Adaptive Mesh Refinement of a Changing 1D Advection-Diffusion Problem

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Motivation

Solving time dependant PDEs requires a high resolution mesh over the domain of interest. This leads to large mesh sizes, which increase computation time. For time dependant equations, the areas of interest vary over time and having a high resolution mesh in areas where it is not needed is inefficient. In order to reduce the computation time, we investigate the possibility of a continuous mesh refinement-derefinement method where low-gradient areas' elements are coarsened to the limits of the error bound.

Contribution

- Derefinement method to mark elements with error below a minimum tolerance and merge with nearby elements of same property.
- Comparison of number of nodes after refinement and derefinement.
- Proof of error tolerances band reduction.

Mesh Generation

In order to optimize the mesh, we need to know which elements to refine and derefine. Starting from any uniform mesh, this is done in pseudocode by:

```
mark_deref = err < tol_min
mark_ref   = err > tol_max

iff mark_ref(i)=1
        refine_element(n(i))

iff mark_deref(i)=1 & mark_deref(i+1)=1
        derefine_elements(n(i),n(i+1))
```

- All elements marked for refinement are split into two equisized elements.
- Derefine if two elements in sequence are marked for it, merging the two elements.
- Uneven selection criteria due to chance of infinite looping.

Results in a 1D mesh can be seen below. The top figure 1a is an example of the selection criteria, and below in figure 1b is the resulting mesh.

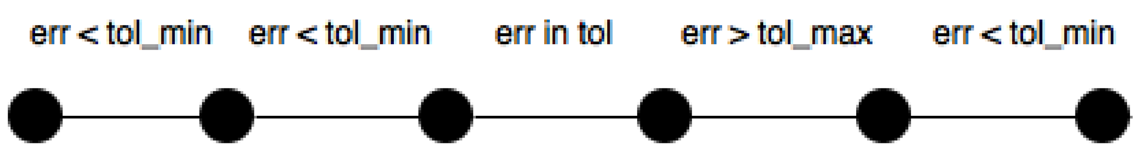

Figure 1a.

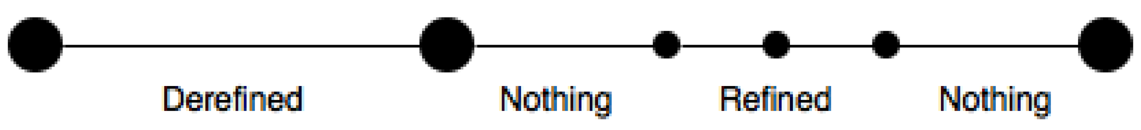

Figure 1b.

Stationary Case

The finite element solution for a stationary problem can be seen below in figure 2a for the refined mesh, and in figure 2b for both refinement and derefinement.

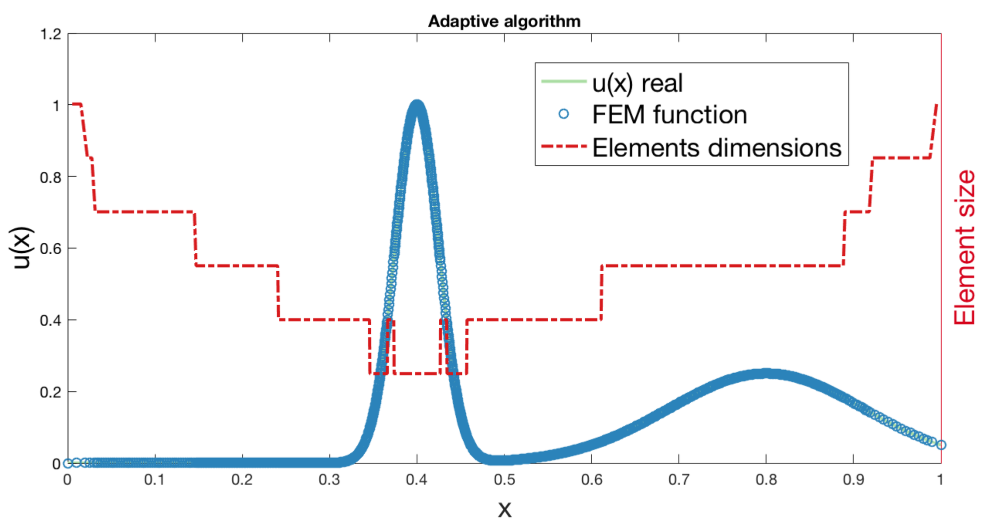

Figure 2a: Fit and element size after refinement.

