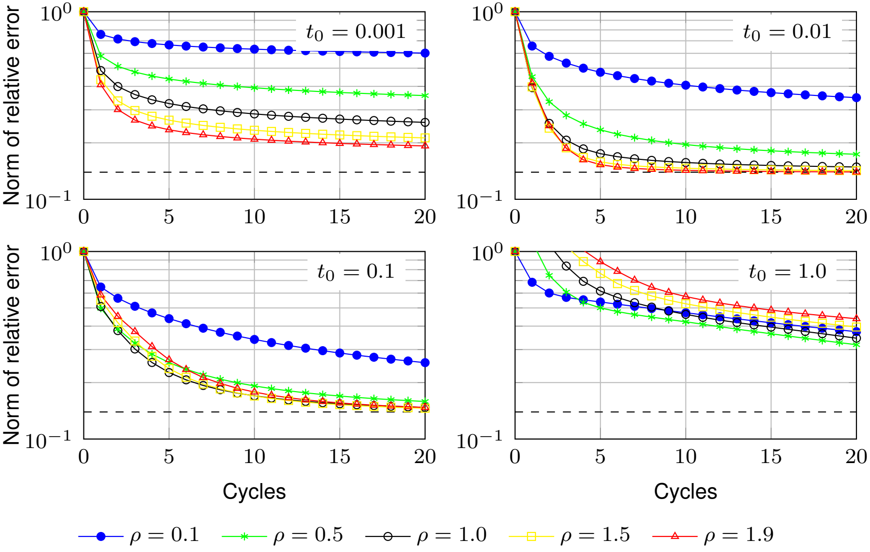
Original



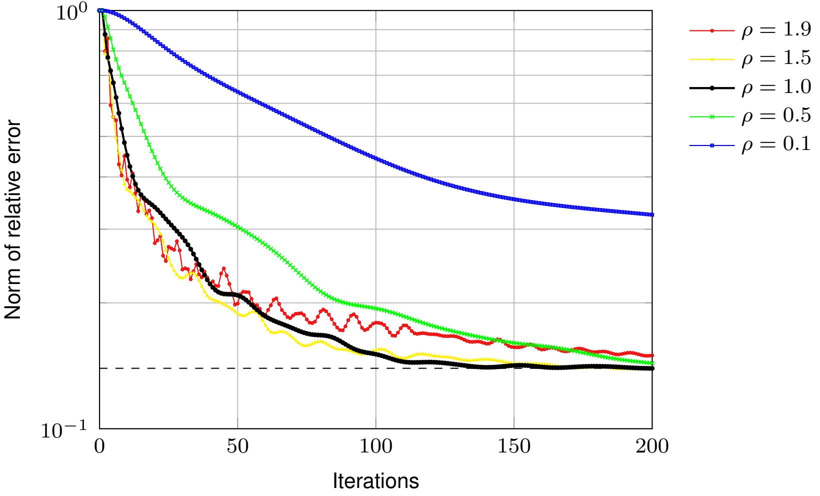
Filtered backprojection

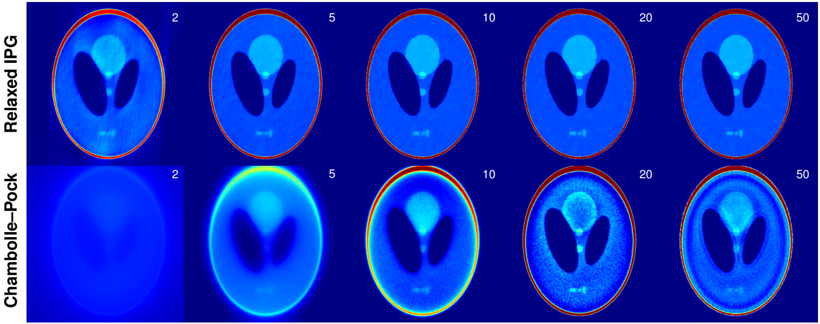
Total variation regularized LS

Relaxed incremental proximal subgradient method



Primal–dual first-order method (Chambolle & Pock)




Relaxed IPG
Chambolle–Pock
2
5
10
20
50
2
5
10
20
50

Summary

- relaxed incremental proximal gradient methods
- slow global rate of convergence
- often fast initial rate rate of convergence
- hybrid methods that transition from incremental to non-incremental method

Thank you for listening!

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