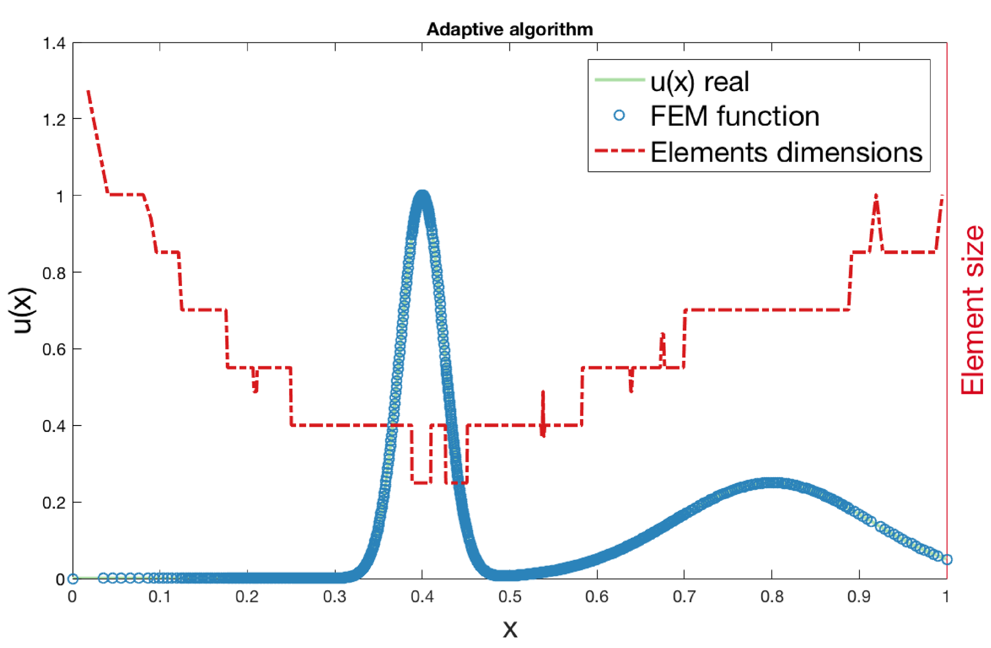

Figure 2b: Fit and element size after refinement and derefinement.

The correct working of the refinement-derefinement loop can be tested using a stationary case of FEM analysis. To perform this test we used the FEM function and error estimator calculated in exercise 1.7 of the course material. Figure 2a and 2b show the difference between a refinement and a refinement-derefinement loop respectively.

- The resolution of the function is similar in the two cases
- The number of elements in the refined mesh is 1110 while in the refined-derefined mesh is 880.

The comparison between the error for each element in the two cases is shown in Figure 3.

- Both the errors are below $\mathtt{tol}_{\max}$.
- In the refined-derefined mesh the error is bigger and more constant between the elements than in the refined mesh.

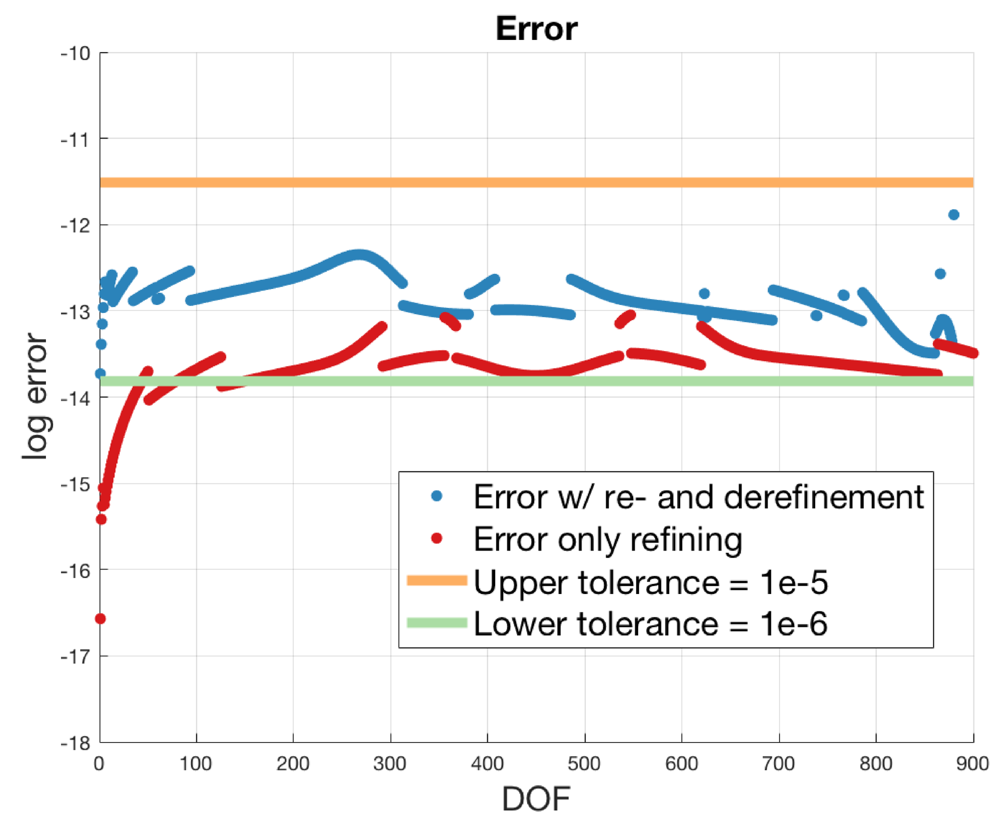

Figure 3: Error of the same function plotted with error bands.

Adaptive Mesh for Time Dependant Linear Interpolation

In order to test the adaptation of the mesh to a time dependent function, the refinement-derefinement loop has been tested with a periodic function, using the error estimator obtained in exercise 1.6 of the course material. This function estimates the error, at the same time-step, between the interpolated function with a coarse mesh and fine mesh obtained by splitting each element in two equisized elements. Figure 4a and 4b show the mesh at two different time-steps.

- The adaptive loop updates the mesh correctly with the moving function, decreasing the size of the elements in the position where more resolution is needed and increasing the elements where the function is almost linear.
- The number of elements is similar in each time-step ≈ 140.

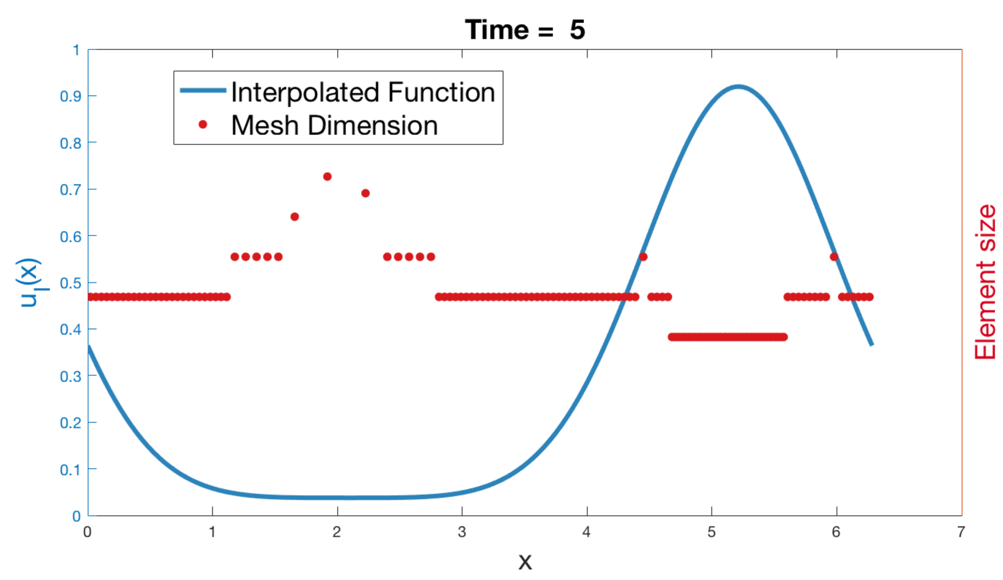

Figure 4a: Mesh size at timestep 5.

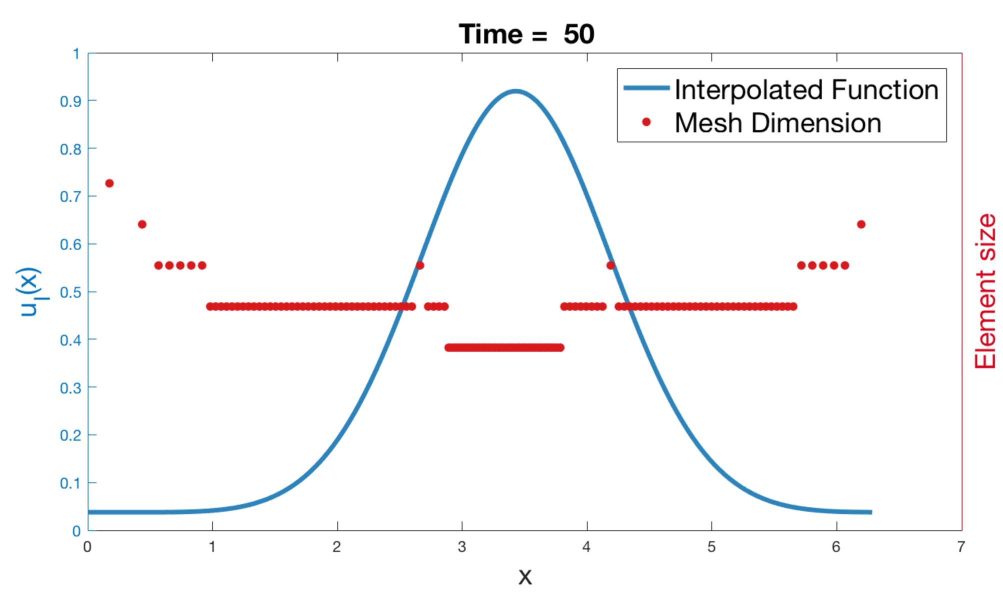

Figure 4b: Mesh size at timestep 50.

Adaptive Mesh for Time Dependant FEM Approximation

Finally, to test the performance of the adaptive loop, we have used a FEM function to solve the linear advection-diffusion problem given in exercise 1.5 of the course material, with a time dependant factor $\epsilon(t)$:

$$f = -(\epsilon(t)u')' + (\Psi u)' \tag{1}$$

where:

$$\epsilon(t) = \frac{1}{i}, \qquad 0 \le i \le 250 \tag{2}$$

$$\epsilon(t) = \epsilon(i-250), \quad 250 \le i \le 500 \tag{3}$$

The error estimator used is the one defined in exercise 1.7 of the course material. The following figures 5a, 5b and 5c give indication about the approximated FEM solution, the mesh size and the error calculated in 3 different time-steps. The value of ϵ in Figure 5a and 5c is similar, but the mesh in the latter is obtained starting from a mesh adapted to a smaller value of ϵ.

- The mesh adapts correctly in the different time-steps, with smaller elements where the gradient changes rapidly in space, and bigger elements where the gradient is more constant in space.
- The error for each time-step is inside the defined tolerances exept for some elements in which the refine-derefine loop is broken to avoid infinite loops.
- The number of elements in the two cases where ϵ is similar is roughly the same.

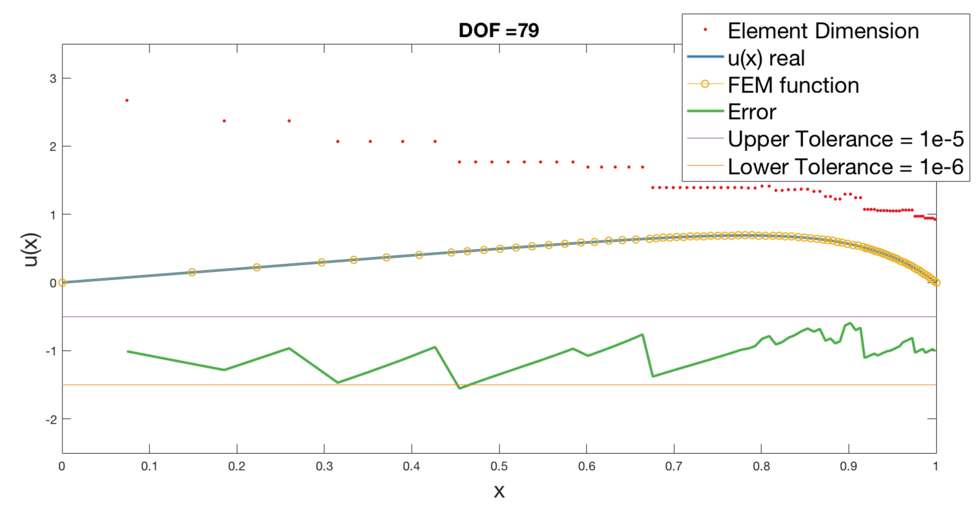

Figure 5a: $\epsilon = 0.0909$.

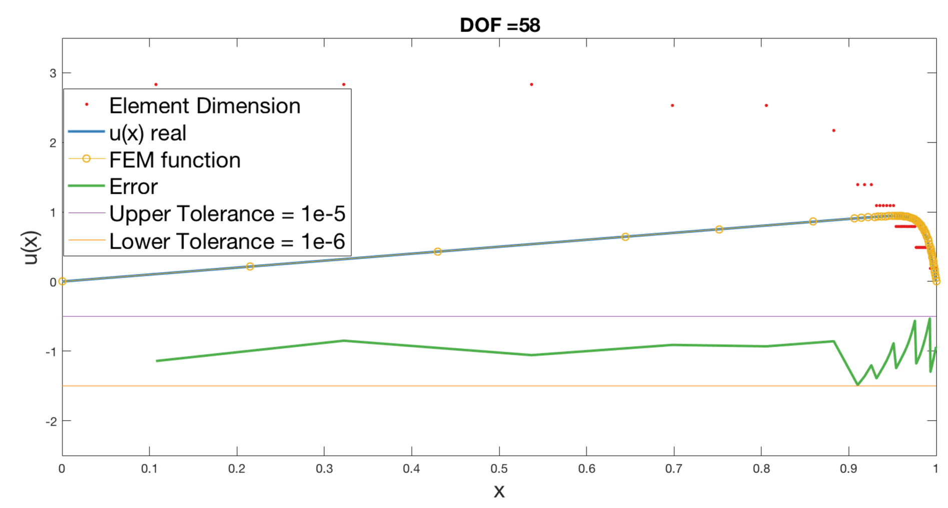

Figure 5b: $\epsilon = 0.0100$.

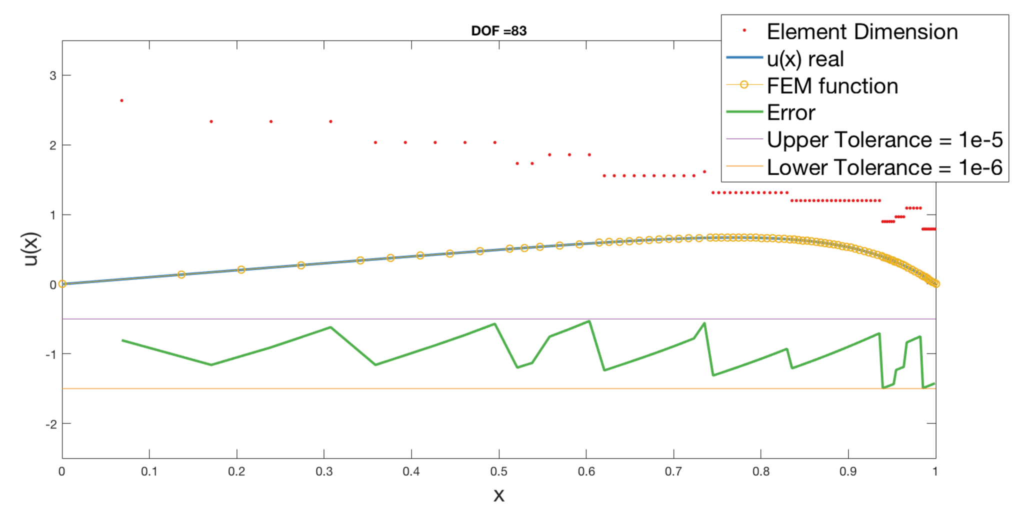

Figure 5c: $\epsilon = 0.1004$ derefined up from figure 5b.

Conclusion and Perspective

- The adaptive mesh with continuous refinement and derefinement has succeeded as a proof of concept.
- Interesting properties for time dependant problems to reduce mesh-size dynamically.
- Increasing complexity of the refinement and derefinement selection criteria, element splitting and concatenating can greatly reduce meshing time.

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